COMMUTATIVE SUBALGEBRA OF A SHUFFLE ALGEBRA ASSOCIATED WITH QUANTUM TOROIDAL $\mathfrak{gl}_{m|n}$

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Abstract. We define and study the shuffle algebra $Sh_{m|n}$ of the quantum toroidal algebra $\mathcal{E}_{m|n}$ associated to Lie superalgebra $\mathfrak{gl}_{m|n}$. We show that $Sh_{m|n}$ contains a family of commutative subalgebras $\mathcal{B}_{m|n}(s)$ depending on parameters $s = (s_1, \dots, s_{m+n})$, $\prod_i s_i = 1$, given by appropriate regularity conditions. We show that $\mathcal{B}_{m|n}(s)$ is a free polynomial algebra and give explicit generators which conjecturally correspond to the traces of the s-weighted R-matrix computed on the degree zero part of $\mathcal{E}_{m|n}$ modules of levels ± 1.

1. Introduction

We continue the study of integrable systems of XXZ-type related to quantum toroidal algebras, see [FHHSY], [FT], [FJMM2], [FJMM3], [FJM1]-[FJM4], [FJMV]. In this paper we make the first steps in the case of quantum toroidal algebra $\mathcal{E}_{m|n}$ of type $\mathfrak{gl}_{m|n}$.

The quantum toroidal algebra $\mathcal{E}_{m|n}$ of type $\mathfrak{gl}_{m|n}$ was introduced and studied in [BM1], [BM2]. The algebra $\mathcal{E}_{m|n}$ depends on parameters q_1, q_2, q_3, $q_1q_2q_3 = 1$, and has several presentations depending on the choice of the Dynkin diagram for $\mathfrak{gl}_{m|n}$. In this paper we use only the standard parity. It is expected that $\mathcal{E}_{m|n}$ has a universal R-matrix and therefore one could construct a quantum integrable system generated by the transfer matrices.

In the even case, (that is in the case of $n = 0$), the corresponding integrals of motion can be written explicitly in the form of multidimensional contour integrals, [FJM3]. These formulas can be used to find interesting generalizations, see [FJMV], [FJM4], to prove dualities, [FJM2], to look for connections to other models, [FJMV]. The spectrum of integrals of motion is known to be given in terms of the Bethe ansatz, [FJMM1], [FJMM2], [AO]. All of that is missing so far in the supersymmetric case.

In this paper we take the shuffle algebra approach developed in the even case in [FT]. The shuffle algebra is a remarkable realization of the nilpotent part of a quantum group by spaces of rational functions symmetric in several groups of variables with explicit poles and satisfying explicit vanishing wheel conditions, see [N1], [N2], [FT], [T1], [T2], Sections A.1 and 2.2. The product of such functions in the shuffle algebra is given by the shuffle star product, see (2.2). The exchange functions in the star product come from the quadratic relations, and the wheel conditions from the Serre relations.

We introduce the shuffle algebra $Sh_{m|n}$ associated to $\mathcal{E}_{m|n}$. The definition $Sh_{m|n}$ is rather standard. However, the isomorphism of the shuffle algebra and the nilpotent part of $\mathcal{E}_{m|n}$ is not trivial, and we do not pursue it here, see [N1], [N2] for the even case. Instead, we find a family of commutative subalgebras $\mathcal{B}_{m|n}(s)$ of $\mathcal{E}_{m|n}$ depending on parameters $s = (s_1, \dots, s_{m+n})$, $s_1 s_2 \cdots s_{m+n} = 1$, which we call the *Bethe algebras*. We use two main ideas. First, it is expected that the Bethe algebra is characterized by the ratio of two limits when some of the variables in the rational function $F \in Sh_{m|n}$

Key words and phrases. shuffle algebra, integrals of motion, Bethe algebra, quantum toroidal $\mathfrak{gl}_{m|n}$.

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go to zero and when the same variables go to infinity, see (4.2). Second, we can compute a few of elements of the Bethe algebra explicitly using the trace of the hypothetical R-matrix, see the Appendix. This happens to be sufficient to conjecture the answer, and then we prove it.

Our arguments often follow the even case of [FT]. However, there are several technical differences worth mentioning.

First, in the bosonic realization of the $\mathcal{E}_{m|n}$ level one modules, part of the generators $E_i(z)$ act not as a single vertex operator but as a sum of two, see [BM1]. As the result, in contrast to the even case, the matrix coefficients of the R-matrix do not factorize. Instead they are given in terms of products of functions $I^c_{M,N}$ of two groups of variables:

$$I^c_{M,N}(y_1,\dots,y_M;z_1,\dots,z_N)=\frac{1}{\prod_{i<j}(y_i-y_j)}\prod_{a=1}^{M}\Big(\frac{q^cT_{q,y_a}-q^{-c}T^{-1}_{q,y_a}}{q-q^{-1}}\Big)\Big(\frac{\prod_{i<j}(y_i-y_j)}{\prod_{i,j}(y_i-z_j)}\Big),$$

where $(T_{q,y_a}f)(y_1,\dots,y_M)=f(y_1,\dots,qy_a,\dots,y_M)$. The functions $I^c_{M,N}$ satisfy a number of identities and curious properties, see Lemmas 5.1–5.4, (A.4), and seem to be an important ingredient.

Second, in the proof we make use of the fusion homomorphism which maps the subalgebra of $Sh_{m|n}$ of weight zero to a subalgebra of weight zero in $Sh_{m|n-1}\otimes Sh_{1|0}$ (with different parameters q_1,q_2), see Proposition 3.2, cf. [FJM1]. It allows us to use the induction on the rank of the algebra. This homomorphism is not present in [FT].

Third, unlike [FT], we are not able to show that our commutative subalgebra coincides with the algebra of transfer matrices restricted to degree zero. We do expect this to be true, see the Appendix.

Fourth, similar to [FT], we show that the Bethe algebra $\mathcal{B}_{m|n}$ is a free polynomial algebra with explicit generators which we expect to correspond to the first m skew-symmetric powers of the vector representations and first n skew-symmetric powers of the covector representations of $\mathfrak{gl}_{m|n}$.

Finally, it is desirable to give the elliptic deformation of $\mathcal{B}_{m|n}$ corresponding to the full trace of the R-matrix (not restricted to the degree zero) as in [FT] for the even case and translate this knowledge into explicit integrals of motion as in [FJM3] in the even case.

Note also, that the case $m=n$ is excluded from our considerations. While many definitions make sense in that case too, the proofs seem to fall apart. This case was not treated in [BM1], [BM2] either.

The paper is constructed as follows. In Section 2 we describe the generalities about the $\mathfrak{gl}_{m|n}$ shuffle algebra. In Section 3 we give the fusion homomorphism of shuffle algebras which is used later for the induction with respect to the rank. The main results of this section are Proposition 3.2 and Proposition 3.4. In Section 4 we define the Bethe subalgebra of the shuffle algebra. We prove an upper estimate for the size of the Bethe algebra, Proposition 4.4. In Section 5, we give explicit generators of the Bethe algebra. We finally prove that the Bethe algebra is commutative in Theorem 5.15. In the Appendix, we give examples of computation with traces of R-matrices which illustrates the origin of our formulas.

Notation. Throughout the paper we fix $q,d\in\mathbb{C}^\times$ and set

$$q_1=q^{-1}d\,,\quad q_2=q^2\,,\quad q_3=q^{-1}d^{-1}\,.$$

We have $q_1q_2q_3=1$. We assume that $q_1^{n_1}q_2^{n_2}q_3^{n_3}=1$, $n_1,n_2,n_3\in\mathbb{Z}$, holds only if $n_1=n_2=n_3$. Fixing a branch of logarithm $\log q_i$ we set $q_i^r=e^{r\log q_i}$ for $r\in\mathbb{Q}$.

If a statement P is true we set $\theta(P)=1$, and $\theta(P)=0$ otherwise.

2. Shuffle algebras

2.1. **Algebra $Sh_{m|n}$.** Let m, n be positive integers with $m \neq n$. In this section we define the shuffle algebra $Sh_{m|n}$ for the quantum toroidal algebra of type $\mathfrak{gl}_{m|n}$.

