Statistics and Applications {ISSN 2452-7395(online)}
Volume 18, No. 2, 2020 (New Series), pp 1-14

Wanted! A Mathematician

Jyotirmoy Sarkar
Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, USA

Received: 21 February 2020; Revised 11 March 2020; Accepted: 14 March 2020

Abstract

How will a mathematician identify a single poisonous bottle of wine from among 1000, if she is permitted only one opportunity to make the fewest number of test subjects drink small extracts from these bottles?

Key words: Optimization; Precision; Resource allocation; Duality principle; Binary numbers; Design of experiments.

PREAMBLE

I revisit a puzzle that has proliferated the Internet in its many different incarnations. Not all sites report the solution to the puzzle; and those that do, do so in a matter-of-fact manner, without explaining how the solution was discovered or why it is optimal. This includes Coldwell (2019), which I liked the most. My objective here is to derive the optimal solution starting from first principle. Additionally, I adopt a story-telling style in hope of exposing a vast array of readers to the secrets of how a mathematician goes about practicing the creative art of Mathematical Sciences. I conclude the paper inviting the reader to solve another optimization problem.

To my family and friends, a reassurance: The story here is entirely fictitious, with no hidden agenda to promote either wine drinking, gambling or calculated killings.

WHAT'S THE PROBLEM?

1. Travel to 20 CE

Hop on a time machine, travel back to 20 CE (common era), and visit the kingdom of the mythical Irish King Conchobar mac Nessa of Ulster. The king is facing an unprecedented predicament. Consequently, he has made an edict inviting all and sundry to participate in a contest in which the winner (to be determined if no one else beats the participant's performance within the next 24 hours) will receive as reward ten thousand gold coins, and any loser (beaten by someone else within the 24 hours limit) will not only lose face, but also lose his head. Will you join the contest?

I think you should not forgo this golden opportunity - after you have derived the optimal solution (with proof) or you have carefully read this paper.

Corresponding Author: Jyotirmoy Sarkar
Email: jsarkar@iupui.edu

2. The King's Conundrum

King Conchobar amassed 1000 bottles of exotic wine, which he had collected from lands far and near, and preserved in a heavily guarded cellar. He had curated the bottles for the express purpose of indulging and impressing his select guests at the *Coronation Anniversary Celebration* coming up in five weeks' time. Unfortunately, his treasured possession was stealthily invaded by a neighboring queen's clever spy, who managed to inject one bottle with poison so lethal that anyone who drinks just a single drop will surely die — though not immediately, but in exactly 30 days. As fate would have it, the spy was quickly caught by the king's elite guards who demanded to know which bottle he had poisoned. However, the spy was unwilling to identify the contaminated bottle, even when offered one thousand gold coins as reward, for he could not trust the guards' offer. Moreover, preferring to demonstrate his total loyalty to his queen even unto death, he swallowed a fatal pill which he had brought with him for a situation just as this one and committed instant suicide — hurling the king in a conundrum.

This suicidal death of the only person who knew which single wine bottle was contaminated with poison left the king first to ponder about how to identify the offending bottle and save the remaining 999 bottles for his prestigious party; then to become progressively puzzled, bewildered and hopelessly perplexed; and eventually to write an edict offering ten thousand gold coins to anyone who could identify the poisonous bottle. He would let the identifier devise a clever experiment in which a few of his 1000 prisoners of war, whom he had captured a year ago when he had invaded the neighboring kingdom, would be forced to drink a concoction extracted from one or more bottles at least 31 days before his Anniversary. He would reward the proposer who is properly trained in the science of mathematics and in the art of exposition who could explain to him, though he himself was not a mathematician by any stretch of the imagination, that indeed the experiment would involve as few prisoners as absolutely necessary. You see, the king wanted to save as many prisoners as possible to serve as slaves, and yet with a very high probability identify the poisonous bottle. In fact, the king had resolved in his mind that on the eve of his celebration, avoiding any spectacle and arousing no suspicion from his subjects and guests, every single experimental prisoner who would survive the forced drinking would be put to death in complete secrecy.

The king's edict also included a rejoinder: Within 24 hours of a proposed solution, if someone else would discover a better solution, which would either increase the probability of correctly identifying the poisonous bottle with the same number of experimental prisoners or fewer, or reduce the number of prisoners without lowering the probability of identification, then the prize would go to the latter solver; and the former proposer would be taunted, humiliated and publicly beheaded in the infamous *Field of Gallows*.

The king sent his emissaries all over the kingdom proclaiming his edict and inviting potential contestants who would design for him the most ideal solution to identify the poisonous bottle with a high probability subject to minimizing the number of experimental subjects. Posters proliferated the marketplace, public squares and sports arena: "Wanted! A Mathematician."

On arrival at Ulster, you learn about this edict from your host family who do their best to dissuade you from participating; but you, who has the benefit of two thousand more years of accumulated human knowledge than the then citizens of Ulster, are not going to give up so easily, are you? Having realized that in order to earn the reward and to save your head (along with your face) you must not only find a solution to the puzzle, but also have the utmost confidence (via a mathematical proof) that no one else will beat your solution either by lowering the number of experimental subjects or by increasing the probability of correct identification, will you accept the king's challenge?