Let $Sh_{m|n}$ be a graded complex vector space:

$$Sh_{m|n} = \bigoplus_{\mathbf{N} \in \mathbb{Z}_{\geq 0}^{m+n}} Sh_{m|n,\mathbf{N}}\,,$$

where each graded component $Sh_{m|n,\mathbf{N}}$ of degree $\mathbf{N} = (N_1, \dots, N_{m+n})$ is the space of all rational functions $F(x_1; \cdots; x_{m+n})$ in the variables $x_i = (x_{i,1}, \dots, x_{i,N_i})$, $1 \leq i \leq m+n$, satisfying the following conditions (i)–(iii) below.

Here and after the index i of x_i is to be read modulo $m+n$, i.e. $x_{i+m+n,a} = x_{i,a}$. We call this index the color of the variable. We call variables $x_{m,a}, x_{m+n,a}$ fermionic, all other variables bosonic. We say that a bosonic variabel $x_{i,a}$ is before the equator if $i = 1, \dots, m-1$ and after the equator if $i = m+1, \dots, m+n-1$.

(i) F has the form

$$F(x_1; \cdots; x_{m+n}) = \frac{f(x_1; \cdots; x_{m+n})}{\prod_{i=1}^{m+n} \prod_{\substack{1 \leq a \leq N_i \\ 1 \leq b \leq N_{i+1}}} (x_{i,a} - x_{i+1,b})}\,,$$

where $f(x_1; \cdots; x_{m+n})$ is a Laurent polynomial.

(ii) F is symmetric in bosonic variables x_i, $i \neq m, m+n$, and skew-symmetric in fermionic variables x_m, x_{m+n}.

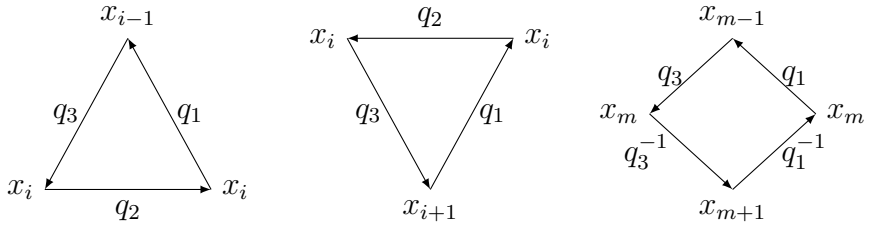
(iii) F vanishes if one of the following holds for some a, b, c, d:
- (a) $x_{i,b} = q_3 x_{i-1,a}$ and $x_{i,c} = q_2 x_{i,b}$, $1 \leq i \leq m-1$;
- (b) $x_{i,b} = q_2 x_{i,a}$ and $x_{i+1,c} = q_3 x_{i,b}$, $1 \leq i \leq m-1$;
- (c) $x_{m,b} = q_3 x_{m-1,a}$, $x_{m+1,c} = q_3^{-1} x_{m,b}$, and $x_{m,d} = q_1^{-1} x_{m+1,c}$;
- (d) $x_{i,b} = q_3^{-1} x_{i-1,a}$ and $x_{i,c} = q_2^{-1} x_{i,b}$, $m+1 \leq i \leq m+n-1$;
- (e) $x_{i,b} = q_2^{-1} x_{i,a}$ and $x_{i+1,c} = q_3^{-1} x_{i,b}$, $m+1 \leq i \leq m+n-1$;
- (f) $x_{m+n,b} = q_3^{-1} x_{m+n-1,a}$, $x_{1,c} = q_3 x_{m+n,b}$, and $x_{m+n,d} = q_1 x_{1,c}$.

We call (iii) *wheel conditions*. The wheel conditions are schematically shown in Figure 1 below. In that figure, for simplicity, we omitted the second indices of variables.

Note that all cubic wheel conditions contain two bosonic variables of the same color. The quartic relations contain two fermionic variables of the same color.

Note also that, due to skew-symmetry, F has obvious zeroes at

$$x_{m,a} = x_{m,b} \quad \text{or} \quad x_{m+n,a} = x_{m+n,b}\,, \quad a \neq b\,. \tag{2.1}$$



(a) (b) (c)

FIGURE 1. Wheel conditions (a), (b), (c). Here i is bosonic before the equator. Conditions (d), (e), (f) are obtained by reversing all arrows. In these conditions m is replaced with $m+n$ and i is bosonic after the equator.

Define *structure functions* $\omega_{i,j}(x,y)$ by

$$\omega_{i,i}(x,y) = \begin{cases} \dfrac{x-q_2y}{x-y} & \text{if } 1 \le i \le m-1, \\ \dfrac{x-q_2^{-1}y}{x-y} & \text{if } m+1 \le i \le m+n-1, \end{cases}$$

$$\omega_{i-1,i}(x,y) = \begin{cases} \dfrac{x-q_1y}{x-y} & \text{if } 1 \le i \le m, \\ \dfrac{x-q_1^{-1}y}{x-y} & \text{if } m+1 \le i \le m+n, \end{cases}$$

$$\omega_{i,i-1}(x,y) = \begin{cases} d\dfrac{x-q_3y}{x-y} & \text{if } 1 \le i \le m, \\ d^{-1}\dfrac{x-q_3^{-1}y}{x-y} & \text{if } m+1 \le i \le m+n. \end{cases}$$

We set $\omega_{i,j}(x,y) = 1$ in all other cases.

For $F \in Sh_{m|n,\mathbf{M}}$ and $G \in Sh_{m|n,\mathbf{N}}$, we define their *shuffle product* $F * G$ by setting

$$(F*G)(x_1;\cdots;x_{m+n}) = \sum_{I_1,J_1} \cdots \sum_{I_{m+n},J_{m+n}} \operatorname{sgn}(I_m,J_m)\operatorname{sgn}(I_{m+n},J_{m+n}) \tag{2.2}$$

$$\times F(x_{1,I_1};\cdots;x_{m+n,I_{m+n}})G(x_{1,J_1};\cdots;x_{m+n,J_{m+n}}) \prod_{i,j=1}^{m+n} \prod_{a\in I_i, b\in J_j} \omega_{i,j}(x_{i,a},x_{j,b})\,.$$

Here the sum is taken over all subsets I_i, J_i of $\{1,\dots,M_i+N_i\}$ such that

$$I_i \sqcup J_i = \{1,\dots,M_i+N_i\}\,, \quad |I_i| = M_i\,,\ |J_i| = N_i\,, \quad i = 1,\dots,m+n\,.$$

For a subset $I = \{a_1,\dots,a_k\}$, $a_1 < \cdots < a_k$, we write $x_{i,I} = (x_{i,a_1},\dots,x_{i,a_k})$. If $I = \{a_1,\dots,a_k\}$, $a_1 < \cdots < a_k$, and $J = \{b_1,\dots,b_l\}$, $b_1 < \cdots < b_l$, then $\operatorname{sgn}(I,J)$ stands for the signature of the permutation $(a_1,\dots,a_k,b_1,\dots,b_l)$.

Then we have the following standard lemma.

Lemma 2.1. *The space $Sh_{m|n}$ with the shuffle product is an associative $\mathbb{Z}_{\ge 0}^{m+n}$-graded algebra.*

Proof. Associativity is obvious from the definition. It is straightforward to check that if $F \in Sh_{m|n,\mathbf{M}}$ and $G \in Sh_{m|n,\mathbf{N}}$, then the shuffle product $F * G$ also satisfies the wheel conditions and therefore $F * G \in Sh_{m|n,\mathbf{M}+\mathbf{N}}$. □

As necessary, the dependence of shuffle algebras on parameters will be exhibited explicitly as $Sh_{m|n}(q_1, q_2, q_3)$.

We have an isomorphism

$$\begin{aligned} &Sh_{m|n}(q_1, q_2, q_3) \longrightarrow Sh_{n|m}(q_1^{-1}, q_2^{-1}, q_3^{-1}), \\ &F(x_1; \cdots; x_{m+n}) \mapsto F(x_{m+1}; \cdots; x_{m+n}; x_1; \cdots; x_m)\,. \end{aligned} \tag{2.3}$$

We expect that as algebra $Sh_{m|n}$ is generated by $\bigoplus_{i=1}^{m+n} Sh_{m|n,\mathbf{1}_i}$, where $\mathbf{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th position.

The quantum toroidal algebras $\mathcal{E}_{m|n}$ associated to $\mathfrak{gl}_{m|n}$ were defined and studied in [BM1], [BM2]. These algebras have generators $E_{i,k}, F_{i,k}$, $K_{i,0}^{\pm 1}$, $H_{i,j}$, C, $i = 1, \ldots, m+n$, $k \in \mathbb{Z}, j \in \mathbb{Z} \setminus \{0\}$.