THINK LIKE A MATHEMATICIAN

3. Put Your Thinking Cap On

Not wishing to give up the great, albeit dangerous, opportunity, you put your thinking cap on and start to ponder over the challenge: You begin with a naïve solution that matches each wine bottle with a unique prisoner, and makes each prisoner drink one shot from the bottle allocated to him. You even think of assigning a different bottle to 999 prisoners and leaving one bottle unassigned, since if no prisoner dies, then the unassigned bottle must be the poisonous one. However, within a short time you rule out this solution because although it would identify the poisonous bottle with 100% certainty, it would also engage too many prisoners in the experiment and expose you to a risk that someone else would easily reduce the number of prisoners. Likewise, you also must discard a second solution which uses only half as many prisoners and makes each experimental prisoner drink a concoction made of one-half shot from each of the two bottles allocated to him. For in this case, while on the eve of the Anniversary you will know for sure which pair of bottles includes the contaminated one, you will not know for sure which one of this pair is the truly poisonous one. Admittedly, compared to detecting one poisonous bottle from among 1000 bottles, it is a much simpler task to detect one bottle out of two. Nonetheless, it is impossible to do so with probability exceeding 1/2, for there remains only one night before the celebration party, rendering it unfeasible to conduct a follow-up experiment!

Proceeding in this manner, you reject a whole family of designs which allocate disjoint batches of b bottles to each of $\lceil 1000/b \rceil$ prisoners (with the last prisoner perhaps being allocated fewer than b bottles), and make each experimental prisoner drink one shot made by mixing $1/b$ fraction of a shot extracted from each of the bottles allocated to him, for while the number of experimental prisoners decreases as the batch size b increases, the probability of correctly identifying the poisonous bottle decreases to only $1/b$, since you will only identify the batch that contains the poisonous bottle, but not the poisonous bottle itself.

As you ponder more over the above family of designs, all at once it dawns on you that you have inadvertently imposed an additional constraint over the solution that was neither explicitly mentioned in the king's edict, nor implied by it: While you permitted a prisoner to drink from multiple bottles, you have allowed only one prisoner to drink from each bottle! Surely someone must necessarily drink from each bottle, save perhaps one (so that at most one bottle is excluded from the experiment); but there was no requirement to restrict each bottle to only one prisoner. How can you construct a more efficient experiment (that is, involve fewer prisoners) that allocates each bottle to a multiplicity of prisoners allowing each prisoner to drink a small extract from that bottle along with extracts from all other bottles allocated to that prisoner and still identify the poisonous bottle?

4. A Sudden Inspiration

While you keep pondering over how to allocate "bottles to prisoners" and "prisoners to bottles," you hear some commotion out in the street caused by people going to the *Field of Gallows* to witness two prisoners who would be hanged, for they had broken into the king's cellar and during the chase that followed to catch them they had knocked off one bottle of wine — shattering it into a thousand pieces and ruining its content. Although curious as a cat, you resist the urge to follow the mob to the *Gallows*. Instead, you put multiple thinking caps on and come to realize two features that would affect your solution: (1) You no longer have 1000 prisoners to engage in your experimental study — your precious resource has depleted to 998 prisoners; and (2) either the contaminated bottle is among the 999 bottles still intact, or it has been already destroyed! That is, *at most one* bottle among 999 is poisonous. You say to yourself: "The number of bottles and the number of prisoners have changed; and these numbers might change again! Therefore, I must be prepared to solve the king's conundrum not only for 1000 bottles and 1000 prisoners (or for 999 bottles and 998 prisoners), but also for any number of bottles B and any number of prisoners P."

With these realizations, should you feel happy or sad? On the surface, it looks like your task has exploded out of proportion compared to the one you began with — as if the challenge has become almost insurmountable. However, on deeper reflection, a light bulb goes on over your head (this is a purely fictitious idiomatic construction, since there wasn't any light bulb around in the first century; but remember you have time traveled from the twenty-first century): "Perhaps I can solve the problem for small values first, then detect a pattern among the solutions, and eventually extend the solution to any pair (B, P)." A much harder challenge seems to have given birth to a wonderful new opportunity!!

5. Solve Some Simpler Problems First

Suppose that among B bottles *exactly one* is poisonous. You can identify the poisonous bottle for small values of B, say for 1, 2 and 3. Then if you notice a systematic pattern among the solutions, perhaps you can conjecture the solution for an arbitrary value of B, and thereafter prove that conjecture.

In fact, for $B = 1$, the problem is already solved: The only available bottle is poisonous.

For $B = 2$, hopefully the king himself could solve the problem based on his own daily experience, without having to pay a mathematician! At every meal, as the king cautiously watches, his butler takes a portion from the king's plate and eats, ensuring the king that his food is safe to eat. Translated to the problem at hand: If $B = 2$, it suffices to enlist $P = 1$ prisoner and have him drink a shot from Bottle 1. If he dies (in 30 days), Bottle 1 is poisonous and the other bottle (labelled as Bottle 0) is safe; if he survives, Bottle 1 is safe, and Bottle 0 must be poisonous.