We expect that the shuffle algebra $Sh_{m|n}$ is isomorphic to the subalgebra of $\mathcal{E}_{m|n}$ generated by $F_{i,k}$. Under this isomorphism $F_{i,k}$ is related to $x_{i,1}^k \in Sh_{m|n,\mathbf{1}_i}$.

In this text we work with the standard parity, where we have only two fermionic colors m and $m+n$. The shuffle algebras in other parities are defined similarly.

2.2. **Shuffle algebra Sh_m.** We shall use also the shuffle algebra Sh_m for the quantum toroidal algebra of type $\mathfrak{gl}_m$. We recall the definition below. Further information can be found in the literature [N1], [N2], [FT].

Assume $m \geq 3$.

Let Sh_m be the graded complex vector space

$$Sh_m = \bigoplus_{\mathbf{N} \in \mathbb{Z}_{\geq 0}^m} Sh_{m,\mathbf{N}}\,,$$

where $Sh_{m,\mathbf{N}}$ is the space of all rational functions $F(x_1; \cdots; x_m)$, $x_i = (x_{i,1}, \ldots, x_{i,N_i})$, such that

(i) F has the form

$$F(x_1; \cdots; x_m) = \frac{f(x_1; \cdots; x_m)}{\prod_{i=1}^m \prod_{\substack{1 \leq a \leq N_i \\ 1 \leq b \leq N_{i+1}}} (x_{i,a} - x_{i+1,b})}\,, \tag{2.4}$$

where $f(x_1; \cdots; x_m)$ is a Laurent polynomial;

(ii) F is symmetric in x_i for all $1 \leq i \leq m$;

(iii) F vanishes if one of the following holds for some a, b, c:

(a) $x_{i,b} = q_3 x_{i-1,a}$ and $x_{i,c} = q_2 x_{i,b}$, $1 \leq i \leq m$;

(b) $x_{i,b} = q_2 x_{i,a}$ and $x_{i+1,c} = q_3 x_{i,b}$, $0 \leq i \leq m-1$.

Here $x_{i+m,a} = x_{i,a}$. The shuffle product is defined by the same formula (2.2), wherein the sgn factors are dropped and the structure functions $\omega_{i,j}(x,y)$ are replaced by

$$\omega_{i,i}(x,y) = \frac{x-q_2y}{x-y}\,,$$
$$\omega_{i-1,i}(x,y) = \frac{x-q_1y}{x-y}\,,$$
$$\omega_{i,i-1}(x,y) = d\frac{x-q_3y}{x-y}\,,$$

and $\omega_{i,j}(x,y) = 1$ in all other cases.

In this case all variables are bosonic. The wheel conditions correspond to triangles in Figure 1.

When $m = 1, 2$ the definition of Sh_m needs a slight modification.
In both cases the wheel conditions (a),(b) apply under the convention $x_{i+m,a} = x_{i,a}$ $(m = 1, 2)$.
For $m = 2$, the structure functions are changed to

$$\omega_{i,i}(x,y) = \frac{x-q_2y}{x-y}\,,$$
$$\omega_{i,i-1}(x,y) = \frac{(x-q_1y)(x-q_3y)}{(x-y)^2}\,.$$

For $m = 1$, the pole condition (2.4) reads

$$F(x) = \frac{f(x)}{\prod_{1\le a<b\le N}(x_a-x_b)^2},$$

where $f(x)$ is a symmetric Laurent polynomial in $x = (x_1, \dots, x_N)$. The structure functions are modified to

$$\omega_{1,1}(x,y) = \frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}\,.$$

In what follows we write $Sh_m = Sh_{m|0}$.

3. Fusion homomorphism

Consider the subalgebra of $Sh_{m|n}$ consisting of subspaces of degree $N_1 = \cdots = N_{m+n}$:

$$Sh^0_{m|n} = \bigoplus_{N\ge 0} Sh^0_{m|n,N}\,, \quad Sh^0_{m|n,N} = Sh_{m|n,N,\dots,N}\,.$$

The aim of this section is to introduce homomorphisms of algebras of the form

$$Sh^0_{m|n}(q_1,q_2,q_3) \longrightarrow Sh^0_{m|n-1}(\bar q_1,q_2,\bar q_3)\otimes Sh_1(q_1^{-1}q_3^{-m+n-1},q_2^{-1},q_3^{m-n})\,,$$
$$Sh^0_{m}(q_1,q_2,q_3) \longrightarrow Sh^0_{m-1}(\tilde q_1,q_2,\tilde q_3)\otimes Sh_1(q_1q_3^{-m+1},q_2,q_3^{m})$$

with appropriate parameters $\bar q_s, \tilde q_s$. We call them *fusion homomorphisms*.

We are motivated by the constructions of [FJMM1]. In that work, fused currents were used to construct subalgebras $\mathcal{E}_{m-1}\otimes\mathcal{E}_1 \subset \mathcal{E}_m$ which is the quantum toroidal version of the Lie subalgebra inclusion $\mathfrak{gl}_{m-1}\oplus\mathfrak{gl}_1 \subset \mathfrak{gl}_m$.

3.1. **Fusion homomorphism for $Sh_{m|n}$.** In this subsection we assume that $m \geq n+1$, $n \geq 1$. We set

$$\bar{q}_1 = q_1 q_1^{-\frac{1}{m-n+1}}, \quad \bar{q}_3 = q_3 q_1^{\frac{1}{m-n+1}}. \tag{3.1}$$

Fix L, N with $0 \leq L \leq N$. We introduce new variables $y_i = (y_{i,a})_{1\leq a\leq L}$, $1 \leq i \leq m+n-1$, $z = (z_b)_{L+1\leq b\leq N}$, and consider the following specialization of the variables $x_i = (x_{i,1}, \dots, x_{i,N})$, see Figure 2 below:

$$1 \leq a \leq L: \qquad x_{i,a} = \begin{cases} q_1 \mathsf{q}^i y_{i,a} & \text{if } 1 \leq i \leq m, \\ q_1 \mathsf{q}^{2m-i} y_{i,a} & \text{if } m+1 \leq i \leq m+n-1, \\ q_1 y_{m+n-1,a} & \text{if } i = m+n, \end{cases} \tag{3.2}$$

$$L+1 \leq b \leq N: \qquad x_{i,b} = \begin{cases} q_3^{i-1} z_b & \text{if } 0 \leq i \leq m-1, \\ q_3^{2m-i-1} z_b & \text{if } m \leq i \leq m+n-1. \end{cases} \tag{3.3}$$

where $\mathsf{q} = q_1^{-1}\bar{q}_1 = q_1^{-1/(m-n+1)}$. We have in particular

$$1 \leq a \leq L: \qquad x_{1,a} = \bar{q}_1 y_{1,a}, \quad x_{m+n-1,a} = y_{m+n-1,a}, \quad x_{m+n,a} = q_1 y_{m+n-1,a},$$
$$L+1 \leq b \leq N: \qquad x_{1,b} = z_b, \quad x_{m+n-1,b} = q_3^{m-n} z_b, \quad x_{m+n,b} = q_3^{-1} z_b.$$

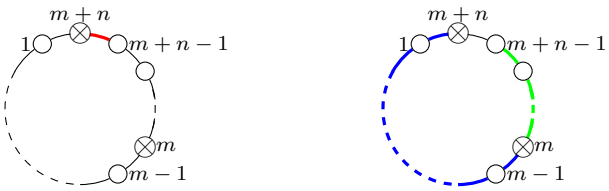


FIGURE 2. Specialization (3.2), (3.3).

For a function $F(x_1; \cdots; x_{m+n})$, let $\rho_{N,L}(F)(y_1; \cdots; y_{m+n-1}; z)$ denote the function obtained by the substitution (3.2)–(3.3).

Lemma 3.1. *For an element*

$$F(x_1; \cdots; x_{m+n}) = \frac{f(x_1; \cdots; x_{m+n})}{\prod_{i=1}^{m+n} \prod_{1\leq a,b\leq N} (x_{i,a} - x_{i+1,b})} \in Sh^0_{m|n,N},$$

define

$$\bar{F}_{N,L}(y_1; \cdots; y_{m+n-1}|z) = \frac{\bar{f}_{N,L}(y_1; \cdots; y_{m+n-1}|z)}{\prod_{i=1}^{m+n-2} \prod_{1\leq a,b\leq N} (y_{i,a} - y_{i+1,b}) \cdot \prod_{1\leq a,b\leq N} (y_{m+n-1,a} - y_{1,b}) \cdot \prod_{1\leq a<b\leq N} (z_a - z_b)^2}$$

as follows.

If $n > 1$ we define $\bar{f}_{N,L}$ by

$$\begin{aligned}
&\rho_{N,L}\big(f(x_1;\cdots;x_{m+n})\big) = const.\bar{f}_{N,L}(y_1;\cdots;y_{m+n-1}|z)\\
&\times \prod_{1\le a,b\le L}(y_{m+n-1,a}-q_2y_{m+n-1,b})\cdot \prod_{L+1\le a,b\le N}(z_a-q_2z_b)^{m+n-2}\prod_{L+1\le a<b\le N}(z_a-z_b)^2\\
&\times \prod_{i=1}^{m-1}\prod_{\substack{1\le a\le L\\ L+1\le b\le N}}(q_1\boldsymbol{q}^i y_{i,a}-q_2q_3^{i-1}z_b)(q_1\boldsymbol{q}^i y_{i,a}-q_2^{-1}q_3^{i-1}z_b)\\
&\times \prod_{i=m+1}^{m+n-1}\prod_{\substack{1\le a\le L\\ L+1\le b\le N}}(q_1\boldsymbol{q}^{2m-i} y_{i,a}-q_2q_3^{2m-i-1}z_b)(q_1\boldsymbol{q}^{2m-i} y_{i,a}-q_2^{-1}q_3^{2m-i-1}z_b)\\
&\times \prod_{\substack{1\le a\le L\\ L+1\le b\le N}}(q_1\boldsymbol{q}^m y_{m,a}-q_2q_3^{m-1}z_b)(q_1\boldsymbol{q}^m y_{m,a}-q_3^{m-1}z_b)(y_{m+n-1,a}-z_b)(y_{m+n-1,a}-q_2z_b)\,.
\end{aligned}$$

If $n = 1$, we replace the factor $\prod_{1\le a,b\le L}(y_{m+n-1,a}-q_2y_{m+n-1,b})$ in the second line of the right hand side by $\prod_{a=1}^{L}y_{m,a}\prod_{1\le a<b\le L}(y_{m,a}-y_{m,b})^2$.

Then $\bar{F}_{N,L}(y_1;\cdots;y_{m+n-1}|z)$ belongs to

$$Sh^0_{m|n-1,L}(\bar{q}_1,q_2,\bar{q}_3)\otimes Sh_{1,N-L}(q_1^{-1}q_3^{-m+n-1},q_2^{-1},q_3^{m-n})\,.$$

Proof. The zero conditions (a)–(f) along with (2.1) imply that $\bar{f}_{N,L}(y_1;\cdots;y_{m+n-1}|z)$ defined as above is a Laurent polynomial. It is straightforward to check that the latter has the (skew-)symmetry in y_i or z, and that it satisfies the wheel conditions for $Sh_{m|n-1}(\bar{q}_1,q_2,\bar{q}_3)$ in $(y_1,\dots,y_{m+n-1})$ and for $Sh_1(q_1^{-1}q_3^{-m+n-1},q_2^{-1},q_3^{m-n})$ in z. □

In the definition of $\bar{f}_{N,L}$ the overall constant multiple is at our disposal. We choose it so that in terms of $\bar{F}_{N,L}$ the formula reads as follows.

$$\rho_{N,L}\big(F(x_1;\cdots;x_{m+n})\big) = \bar{F}_{N,L}(y_1;\cdots;y_{m+n-1}|z)\times ABC\,,$$

where for $n>1$

$$A=\prod_{i=0}^{m-1}\prod_{1\le a,b\le L}\frac{y_{i,a}-y_{i+1,b}}{y_{i,a}-\mathsf{q}y_{i+1,b}}\prod_{i=m}^{m+n-2}\prod_{1\le a,b\le L}\frac{y_{i,a}-y_{i+1,b}}{y_{i,a}-\mathsf{q}^{-1}y_{i+1,b}}$$
$$\times\prod_{1\le a<b\le L}d^{-1}\frac{(y_{m+n-1,a}-q_2y_{m+n-1,b})(y_{m+n-1,a}-q_2^{-1}y_{m+n-1,b})}{(y_{m+n-1,a}-q_1y_{m+n-1,b})(y_{m+n-1,a}-q_1^{-1}y_{m+n-1,b})},$$

$$B=\prod_{L<a<b\le N}d^{m-n}\left(\frac{z_a-q_2z_b}{z_a-q_3z_b}\frac{z_a-q_2^{-1}z_b}{z_a-q_3^{-1}z_b}\right)^{m+n-2}\frac{(z_a-z_b)^2}{(z_a-q_3z_b)(z_a-q_3^{-1}z_b)}\frac{(z_a-z_b)^2}{(z_a-q_3^{m-n+1}z_b)(z_a-q_3^{-m+n-1}z_b)},$$

$$C=\prod_{i=1}^{m-1}\prod_{\substack{1\le a\le L\\ L<b\le N}}d\frac{(q_1\mathsf{q}^iy_{i,a}-q_2q_3^{i-1}z_b)(q_1\mathsf{q}^iy_{i,a}-q_2^{-1}q_3^{i-1}z_b)}{(q_1\mathsf{q}^iy_{i,a}-q_3^iz_b)(q_1\mathsf{q}^iy_{i,a}-q_3^{i-2}z_b)}$$
$$\times\prod_{\substack{1\le a\le L\\ L<b\le N}}d^{-1}\frac{(q_1\mathsf{q}^my_{m,a}-q_3^{m-1}z_b)(q_1\mathsf{q}^my_{m,a}-q_2q_3^{m-1}z_b)}{(q_1\mathsf{q}^my_{m,a}-q_3^{m-2}z_b)(q_1\mathsf{q}^my_{m,a}-q_3^{m-2}z_b)}$$
$$\times\prod_{i=m+1}^{m+n-2}\prod_{\substack{1\le a\le L\\ L<b\le N}}d^{-1}\frac{(q_1\mathsf{q}^{2m-i}y_{i,a}-q_2q_3^{2m-i-1}z_b)(q_1\mathsf{q}^{2m-i}y_{i,a}-q_2^{-1}q_3^{2m-i-1}z_b)}{(q_1\mathsf{q}^{2m-i}y_{i,a}-q_3q_3^{2m-i-1}z_b)(q_1\mathsf{q}^{2m-i}y_{i,a}-q_3^{-1}q_3^{2m-i-1}z_b)}$$
$$\times\prod_{\substack{1\le a\le L\\ L<b\le N}}\frac{(y_{m+n-1,a}-q_2q_3^{m-n}z_b)(y_{m+n-1,a}-q_2^{-1}q_3^{m-n}z_b)}{(y_{m+n-1,a}-q_3q_3^{m-n}z_b)(y_{m+n-1,a}-q_1^{-1}q_3^{m-n}z_b)}\frac{(y_{m+n-1,a}-q_2z_b)(y_{m+n-1,a}-z_b)}{(y_{m+n-1,a}-q_3^{-1}z_b)(y_{m+n-1,a}-q_1^{-1}z_b)}.$$

For $n=1$, B is the same and A,C are modified as

$$A=\prod_{i=0}^{m-1}\prod_{1\le a,b\le L}\frac{y_{i,a}-y_{i+1,b}}{y_{i,a}-\mathsf{q}y_{i+1,b}}\times\prod_{1\le a<b\le L}\frac{d^{-1}(y_{m,a}-y_{m,b})^2}{(y_{m,a}-q_1y_{m,b})(y_{m,a}-q_1^{-1}y_{m,b})},$$
$$C=\prod_{i=1}^{m-1}\prod_{\substack{1\le a\le L\\ L<b\le N}}d\frac{(q_1\mathsf{q}^iy_{i,a}-q_2q_3^{i-1}z_b)(q_1\mathsf{q}^iy_{i,a}-q_2^{-1}q_3^{i-1}z_b)}{(q_1\mathsf{q}^iy_{i,a}-q_3^iz_b)(q_1\mathsf{q}^iy_{i,a}-q_3^{i-2}z_b)}$$
$$\times\prod_{\substack{1\le a\le L\\ L<b\le N}}\frac{(y_{m,a}-q_1^{-1}q_3^{m-2}z_b)(y_{m,a}-q_3^{m-1}z_b)}{(y_{m-1,a}-q_3^{m-2}z_b)(y_{m,a}-q_1^{-1}q_3^{m-1}z_b)}\frac{(y_{m,a}-q_2z_b)(y_{m,a}-z_b)}{(y_{m,a}-q_3^{-1}z_b)(y_{m,a}-q_1^{-1}z_b)}.$$

Denoting $\bar F_{N,L}$ by $\pi_{N,L}(F)$, we set $\pi_{m|n}^{m|n-1}=\sum_{L=0}^N\pi_{N,L}$.