Had the king made one prisoner drink a little from each of the two bottles, then surely the prisoner would die; and the king would not know which bottle killed him. On the other hand, if the king had enrolled two prisoners and made each prisoner drink a little from a different bottle and kept track of who drank from which bottle, he would have surely identified the poisonous bottle, but he would have acted sub-optimally according to the terms of his own edict.

What if there is *exactly one* poisonous bottle among $B = 3$ bottles? Then one prisoner is not enough; but $P = 2$ prisoners suffice. Label the bottles with serial numbers 1, 2, 3. Assign Bottle 1 to Prisoner 1, Bottle 2 to Prisoner 2, and Bottle 3 to both prisoners. Let each prisoner drink from the two bottles assigned to him. Surely, at least one prisoner must die. If both prisoners die, then Bottle 3 is poisonous; otherwise, if only Prisoner 1 dies, then Bottle 1 is poisonous; and if only Prisoner 2 dies, then Bottle 2 is poisonous.

For $B = 4$ bottles, the same reasoning above shows that two prisoners suffice to detect the single poisonous bottle with 100% certainty: Just label the newest bottle as 0 and assign it to neither prisoner. If both prisoners survive, Bottle 0 must be poisonous. Thus, in the presence of three bottles, an additional fourth bottle did not make the problem more complex: We simply do nothing to the fourth bottle. Alternatively, having learned the solution to $B = 4$, we can construct the solution to $B = 3$ simply by eliminating any one of the four bottles. Thus, we discover a multiplicity of solutions for $B = 3$. For instance, we could assign one bottle to each of the two prisoners, and set aside the third bottle, assigning it to neither prisoner. Now at most one prisoner may die. If neither prisoner dies, then the bottle that was set aside is poisonous; otherwise, whichever bottle the dead prisoner had drunk from is poisonous. Although there are multiple solutions to $B = 3$ bottles and $P = 2$ prisoners, the solution to $B = 4$ is unique.

How are the solutions to $B = 2$ and $B = 4$ interrelated? Starting from the solution to either problem, can we construct the solution to the other problem? Notice that for $B = 2$, we set aside one bottle and make one prisoner drink from the other bottle. Likewise, for $B = 4$, we set aside one bottle and make each of the two prisoners drink from exactly two bottles, giving them a common bottle to drink from and then another bottle unique to each. In the next paragraph we describe an alternative way to understand this allocation of bottles to the two prisoners that will reveal how the solution for $B = 4$ can arise out of the solution for $B = 2$.

Imagine that the four bottles are rearranged into two bundles of two bottles each — very much like two bottles are packaged together to promote a buy-one-get-one-free deal in a twenty-first century grocery store. Set aside one bundle and assign the other bundle to Prisoner 1. Then the fate of Prisoner 1 will detect which bundle contains the contaminated bottle. This is exactly the solution to the $B = 2$ case. Next, to determine which member of the suspected bundle is the contaminated bottle, we need to experiment again using a second prisoner, except that such sequential experimentation is expressly disallowed. Fortunately, we can pick one bottle from each bundle and assign the two chosen bottles to Prisoner 2 at the same time we start to experiment with Prisoner 1, and then the responses from the two prisoners will be available at the same time. Thus, each of the 4 bottles is matched to *a unique subset* of the two prisoners. Accordingly, the death of a specific subset of prisoners ($\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$) uniquely identifies the poisonous bottle.

Now we are ready to move on to the next step in the generalization: Among $B = 8$ bottles, *exactly one* is poisonous. In this case, simply form 4 pairs; allocate the pairs to two prisoners using the above solution to the $B = 4$ case, by bundling two pairs together, etc. Remember that assigning a bundle to a prisoner is the same as assigning all bottles within the bundle to that prisoner. Their fate will determine which pair contains the contaminated bottle. Simultaneously, allocate one bottle from each pair to Prisoner 3, whose fate will determine which member of the detected bundle is the contaminated bottle. More specifically, pair up Bottles 1-2, 3-4, 5-6, 7-8. To Prisoner 1 assign Bottles 5-6-7-8, to Prisoner 2 assign Bottles 3-4, 7-8, and to Prisoner 3 assign the even-numbered Bottles 2, 4, 6, 8. You may permute the bottles and/or permute the prisoners any way you like.

In this manner, for any value of $B = 2^k$, a power of 2, we can extend the above method of allocating $B = 2^k$ bottles to k prisoners.

If B is not a power of 2, simply augment some more bottles, filled with harmless water (or even keeping them empty), until there is a total of $B = 2^k$ bottles. For example, suppose that there were 15 bottles, one of which is poisonous. How will you conduct the experiment to detect the poisonous bottle? Augment a bottle of water; label it 1; and label the other bottles with serial numbers 2 through $16 = 2^4$. You enroll four prisoners, labelled 1-4. Give Prisoner 1 extracts from even numbered bottles; that is, alternately skip a bottle, include a bottle. For Prisoner 2, alternately skip two bottles, then include two bottles; that is, give Bottles 3-4, 7-8, 11-12, 15-16. To Prisoner 3, alternately skip four bottles, then include four bottles; that is, give Bottles 5-8, 13-16. To Prisoner 4, give extracts from the last eight bottles 9-16. Note that no prisoner got anything from Bottle 1, which you had augmented, and is surely not poisonous. It is straightforward to verify that depending on which bottle is poisonous, the subset of dead prisoners after 30 days will be different.