Proposition 3.2. *The linear map*

$$\pi_{m|n}^{m|n-1}:Sh^0_{m|n}(q_1,q_2,q_3)\longrightarrow Sh^0_{m|n-1}(\bar q_1,q_2,\bar q_3)\otimes Sh_1(q_1^{-1}q_3^{-m+n-1},q_2^{-1},q_3^{m-n}),$$
$$\bar q_1=q_1q_1^{-\frac{1}{m-n+1}},\quad \bar q_3=q_3q_1^{\frac{1}{m-n+1}},$$

is a homomorphism of $\mathbb{Z}_{\ge0}$-graded algebras.

Proof. Let $F\in Sh^0_{m|n,M}$, $G\in Sh^0_{m|n,N}$. First we examine the effect of the specialization (3.2),(3.3) in the shuffle product (2.2). Let $0\le L\le M+N$. For each partition $I_i\sqcup J_i=\{1,\dots,M+N\}$ such

that $|I_i| = M$, $|J_i| = N$, set

$$I_i' = I_i \cap \{1, \dots, L\}\,, \quad I_i'' = I_i \cap \{L+1, \dots, M+N\}\,,$$
$$J_i' = J_i \cap \{1, \dots, L\}\,, \quad J_i'' = J_i \cap \{L+1, \dots, M+N\}\,,$$

so that

$$I_i' \sqcup J_i' = \{1, \dots, L\}\,, \quad I_i'' \sqcup J_i'' = \{L+1, \dots, M+N\}\,,$$
$$|I_i'| + |I_i''| = M\,, \quad |J_i'| + |J_i''| = N\,.$$

In order that the factor

$$\prod_{a \in I'_{m+n-1}, b \in J'_{m+n}} \omega_{m+n-1,m+n}(x_{m+n-1,a}, q_1 x_{m+n-1,b})$$

be non-vanishing, we must have $I'_{m+n-1} \cap J'_{m+n} = \emptyset$, which implies

$$I'_{m+n-1} \subset I'_{m+n}\,.$$

Similarly,

$$\prod_{a \in I_i'', b \in J''_{i-1}} \omega_{i,i-1}(q_3^{i-1} z_a, q_3^{i-2} z_b) \neq 0, \quad 1 \le i \le m\,,$$
$$\prod_{a \in I_i'', b \in J''_{i-1}} \omega_{i,i-1}(q_3^{2m-i-1} z_a, q_3^{2m-i} z_b) \neq 0, \quad m+1 \le i \le m+n-1\,,$$

implies

$$I''_{m+n-1} \subset I''_{m+n-2} \subset \cdots \subset I''_1 \subset I''_{m+n}\,.$$

Since $|I_i'| + |I_i''| = M$ for all i, we must have that

$$I'_{m+n-1} = I'_{m+n}\,, \quad I''_1 = \cdots = I''_{m+n}\,.$$

Writing $I'' = I_i''$ and $J'' = J_i''$, we see that $\rho_{M+N,L}\big(F * G\big)$ is reduced to the sum over partitions

$$I_i' \sqcup J_i' = \{1, \dots, L\}\,, \quad i = 1, \dots, m+n-1\,,$$
$$I'' \sqcup J'' = \{L+1, \dots, M+N\}\,,$$
$$|I_i'| = k\,, \quad |J_i'| = L-k\,, \quad |I''| = M-k\,, \quad |J''| = N-L+k\,.$$

It remains to show that, after cancelling out the factors ABC in the Lemma from both sides, this sum reduces to the shuffle product $\pi_{M,k}(F) * \pi_{N,L-k}(G)$. This is a lengthy but straightforward computation. □

Iterating the map in Proposition 3.2, we obtain

Corollary 3.3. *There exists a homomorphism of $\mathbb{Z}_{\ge 0}$-graded algebras*

$$\pi_{m|n} : Sh^0_{m|n}(q_1, q_2, q_3) \longrightarrow Sh^0_m(q_1 q_1^{-\frac{n}{m}}, q_2, q_3 q_1^{\frac{n}{m}}) \otimes \Big(\bigotimes_{j=0}^{n-1} Sh_1(q_2^{j+1} q_3^{-m+n}, q_2^{-1}, q_2^{-j} q_3^{m-n}) \Big)\,. \tag{3.4}$$

3.2. **Fusion homomorphism for Sh_m.** The fusion homomorphism can be defined also for Sh_m. Since the argument is quite similar, we state only the result.

Let $m \geq 2$. As before, we fix N, L with $0 \leq L \leq N$. Consider the specialization of variables from $x_i = (x_{i,a})_{1\leq a\leq N}$, $1 \leq i \leq m$, to $y_i = (y_{i,a})_{1\leq a\leq L}$, $1 \leq i \leq m-1$, and $z = (z_a)_{L+1\leq a\leq N}$, given by

$$1 \leq a \leq L : \qquad x_{i,a} = q_1^{-\frac{m-i}{m-1}} y_{i,a} \quad \text{for } 1 \leq i \leq m, \tag{3.5}$$

$$L+1 \leq b \leq N : \qquad x_{i,b} = q_3^{i-1} z_b \quad \text{for } 1 \leq i \leq m. \tag{3.6}$$

Here we set $y_{m,a} = y_{1,a}$. Let $\tilde{\rho}_{N,L}(F)$ denote the substitution given by (3.5) and (3.6). For $F \in Sh^0_{m,N}$ we define $\tilde{F}_{N,L}$ by

$$\begin{aligned}
\tilde{\rho}_{N,L}(F) &= \tilde{F}_{N,L}(y_1;\cdots;y_{m-1}|z) \\
&\times \prod_{1\leq a<b\leq L} \Big(d\frac{(y_{1,a}-q_2y_{1,b})(y_{1,a}-q_2^{-1}y_{1,b})}{(y_{1,a}-q_1y_{1,b})(y_{1,a}-q_1^{-1}y_{1,b})} \Big) \times \prod_{1\leq a,b\leq L} \prod_{i=1}^{m-1} \frac{y_{i,a}-y_{i+1,b}}{y_{i,a}-q_1^{1/(m-1)}y_{i+1,b}} \\
&\times \prod_{L+1\leq a<b\leq N} \Big(d\frac{(z_a-q_2z_b)(z_a-q_2^{-1}z_b)}{(z_a-q_3z_b)(z_a-q_3^{-1}z_b)} \Big)^{m-1} \prod_{L+1\leq a,b\leq N} \frac{d(z_a-z_b)^2}{(z_a-q_3^{m-1}z_b)(z_a-q_3^{-m+1}z_b)} \\
&\times \prod_{\substack{1\leq a\leq L \\ L+1\leq b\leq N}} \Big(\prod_{i=2}^{m-1} d\frac{(q_1^{-(m-i)/(m-1)}y_{i,a}-q_1q_3^iz_b)}{(q_1^{-(m-i)/(m-1)}y_{i,a}-q_3^iz_b)} \frac{(q_1^{-(m-i)/(m-1)}y_{i,a}-q_1^{-1}q_3^{i-2}z_b)}{(q_1^{-(m-i)/(m-1)}y_{i,a}-q_3^{i-2}z_b)} \Big) \\
&\times \prod_{\substack{1\leq a\leq L \\ L+1\leq b\leq N}} d^2\frac{(y_{1,a}-q_1^2q_3z_b)(y_{1,a}-q_3^{-1}z_b)}{(y_{1,a}-q_1q_3z_b)(y_{1,a}-z_b)} \frac{(y_{1,a}-q_1^{-1}q_3^{m-2}z_b)(y_{1,a}-q_1q_3^mz_b)}{(y_{1,a}-q_3^{m-2}z_b)(y_{1,a}-q_1q_3^{m-1}z_b)} .
\end{aligned}$$

Proposition 3.4. *The map sending $F \in Sh^0_{m,N}$ to $\sum_{L=0}^N \tilde{F}_{N,L}$ gives a homomorphism of $\mathbb{Z}_{\geq 0}$-graded algebras*

$$\pi_m^{m-1} : Sh^0_m(q_1,q_2,q_3) \longrightarrow Sh^0_{m-1}(\tilde{q}_1,q_2,\tilde{q}_3) \otimes Sh_1(q_1q_3^{-m+1},q_2,q_3^m) ,$$

where

$$\tilde{q}_1 = q_1q_1^{\frac{1}{m-1}} , \quad \tilde{q}_3 = q_3q_1^{-\frac{1}{m-1}} . \tag{3.7}$$

Corollary 3.5. *There exists a homomorphism of $\mathbb{Z}_{\geq 0}$-graded algebras*

$$\pi_m : Sh^0_m(q_1,q_2,q_3) \longrightarrow \bigotimes_{j=0}^{m-1} Sh_1(q_2^jq_1^m,q_2,q_2^{-j-1}q_1^{-m}) . \tag{3.8}$$

4. Subalgebra $\mathcal{B}_{m|n}(s)$

4.1. **Subalgebra $\mathcal{B}_{m|n}(s)$.** We introduce a family of subspaces $\mathcal{B}_{m|n}(s)$ of $Sh_{m|n}$ defined by certain regularity conditions at 0 and ∞ depending on parameters $s = (s_1,\dots,s_{m+n})$. We will show in Section 5 that for each s, $\mathcal{B}_{m|n}(s)$ is a free commutative subalgebra of $Sh_{m|n}$. We call $\mathcal{B}_{m|n}(s)$ the Bethe algebra. We expect that in relation to the quantum toroidal algebras, $\mathcal{B}_{m|n}(s)$ is a deformation

of the Cartan half currents of the affine $U_q\hat{\mathfrak{gl}}_{m|n} \in \mathcal{E}_{m|n}$ which corresponds to $s_1, \dots, s_{m+n-1} \to 0$, cf. Remark 3.11 [FT].

To this end, we prepare some notation.

Quite generally, for an element $F \in Sh_{m|n,\mathbf{N}}$ and $\boldsymbol{k} = (k_1, \dots, k_{m+n}) \in \mathbb{Z}_{\geq 0}^{m+n}$ with $0 \leq k_i \leq N_i$, we set

$$F_\xi^{\mathbf{k}} = F(\xi x_1', x_1''; \cdots ; \xi x_{m+n}', x_{m+n}'')\,,$$

where

$$x_i' = (x_{i,1}, \dots, x_{i,k_i})\,, \quad x_i'' = (x_{i,k_i+1}, \dots, x_{i,N_i})\,.$$

We shall write

$$F_\infty^{\mathbf{k}} = \lim_{\xi\to\infty} F_\xi^{\mathbf{k}}\,, \quad F_0^{\mathbf{k}} = \lim_{\xi\to 0} F_\xi^{\mathbf{k}}\,, \tag{4.1}$$

provided these limits exist.

Further we fix an $(m+n)$-tuple of parameters

$$s = (s_1, \dots, s_{m+n}) \in (\mathbb{C}^\times)^{m+n} \quad \text{satisfying } \prod_{i=1}^{m+n} s_i = 1.$$

We extend the suffix to all $i \in \mathbb{Z}$ by $s_{i+m+n} = s_i$. We shall say that s is *generic* if

$$q_1^{a_1} q_2^{a_2} \prod_{i=1}^{m+n-1} s_i^{b_i} = 1\,, a_i, b_j \in \mathbb{Z}\,, \quad \text{holds only if } a_i = b_j = 0 \text{ for all } i, j.$$

Definition 4.1. *We define the subspace $\mathcal{B}_{m|n,N}(s)$ to be the set of all elements $F \in Sh^0_{m|n,N}$ such that both limits* (4.1) *exist for all $\mathbf{k} = (k_1, \dots, k_{m+n})$ satisfying $0 \leq k_i \leq N$, and are related by*

$$F_\infty^{\mathbf{k}} = F_0^{\mathbf{k}} \times \prod_{i=1}^{m+n} s_i^{k_i} \cdot d^{2N(k_m - k_{m+n})}\,. \tag{4.2}$$

Note that the power of d is absent if $n = 0$.

We set

$$\mathcal{B}_{m|n}(s) = \bigoplus_{N=0}^{\infty} \mathcal{B}_{m|n,N}(s)\,.$$

Examples of elements in $\mathcal{B}_{m|n}(s)$ will be given in Section 5.2 below.

Lemma 4.2. *The subspace $\mathcal{B}_{m|n}(s)$ is a subalgebra of $Sh^0_{m|n}$.*

Proof. Consider the shuffle product (2.2) of $F \in \mathcal{B}_{m|n,M}(s)$ and $G \in \mathcal{B}_{m|n,N}(s)$. In each summand denote

$$F_I = F(x_{1,I_1}; \cdots ; x_{m+n,I_{m+n}})\,, \quad G_J = G(x_{1,J_1}; \cdots ; x_{m+n,J_{m+n}})\,,$$
$$\omega_{I,J} = \prod_{i,j=1}^{m+n} \prod_{\substack{a\in I_i\\ b\in J_j}} \omega_{i,j}(x_{i,a}, x_{j,b})\,.$$

Let $h_i = |I_i \cap \{1, \dots, k_i\}|$, $l_i = |J_i \cap \{1, \dots, k_i\}|$, so that $h_i + l_i = k_i$. Then

$$(F_I)^{\mathbf{k}}_{\infty} = (F_I)^{\mathbf{k}}_0 \cdot \prod_{i=1}^{m+n} s_i^{h_i} \cdot d^{2M(h_m - h_{m+n})}\,, \tag{4.3}$$

$$(G_J)^{\mathbf{k}}_{\infty} = (G_J)^{\mathbf{k}}_0 \cdot \prod_{i=1}^{m+n} s_i^{l_i} \cdot d^{2N(l_m - l_{m+n})}\,. \tag{4.4}$$

On the other hand, it is easy to see that

$$\lim_{\xi\to\infty} \omega_{i,i}(\xi x, y) = \lim_{\xi\to 0} \omega_{i,i}(\xi x, y) \cdot q_2^{-\theta(1\le i\le m-1)+\theta(m+1\le i\le m+n-1)}\,,$$
$$\lim_{\xi\to\infty} \omega_{i,i+1}(\xi x, y) = \lim_{\xi\to 0} \omega_{i,i+1}(\xi x, y) \cdot q_1^{-\theta(0\le i\le m-1)+\theta(m\le i\le m+n-1)}\,,$$
$$\lim_{\xi\to\infty} \omega_{i,i-1}(\xi x, y) = \lim_{\xi\to 0} \omega_{i,i-1}(\xi x, y) \cdot q_3^{-\theta(1\le i\le m)+\theta(m+1\le i\le m+n)}\,,$$

and that

$$\frac{\lim_{\xi\to\infty} \omega_{i,j}(x, \xi y)}{\lim_{\xi\to 0} \omega_{i,j}(x, \xi y)} = \Big(\frac{\lim_{\xi\to\infty} \omega_{i,j}(\xi x, y)}{\lim_{\xi\to 0} \omega_{i,j}(\xi x, y)}\Big)^{-1}\,.$$

This yields

$$(\omega_{I,J})^{\mathbf{k}}_{\infty} = (\omega_{I,J})^{\mathbf{k}}_0 \cdot d^{2N(h_m - h_{m+n}) + 2M(l_m - l_{m+n})}\,. \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that

$$(F * G)^{\mathbf{k}}_{\infty} = (F * G)^{\mathbf{k}}_0 \cdot \prod_{i=1}^{m+n} s_i^{k_i} \cdot d^{2(M+N)(k_m - k_{m+n})}\,.$$

This completes the proof. □

We call the subalgebra $\mathcal{B}_{m|n}(s)$ the Bethe algebra.