Reversing the logic, once you know which prisoners have died 30 days later, you can identify the poisonous bottle X uniquely! For example, suppose that Prisoners 1, 2 and 4 die, but Prisoner 3 is alive. Since Prisoner 4 died, X is among 9-16 (the latter half); since Prisoner 3 is alive, X is among 9-12 (the beginning half of the candidate bottles from the previous step); since Prisoner 2 died, X is among 11-12 (why?); and since Prisoner 1 died, X is 12 (since it must be even). Eureka!

6. Binary Codes to Allocate Bottles to Prisoners

For $B = 2^4$ bottles and $P = 4$ prisoners, to smartly conduct the experiment and to confidently identify the offending bottle, you may want to label the bottles with four-digit binary codes 0000 to 1111 (representing numbers 0 through 15, the previously stated serial numbers 1 through 16 reduced by one). Using these binary codes, assign to Prisoner 1 extracts from all eight bottles that have 1 in the rightmost digit; to Prisoner 2 assign all eight bottles that have a 1 in the second digit from right; etc. After 30 days, when you know the fates of all prisoners, summarize that information by writing a 0 for a live prisoner and a 1 for a dead prisoner, starting from the rightmost digit for Prisoner 1 and moving leftward prisoner by prisoner. This summary code *is* the label of the poisonous bottle!

The above strategy of allocating bottles to prisoners is easily extended to P prisoners and $B = 2^P$ bottles, when exactly one bottle is poisonous.

7. Proving Optimality

Can you prove that indeed four is the fewest number of prisoners needed when there is exactly one poisonous bottle among $B = 15$ bottles? For if you cannot, you will have no confidence that your head will remain in its proper place if King Conchobar is still reigning.

To prove optimality of our proposed solution, we utilize a duality principle at play here. It changes the original problem into an equivalent dual problem, whose solution may be easier.

The Duality Principle: Optimization problems may be viewed from either of two perspectives — the primal problem and the dual problem. It suffices to solve either problem; the other problem is immediately solved. Moreover, the solution to the primal (minimization) problem provides an upper bound to the solution of the dual (maximization) problem; likewise, the solution to the dual (maximization) problem provides a lower bound to the solution of the primal (minimization) problem.

Returning to our detection of the single poisonous bottle out of B bottles, let us state the primal and dual problems.

Primal Problem: Given B bottles, with *exactly one* poisonous among them, to determine the fewest number of prisoners P needed to detect the poisonous bottle with the highest probability.

Dual Problem: Given P experimental prisoners, to find the largest number of bottles B so that the single poisonous bottle from among B can be identified with the highest probability.

8. Solving the Dual Problem

The dual problem can be easily solved for small values of P, say for 1, 2 and 3. Then having noticed a systematic pattern in the solutions, one may conjecture a reasonable solution for an arbitrary value of P, and prove the conjecture. We follow this strategy below.

If $P = 1$ prisoner is available, we can have him drink a shot from one bottle. If he dies (in 30 days), the bottle is poisonous; if he survives, the bottle is safe. If there are 2 bottles and it is known that *exactly one* of them is poisonous, then also $P = 1$ prisoner suffices. Let him drink from one bottle and set aside the other bottle: If he dies in 30 days, then the bottle he drank from is poisonous and the other bottle is safe; if he survives, then the bottle he drank from is safe and the other bottle is poisonous. Making him drink from both bottles is futile: For then, he will surely die; and we would not know which bottle killed him.

Next, we must explain that if there are three bottles with *exactly one* of them poisonous, then $P = 1$ prisoner is not sufficient to detect the poisonous bottle. If the prisoner drinks from two or more bottles and dies, we cannot identify which bottle killed him; if he drinks from only one bottle and survives, we cannot tell which of the remaining two bottles is poisonous. Thus, with $P = 1$ prisoner, we can detect the single poisonous bottle from among at most $B = 2$ bottles.

Now consider the situation when there are two bottles of wine and *at most one* of them is poisonous. In this case, one prisoner will not suffice, you will need two prisoners. Here is why. With only one prisoner available, we have two choices: (1) Make him drink from one bottle. If he dies on the 30th day, we know the bottle he drank from is poisonous; and the other bottle is safe. If he survives beyond the 30 days, we know the bottle he drank from is safe; and the second bottle may be either safe or poisonous, but we will not know the complete truth. (2) Make the prisoner drink a little from each of the two bottles. If he survives, then both bottles are safe. If he dies, then one of the bottles is poisonous; but we do not know which one. Thus, in each case, we fail to discover complete information about the two bottles. Therefore, we must enroll a second prisoner in the experiment; assign one bottle to each; make them drink a portion from the assigned bottle. If both prisoners survive beyond 30 days, then both bottles are safe. If not, the dead prisoner must have drunk from the poisonous bottle and the surviving one from the safe bottle. Note that both prisoners cannot die since *at most one* bottle is poisonous.