Remark 4.3. *For $n = 0$, the subspace $\mathcal{B}_{m|0}(s)$ is a slight modification of $\mathcal{A}(\bar{s})$ studied in* [FT]. *Though they turn out to coincide for generic s, it is a priori not obvious that the latter is a subalgebra.* □

4.2. **Gordon filtration.** Gordon filtration is a standard method used to study spaces of polynomials with vanishing conditions on the planes of codimension higher than one. It is designed in such a way that associated graded spaces become spaces of polynomials without conditions.

In this section we apply the Gordon filtration technique to obtain an estimate from above to the dimensions of the graded components $\mathcal{B}_{m|n,N}(s)$ of the Bethe algebra. Our proof is similar to the proof of Lemma 2.4 in [FT].

Let $\mathcal{R}_{m|n}$ denote the graded polynomial algebra in variables $\{T_{i,\nu} \mid 1 \le i \le m+n, \nu \in \mathbb{Z}_{>0}\}$ with $\deg T_{i,\nu} = \nu$. Let $\mathcal{R}_{m|n,N}$ denote the degree N component of $\mathcal{R}_{m|n}$.

Proposition 4.4. *Assume s is generic and $m \neq n$. Then* $\dim \mathcal{B}_{m|n,N}(s) \le \dim \mathcal{R}_{m|n,N}$.

Proof. Fix $N \in \mathbb{Z}_{\geq 0}$ and denote $B = \mathcal{B}_{m|n,N}$ and $Sh = Sh^0_{m|n,N}$.

We will define a finite partially ordered set P and for each $p \in P$ we will give a subspace $B_p \subset B$ and show that $\cap_p B_p = 0$. That will produce a bound:

$$\dim B \leq \sum_{p \in P} \dim \operatorname{gr} B_p, \qquad \operatorname{gr} B_p = \Big(\bigcap_{p'>p} B_{p'}\Big) / \Big(\bigcap_{p' \geq p} B_{p'}\Big).$$

Here, if $p \in P$ is a maximal element, then we set $\cap_{p'>p} B_{p'} = B$.

Indeed, refine the partial order $>$ on P to a total order $\succ$ and set $\tilde{B}_p = \cap_{p' \succ p} B_{p'}$. Then $p \succ p'$ implies $\tilde{B}_p \supset \tilde{B}_{p'}$. And, therefore, we have a linear filtration $\tilde{B}_p$ on B and therefore as a vector space

$$B \simeq \oplus \operatorname{gr} \tilde{B}_p, \qquad \operatorname{gr} \tilde{B}_p = \Big(\bigcap_{p' \succ p} B_{p'}\Big) / \Big(\bigcap_{p' \geq p} B_{p'}\Big).$$

For any subspaces A_1, A_2, C, with $A_2 \subset A_1$, we have an injection $(A_1 \cap C)/(A_2 \cap C) \to A_1/A_2$. Therefore for every p we have an injective map $\operatorname{gr} \tilde{B}_p \to \operatorname{gr} B_p$ which corresponds to $A_1 = \cap_{p'>p} B_{p'}$, $A_2 = \cap_{p' \geq p} B_{p'}$, $C = \cap_{p' \succ p} B_{p'}$. Thus we have the upper bound on the dimension of B.

After that we will argue that $\operatorname{gr} B_p$ are at most one dimensional, and in many cases are trivial which will result in the proof of the proposition.

We start by defining the set P.

For $a, b \in \mathbb{Z}$, denote the interval $[a, b] = \{a, a+1, \dots, b\} \subset \mathbb{Z}$. The interval $[a, b]$ is non-empty if and only if $a \leq b$. For $i = 1, \dots, m+n$ denote $[a, b]_i$ the number of elements of $[a, b]$ congruent to i modulo $m+n$. Thus modulo $m+n$ we have

$$[a, b] \equiv \{\underbrace{1, \dots, 1}_{[a,b]_1}, \underbrace{2, \dots, 2}_{[a,b]_2}, \dots, \underbrace{m+n, \dots, m+n}_{[a,b]_{m+n}}\}.$$

Intervals $[a, b]$ and $[c, d]$ are called equivalent if they are shifts by a multiple of $m+n$: $a-c = b-d = (m+n)k$, for some $k \in \mathbb{Z}$. If $[a, b]$ and $[c, d]$ are equivalent then $[a, b]_i = [c, d]_i$ for all i. The converse is not true, for example $[1, m+n]$ and $[2, m+n+1]$ are not equivalent but $[1, m+n]_i = [2, m+n+1]_i = 1$ for all i.

An interval $[a, b]$ is called cyclic if its number of elements is a multiple of $m+n$: $b-a+1 = (m+n)k$, $k \in \mathbb{Z}_{>0}$. In other words, an interval $[a, b]$ is cyclic if and only if $[a, b]_i$ does not depend on i.

For $\boldsymbol{a} = (a_1, \dots, a_s)$, $\boldsymbol{b} = (b_1, \dots, b_s)$, $a_j \leq b_j$, denote $[\boldsymbol{a}, \boldsymbol{b}]$ the unordered set of intervals $[a_j, b_j]$.

The set of intervals $[\boldsymbol{a}, \boldsymbol{b}]$ is called N-admissible if intervals $[a_j, b_j]$ are non-empty for all j and $\sum_{j=1}^s [a_j, b_j]_i = N$ for all i.

Two sets of intervals $[\boldsymbol{a}, \boldsymbol{b}]$ and $[\boldsymbol{c}, \boldsymbol{d}]$ are called equivalent if there exists an ordering of intervals in $[\boldsymbol{a}, \boldsymbol{b}]$ such that $[a_j, b_j]$ is equivalent to $[c_j, d_j]$ for all j.

The set of intervals $[\boldsymbol{a}, \boldsymbol{b}]$ is called cyclic if intervals $[a_j, b_j]$ are cyclic for all j. Note that $\dim \mathcal{R}_{m|n,N}$ equals the number of equivalence classes of N-admissible cyclic sets of intervals.

The set P is the set of all equivalence classes of N-admissible intervals $[\boldsymbol{a}, \boldsymbol{b}]$.

Next we define an order on P.

A partition of M is an ordered set of non-negative integers called parts which sum up to M: $\lambda = (\lambda_1, \dots, \lambda_s)$, $\lambda_1 \geq \dots \geq \lambda_s \geq 0$, $\sum_{j=1}^s \lambda_j = M$. Define the lexicographical partial ordering on set of partitions: $\lambda > \mu$ if for some j, $\lambda_j > \mu_j$ and $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\dots$, $\lambda_{j-1} = \mu_{j-1}$.

Given a set of intervals $[\boldsymbol{a}, \boldsymbol{b}]$, let $\lambda_{[\boldsymbol{a},\boldsymbol{b}]}$ be the partition of $(m+n)N$ with parts $b_j - a_j + 1$ written in the non-decreasing order. We give a partial order on P according to the order on the corresponding

partitions: we set $[\boldsymbol{a},\boldsymbol{b}] > [\boldsymbol{c},\boldsymbol{d}]$ if $\lambda_{[\boldsymbol{a},\boldsymbol{b}]} > \lambda_{[\boldsymbol{c},\boldsymbol{d}]}$. If $\lambda_{[\boldsymbol{a},\boldsymbol{b}]} = \lambda_{[\boldsymbol{c},\boldsymbol{d}]}$ and $[\boldsymbol{a},\boldsymbol{b}] \neq [\boldsymbol{c},\boldsymbol{d}]$, then $[\boldsymbol{a},\boldsymbol{b}]$ and $[\boldsymbol{c},\boldsymbol{d}]$ are not comparable.

Now we define the subspaces $B_{[\boldsymbol{a},\boldsymbol{b}]}$, $[\boldsymbol{a},\boldsymbol{b}] \in P$. Given $[\boldsymbol{a},\boldsymbol{b}] \in P$, we define the substitution of variables $(x_{1,1},\dots,x_{1,N},\dots,x_{m+n,1},\dots,x_{m+n,N})$ appearing in Sh to the variables $y_1,\dots,y_s$ multiplied by some constants as follows.

For $a \in \mathbb{Z}$ define recursively constants α_a by the rule $\alpha_0 = 1$, $\alpha_{a+1} = q_3\alpha_a$ if $a \equiv i \pmod{m+n}$ with $0 \leq i \leq m-1$ and $\alpha_{a+1} = q_3^{-1}\alpha_a$ if $a \equiv i \pmod{m+n}$ with $m \leq i \leq m+n-1$. Thus α_a is a power of q_3, and $\alpha_{k(m+n)} = q_3^{(m-n)k}$. In particular, since $m \neq n$, $a \equiv b$ together with $a \neq b$ implies $\alpha_a \neq \alpha_b$.