Let us return to the case when *exactly one* of the bottles is poisonous. If $P = 2$ prisoners are available, we can double the number of bottles to $B = 4$. Pair up the bottles to form two bundles. Simply use Prisoner 1 to detect the bundle with the poisonous bottle (ensuring that the prisoner drinks from both bottles within the bundle assigned to him). Simultaneously, choose one bottle from each bundle and assign them to Prisoner 2 to detect which member of the bundle is the poisonous bottle. Label the bottles with binary codes 00, 01, 10, 11. Then let Prisoner 1 drink from the second and the fourth bottles (which in binary code have 1 in the rightmost digit), and Prisoner 2 from the third and the fourth bottles (which have 1 in the leftmost digit). If both prisoners die, then the fourth bottle is poisonous; otherwise, if only Prisoner 1 dies, then the second bottle is poisonous; if only Prisoner 2 dies, then the third bottle is poisonous; finally, if none of the prisoners dies, then the first bottle, from which neither prisoner drank, is poisonous.

We should also check that with $P = 2$ prisoners available, it is not possible to detect the single poisonous bottle from among five bottles. To prove this impossibility, for each bottle, ask yourself: "To whom is the bottle assigned?" There are exactly four possible answers: The bottle is assigned to both prisoners, only to Prisoner 1, only to Prisoner 2, to neither prisoner. Therefore, by the pigeonhole principle [see Wikipedia (2019)], at least two bottles must be assigned to the exact same subset of prisoners. Should every member of that subset of prisoners die and no other prisoner die, then we would not know which of the two or more bottles assigned to that subset of prisoners is poisonous.

We leave to the reader to study the situation when there are four bottles of wine and *at most one* of them is poisonous. Two prisoners will not suffice, you will need a third prisoner.

By now a clear pattern has emerged, which we state as a Theorem.

Theorem 1: Exactly P prisoners suffice to detect the single poisonous bottle from among $2^{P-1} < B \leq 2^P$ bottles; but fewer than P prisoners do not suffice. If among 2^P bottles *at most one* is poisonous, then we must enroll $(P+1)$ prisoners.

Proof: Suffices it to prove the theorem for the largest value of B, namely, 2^P. (For fewer than 2^P bottles, fill $(2^P - B)$ additional bottles with safe-to-drink water and conduct the experiment for 2^P bottles.) Label the bottles (after permuting them randomly) with serial numbers 0 through $(2^P - 1)$ written in binary codes consisting of P digits ranging from $(000 \ldots 0)$ to $(111 \ldots 1)$. Then assign to Prisoner j all those bottles that have 1 in the j-th digit from right. In other words, there is a one-to-one correspondence between the bottles and all possible subsets of P prisoners. Therefore, the subset of prisoners who die in 30 days identifies the poisonous bottle with 100% accuracy: The binary code for the poisonous bottle has in digit j from right the value 1 if Prisoner j is dead, and the value 0 if Prisoner j is alive.

If fewer than P prisoners are available, by the pigeonhole principle multiple bottles will have to be assigned to the same subset of (fewer than P) prisoners. Should that subset of prisoners and no other prisoner die, then we would not know which one of these multiple bottles is poisonous.

When there are 2^P bottles of which *at most one* is poisonous, P prisoners will not suffice: One more prisoner must be enrolled and made to drink a shot from Bottle 000…0 (from which none of the previous P prisoners drank) to determine whether this bottle is safe or poisonous, just in case no other prisoner dies.

This completes the proof of the theorem. □

Applying Theorem 1, we conclude that it suffices to enlist 10 prisoners in King Conchobar's experiment to detect with complete certainty *at most one* poisonous bottle from among 999 bottles, since $2^9 < 999 < 2^{10}$; but 9 prisoners will not do. Go ahead and accept King Conchobar's challenge; just remember to pass on as royalty 15% of your reward to yours truly when you do safely return to the twenty-first century.

9. Executing the Experiment in Practice

To maintain complete secrecy and absolute control over the experiment, the King himself should decide who will drink from which bottle (after he learns the strategy from the mathematician). In complete secrecy of his cellar, he should prepare 10 cups with distinct IDs monogrammed on them so that he would know who drank the cup. He should arrange these cups in random order in 10 positions. Then he should make tags with labels ranging from 0 to 1023, written in ten-digit binary codes such as 0000001101(=13) or 1101110100 (=884), but discard the $22 = \binom{10}{0} + \binom{10}{1} + \binom{10}{9} + \binom{10}{10}$ tags that have 0, 1, 9 or 10 ones in them (that is, discard serial numbers 0 to 10 and 1013 to 1023). Although not necessary, in order to achieve a perfect balance, the king should augment the bottles of wine with a few more bottles of water for a total of 1002 bottles; and assign a unique tag to each of the 1002 bottles in a random order.

The king should prepare what goes into each of the 10 cups, where each cup corresponds to a digit (position) of the binary code. From each bottle, labeled with a unique binary code, he should draw a small amount of wine (say, 1/4 ml if each bottle contains 1 liter) to put into each cup that corresponds to a digit (position) with value 1, and not into the other cups that correspond to digits (positions) with value 0. (He can use syringes to extract wine from the bottle without opening the cork, provided he carefully washes any syringe clean before reusing it.) Each cup will contain the concoction made up of portions drawn from exactly half of the 1002 bottles, thereby containing only 125.25 ml total. This is because every digit (position) has as many 1's as 0's — the balance we referred to earlier.

Thereafter, the king should make a public announcement that he will not only set free but also elevate to nobility ten prisoners on the auspicious occasion of his anniversary — the ten who are judged winners in a series of athletic competitions to be held immediately. This will ensure that the prisoners enrolled in his crafty experiment are healthy, will likely not die of any other cause in the next one month, and will participate willfully and joyfully, oblivious to his devious scheme. The king will invite these 10 athletic winners to a royal dinner, where they will be served the cup with the secret ID matched to each experimental prisoner.

For such an experiment to be successful (from the king's perspective), the king must ensure that none of his experimental subjects dies during these 30 days for any other reason. Perhaps he should invite them to dinner every evening for the next 30 days on pretext of teaching them proper manners of nobility, but truly for keeping attendance and checking on their health. To ensure absolute certainty that no one will kill himself or another participant enrolled in the experiment, he should assign guards and physicians to look after their total wellbeing. It is of paramount importance that he knows exactly which subset of the 10 prisoners died because of unknowingly drinking from the poisoned bottle — for that subset of dead prisoners will uniquely identify the poisonous bottle.

On the 31st day, the king will know which experimental participants have died. Whichever unique bottle was assigned to this subset of dead prisoners is the poisonous one!

While this subset may be of size 2 to 8, on average 5 subjects are expected to die of poisoning. Of course, to rule out any future information leak, the king will very likely renege on his promise; and kill all surviving experimental participants. The king can now enjoy the remaining bottles of wine (minus the 1/2-2 ml drawn out of each) and send the one "special bottle" as a gift to the neighboring queen, with "PEACE" inscribed on it.

ACT LIKE A STATISTICIAN

10. Connection to *Design of Experiments*

We narrated the above fictitious short story hoping to inspire students to learn the beautiful and useful art and science of experimental designs. How is the story of King Conchobar related to *Design of Experiments*?

First, we find it astonishing that back in 20 AD, King Conchobar literally heeded the sage advice of our modern-day statisticians:

"Experimentation is an essential part of any problem of decision-making. Whenever one is faced with the necessity of accepting one out of a set of alternative decisions, one has to undertake some experiments to collect observations on which the decision has to be made."

— Shah and Sinha (2012)

In the story, we can substitute some terminologies from *Design of Experiments*: For instance, each bottle of wine can be thought of as a treatment to be assigned to one or more prisoners, each of whom can be thought of as an experimental unit (on which we can apply as many treatments as we wish).

Since only one treatment is fatal and all other treatments are innocuous, we are essentially conducting a hypothesis test among 1000 hypotheses (each stating one particular bottle is poisonous or all 999 bottles are innocuous), based on data consisting of a single dichotomous response variable — the prisoner is either dead or alive after 30 days. Indeed, since the king has diluted the poisonous drink by a factor of 1 in 501, and each cup either contains ¼ ml of poisonous wine or none at all, the poison remains potent; and it will surely kill any unfortunate soul that drinks it.

In fact, our design is so well thought out that we need no sophisticated analyses: The responses from the ten subjects (almost magically) suffice to identify the poisonous bottle! Thus, the hypothesis test is 100% accurate, with zero probability of Type I error (declaring a bottle poisonous when it is not) and zero probability of Type II error (declaring a bottle safe when it is poisonous), provided that no one dies from a cause other than drinking from the poisoned bottle.

Our story illustrates the following two quotes from leading experts in *Design of Experiments* on the importance of choosing the experimental design carefully:

"If the experimental design is wisely chosen, a great deal of information in a readily extractable form is usually available, and no elaborate analysis may be necessary."

— Box, *et al*. (2005)

"If you do the pre-experiment planning carefully and select a reasonable design, the analysis will almost always be relatively straight-forward. In fact, a well-designed experiment will sometimes almost analyze itself!"

— Montgomery (2013)

Our story also demonstrates some best practices propounded by experts in *Design of Experiments*: An appropriate experimental design is a solution to an optimization problem that expends the least amount of resources and still extracts enough information to resolve an issue with the highest possible precision. One must be mindful of utilizing resources to their maximum potential; practice all kinds of safeguards to reduce biases in the study; and above all, one must not compromise the quality of knowledge one seeks to discover.

A well-known strategy to reduce biases in an experiment is to incorporate proper randomization (that is, to the extent permitted, units must be chosen at random to receive a treatment or a combination of treatments). To accomplish this, we advised the king to randomly assign the binary codes to the wine bottles and to randomly permute the monogrammed cups in positions 1 through 10. Another useful concept in experimental design is balance; for example, each experimental unit must receive the same number of treatments. In the king's experiment, we advocated augmenting two bottles of water to ensure that every cup receives extracts from 501 bottles and therefore contains the same amount of wine (125.25 ml). On the other hand, every treatment (bottle) was applied to 2-8 experimental units (prisoners) according as the binary code assigned to the treatment. Another key concept in implementing a designed experiment is to permit replication (that is, multiple units receive the same treatment combinations) with an aim to reduce associated statistical errors. Since the king's experimental design already achieves a 100% accuracy, there is no need for further reduction of error. Hence, no replication is needed, or recommended.

Lastly, the sanctity of the response variable must be preserved. In the king's experiment, the cause of death must be none other than consumption of poisonous wine. Therefore, we advised the king to identify the healthiest prisoners through athletic competitions, to offer them freedom and a bright future to ensure their cooperation and desire to survive, and to keep them under watch by guards to prevent any homicide and to appoint physicians to treat them of any other ailment.

11. A Variation on the Detection Problem

Recall that the prisoners who drink from the poisonous bottle die not immediately, but 30 days later. Implicitly we are assuming that death can occur at a random time before the 30 days are over; that is, during the time period (0, 30]. Other than knowing the support, the exact probability distribution of the delay time between drinking and death is unknown. This was the reason for restricting the experiment to only one opportunity; that is, make all experimental subjects drink wine at the same time.

However, suppose that death will occur sometime during the period (29½, 30] days after drinking the poisonous wine. Then the experiment can be conducted on four successive days. In such a case, the king can get by with engaging only 8 prisoners in his experiment: On Day 1, he will extract wine from Bottles 1-256 to assign to the 8 prisoners according to the binary rule described in Section 6. On Day 2, he will extract wine from Bottles 257-512 to give to the same 8 prisoners. On Day 3, he will use Bottles 513-768 to assign to the same 8 prisoners. On Day 4, he will use Bottles 769-1000 (plus 24 water bottles) to assign to the same 8 prisoners.

If some prisoners die on Day 31 minus half a day, then using the subset of dead prisoners, the king will identify the poisonous bottle from among 1-256. Otherwise, if all prisoners survive on Day 31, then all these bottles are innocuous, and the king must wait to check the survival status on Day 32. If some prisoners die on Day 32 minus half a day, then using the subset of dead prisoners, the king will identify the poisonous bottle from among 257-512. Otherwise, if all prisoners survive on Day 32, all these bottles are innocuous. And so on.

Referring to *Design of Experiments* literature, we are reminded of a crossover design, in which the same unit receives different treatment combinations in different time periods provided that the response is attributable to the correct treatment combination. For the king's experiment, the response on each prisoner is no longer a binary variable taking values 1 or 0; rather it is a quintenary variable taking values 1, 01, 001, 0001, 0000, according as the time of death is Day 31, 32, 33, 34 or no death at all respectively. Thus, when the time of death after drinking from the poisonous bottle is within half a day of the 30^{th} day mark, we have reduced the number of experimental units to 8, without compromising the inference.

Carrying this argument further, if anyone drinking from the poisonous bottle will surely die within 23 hours, then the king can conduct his devious experiment on 32 nights, requiring only 5 prisoners and utilizing 32 distinct bottles each night. Each prisoner's status will be one of 33 possible outcomes: Death before Day 2, 3, …, 33 or Survival. Thus, with more precise information on the response variable, the sample size can be reduced without sacrificing the quality of inference. Moreover, the experiment can be terminated as soon as at least one prisoner dies.

THINK SOME MORE

12. Further Study

We invite the astute reader to solve another optimization problem.

Exercise

Suppose that a building has 1000 floors above ground. If you drop a marvelous marble from floor N or above, the marble will surely break; but if you drop it from any lower floor, there will be absolutely no effect of the impact. Being as good as new, it can be dropped again (from a higher floor). Every time you want to drop a marble, you must pay ₹10 (with a coupon) to take the elevator to the desired floor. Taking the down-elevator to check whether the marble is intact or broken costs you nothing. At the start of the experiment, you can buy any number of marbles for ₹50 each and any number of coupons for ₹10 each. At any other time, you cannot buy or sell a marble or a coupon. What is the least amount of money you must spend to determine N with complete certainty?

Note that you must minimize the *maximum* amount of money you may spend, and not minimize the expected amount of money.

My answer to the Exercise is ₹330; and I offer this answer in good faith that in case you beat my solution within 24 hours, you won't demand my head. Partial explanation of my answer is given in the Appendix. Can you find a better solution? Or, can you prove that my solution is indeed optimal? Please email me (at jsarkar@iupui.edu) a better solution or a proof that my solution is the best.

Acknowledgements

The author is indebted to Professor George D. Gopen, who through his workshop, articles and books taught him Gopen's *Reader Expectation Approach* to writing science; and who inspired him to incorporate writing into the classroom. The author thanks all students who participated in a writing assignment on this topic and suggested some refinements. The author expresses gratitude to Monica Pena Pardo for giving detailed feedback on an earlier draft. Finally, the author sincerely thanks the Editor for his generous suggestions to improve the paper's impact.

References

Coldwell, N. *A Collection of Quant Riddles with Answers, Riddle 61*. Retrieved on November 07, 2019 from http://puzzles.nigelcoldwell.co.uk/.

Wikipedia, The Free Encyclopedia. *Pigeonhole Principle.* Retrieved on November 07, 2019 from https://en.wikipedia.org/wiki/Pigeonhole_principle.

Shah, K. R. and Sinha, B. K. (2012). *Theory of Optimal Designs*. In: J. Berger, S. Fienberg, J. Gani, K. Krickeberg and B. Singer (Editors), *Lecture Notes in Statistics*, **54**, New York, Springer-Verlag.

Box, G. E. P., Hunter, J. S. and Hunter, W. G. (2005). *Statistics for Experimenters: Design, Innovation and Discovery*. 2nd Edition, Wiley, New York.

Montgomery, D. C. (2013). *Design and Analysis of Experiments*. 8th Edition, Wiley, New York.

APPENDIX

My Answer to the *Exercise* in Section 12

Do not read this Appendix until after you have tried to solve the Exercise.

I will buy four marbles; and I plan to drop the first marble from floors (in order)

$$286, 506, 671, 791, 875, 931, 966, 986, 996, 1000.$$

(To understand where these floor numbers came from, study their successive differences). If the marble does not break at all, then N exceeds 1000; and I will have three unused marbles and three unused coupons. Otherwise, if the marble breaks during any one of the above ten drops, then logic establishes that we need a total of 13 drops to determine N with certainty using the remaining three marbles. Let me illustrate one such situation (leaving all the rest to the reader): Say, the first marble breaks after the 4th drop. Then $671 < N \leq 791$; and the problem reduces to three marbles and 120 floors. In this case, identifying N requires 9 more drops, as explained below (and so a total of $4 + 9 = 13$ drops are needed).

Drop the second marble from floors (in order)

$$707, \quad 735, \quad 756, \quad 771, \quad 781, \quad 787, \quad 790.$$

Say, the second marble breaks after the 3rd drop (all other possibilities are left to the reader). Then $735 < N \leq 756$. So, the problem reduces to two marbles and 21 floors, which requires 6 more drops (which justifies the required 3+6 = 9 drops after the first marble breaks): Drop the third marble from floors $741, 746, 750, 753, 755$. If the third marble breaks during the second drop, then $741 < N \leq 746$ (again, all other possibilities are left to the reader). So, the problem reduces to one marble and 4 floors, requiring 4 more drops (from floors $742, 743, 744, 745$) and justifying the required 2+4 = 6 drops after the second marble breaks). Thus, in the worst case, the total number of drops of all four marbles is 4+9 = 13, and I must be prepared to spend a maximum total of $4 \times ₹50 + 13 \times ₹10 = ₹330$.

I claim that my choice of buying four marbles, followed by the above strategy of sequentially determining which floor to drop the marbles from, is indeed wise. To justify my claim, let me document what my prospect will be if I buy fewer than four marbles. First, if I buy only one marble, I must be prepared to spend at most ₹10,050 (dropping the marble from floors 1, 2, 3, …). I cannot risk skipping any floor: For if I do and the marble breaks, then N can be any one of the floors I have skipped or the one from which I dropped the marble last. However, in this case, I have no marble left to determine N with certainty! Second, if I buy two marbles, then using the best possible strategy, I may require up to 45 drops (why?). Therefore, I must spend ₹550 in the worst case. Third, if I buy three marbles, then using the best possible strategy, I may have to drop the marbles a total of at most 19 times (why?). Hence, I must spend ₹340 in the worst case. All these options lead to spending more than ₹330, which I agreed to spend to buy four marbles and 13 drops.

What if I buy more than four marbles? If I buy five marbles, then using the best possible strategy, I may need a maximum of 12 drops (why?). So, I must spend ₹370 in the worst case. If I buy six marbles, then using the best possible strategy, I may have to drop the marbles up to 12 times (why?). Hence, I must spend ₹420 in the worst case. Thus, compared to the best strategy using five marbles, the best strategy with a sixth marble does not reduce the number of drops! It was a waste to buy the sixth marble. Buying seven or more marbles will already cost me more than ₹330 even before I buy any elevator coupons! Hence, I recommend buying four marbles and 13 coupons. Can you beat my choice or prove that it is the optimal choice?

The above solution is intricately associated with the relative cost of a marble to a coupon for each elevator ride up. When this relative cost changes, the answer may change. For example, if the cost of each marble decreases to ₹10 but the cost of each coupon remains at ₹10, then I have *two* best choices: Either buy four marbles and 13 coupons; or buy five marbles and 12 coupons. For each choice I will incur a total cost of ₹170. On the other hand, if the cost of each marble increases to ₹100 but the cost of each coupon remains at ₹10, then my best choice is to buy three marbles and 19 coupons incurring a total cost of ₹490. What if the cost of each marble is ₹1000, but the cost of each coupon remains at ₹10? I leave the discovery of the best solution(s) to the reader. In every case, I invite the reader to find a better solution or to prove the optimality of my solution.