Since $[\boldsymbol{a},\boldsymbol{b}] \in P$ is N admissible, there exists a bijection τ between the set $[1,m+n]\times[1,N]$ with the union of intervals $[a_j,b_j]$:

$$\tau:\ \{1,\dots,m+n\}\times\{1,\dots,N\} \to \sqcup_{j=1}^{s}\{a_j,a_j+1,\dots,b_j\}.$$

Let $\tau_1(i,j) \in \{1,\dots,s\}$ be the label of the interval of $\tau(i\times j)$, and let $\tau_2(i,j) \in [a_{\tau_1(i,j)},b_{\tau_1(i,j)}]$ be the number in that interval given by $\tau(i\times j)$. Thus $\tau(i\times j) = \tau_2(i,j) \in [a_{\tau_1(i,j)},b_{\tau_1(i,j)}]$.

We choose τ in such a way that the color is preserved: $\tau_2(i,j) \equiv i \pmod{m+n}$. Clearly such a bijection exists, and it is not unique in general.

We view τ as a substitution of variables $x_{i,j}$ appearing in Sh substituting $x_{i,j}$ to $\alpha_{\tau_2(i,j)}\, y_{\tau_1(i,j)}$.

Recall that functions $F \in Sh$ have the form

$$F = \frac{f(x_1;\dots;x_{m+n})}{\Pi} \qquad \Pi = \prod_{i=1}^{m+n}\prod_{r,l=1}^{N}(x_{i,r}-x_{i+1,l}),$$

where the numerator f is a Laurent polynomial of variables $x_{i,j}$ symmetric in bosonic variables with the same first index, skew-symmetric in the fermionic variables with the same first index and satisfying the wheel conditions.

If $F \in \mathcal{B}$, then the Laurent polynomial f is actually a polynomial since the limit as $x_{i,j} \to 0$ exists. The total degree of polynomial f is the same as the total degree of Π, that is $(m+n)N^2$ since the limits when all variables go to infinity, $F_\infty^{(N,\dots,N)}$, or to zero $F_0^{(N,\dots,N)}$, are well-defined. In each variable the degree is at most $2N$ since the limit when this variable goes to infinity is well-defined.

Define an evaluation map

$$\phi_{[\boldsymbol{a},\boldsymbol{b}]}:\ Sh \to \mathbb{C}[y_1,\dots,y_s], \qquad F(x_{i,j}) \mapsto f(\tau(x_{i,j})),$$

substituting all variables $x_{i,k}$ in the numerator of $F \in Sh$ to y_j according to the substitution τ.

Note that because of the symmetry of polynomial $f \in Sh$ the result of substitution up to a sign is independent of the choice of the bijection τ. In particular the kernel of the evaluation map $\operatorname{Ker}\phi_{[\boldsymbol{a},\boldsymbol{b}]} \subset Sh$ does not depend on the choice of τ.

Note that if $[\boldsymbol{a},\boldsymbol{b}]$ and $[\boldsymbol{c},\boldsymbol{d}]$ are equivalent: $a_j - c_j = b_j - d_j = Nk_j$ then $\phi_{[\boldsymbol{a},\boldsymbol{b}]}$ coincides with $\phi_{[\boldsymbol{c},\boldsymbol{d}]}$ followed by the shift $y_j \to q_3^{k_j(m-n)}y_j$ for all j. In particular, $\operatorname{Ker}\phi_{[\boldsymbol{a},\boldsymbol{b}]} = \operatorname{Ker}\phi_{[\boldsymbol{c},\boldsymbol{d}]}$.

Note that $m \neq n$ is important here since $F \in Sh$ vanishes if two fermionic variables are set equal and therefore the evaluation map would be zero in the case when one of the intervals $[a_j,b_j]$ contains two numbers congruent to m or to $m+n$.

Note also that (unlike the purely bosonic case), in some cases, the evaluation applied to the denominator Π can give a zero.

Define

$$B_{[\boldsymbol{a},\boldsymbol{b}]} = \operatorname{Ker} \phi_{[\boldsymbol{a},\boldsymbol{b}]}, \qquad \bar{B}_{[\boldsymbol{a},\boldsymbol{b}]} = \bigcap_{[\boldsymbol{c},\boldsymbol{d}]>[\boldsymbol{a},\boldsymbol{b}]} B_{[\boldsymbol{c},\boldsymbol{d}]}.$$

We claim that $\cap_{[\boldsymbol{a},\boldsymbol{b}]} B_{[\boldsymbol{a},\boldsymbol{b}]} = 0$. Indeed, let $F = f/\Pi \in Sh$ vanish under all evaluation maps. In particular, it has to vanish at $x_{i+1,r} = q_3 x_{i,l}$ if i is before the equator and at $x_{i+1,r} = q_3^{-1} x_{i,l}$ if i is after the equator. Thus the polynomial f of degree $(m+n)N^2$ has $(m+n)N^2$ linear factors and is determined up to a constant. Thus

$$F = \frac{c}{\Pi} \prod_{i=0}^{m-1} \prod_{r,l=1}^{N} (x_{i+1,r} - q_3 x_{i,l}) \prod_{i=m}^{m+n-1} \prod_{r,l=1}^{N} (x_{i+1,r} - q_3^{-1} x_{i,l}),$$

where c is a constant. But such F is clearly not in B.

Next we study $\operatorname{gr} B_{[\boldsymbol{a},\boldsymbol{b}]} = \phi_{[\boldsymbol{a},\boldsymbol{b}]}(\bar{B}_{[\boldsymbol{a},\boldsymbol{b}]})$. We now show that this space is at most one dimensional.

Denote $g = \phi_{[\boldsymbol{a},\boldsymbol{b}]}(F)$. Recall that $\deg g = \deg f = (m+n)N^2$. We claim that g has $(m+n)N^2$ linear factors, one for each factor of Π, and therefore it is unique up to a constant which implies $\operatorname{gr} B_p$ is one-dimensional.

Let us say a variable $x_{i,r}$ has type j if it specializes to a multiple of y_j: $\tau_1(i,r) = j$.

First, g is divisible by $y_j^{r_j}$ where r_j is the number of factors $x_{i,r} - x_{i+1,l}$ in Π such that both $x_{i,r}$ and $x_{i+1,l}$ have type j. This is because the limit of F as all variables of type j go to zero should be well defined.

Next we consider linear factors of g of the form $y_j - \alpha y_{j'}$, $j \neq j'$ and $\alpha \in q_1^{\mathbb{Z}} q_3^{\mathbb{Z}}$. For that we fix an order $y_1, \dots, y_s$ in such a way that the corresponding intervals are ordered: $b_i - a_i \geq b_{i+1} - a_{i+1}$.

Consider a factor $x_{i,r} - x_{i+1,l}$ in Π where the variables $x_{i,r}$, $x_{i+1,l}$ are specialized to two different variables. Suppose $\tau(i \times r)$ is $c \in [a_j, b_j]$ and $\tau((i+1) \times l)$ is $c' \in [a_{j'}, b_{j'}]$. We present a corresponding zero as follows.

If $j > j'$ and c' is bosonic, $c' \not\equiv m, m+n \pmod{m+n}$, and $c < b_j$, then we obtain a zero from the cubic wheel condition corresponding to variables $x_{i,r}, x_{i+1,l}$ and the variable which corresponds to $(c+1) \in [a_j, b_j]$. It guarantees that g is divisible by $y_j - \alpha y_{j'}$ with $\alpha = q_1^{\pm 1} \alpha_{c'}/\alpha_c$. Here we have q_1 if c' is before the equator, that is $c' \equiv 1, \dots, m-1$ modulo $m+n$, and q_1^{-1} if c' is after equator, that is $c' = \equiv m+1, \dots, m+n-1 \pmod{m+n}$.

In the case $c = b_j$, we obtain a factor $y_j - \alpha y_{j'}$ with $\alpha = q_3^{\pm 1} \alpha_{c'}/\alpha_c$ due to the fact that f should vanish under substitutions corresponding to larger sets of intervals in the sense of the order in P.

If $j' > j$ and c is bosonic, $c \not\equiv m, m+n$ and $c' > a_{j'}$, then we similarly obtain a zero from the cubic wheel condition corresponding to variables $x_{i,r}, x_{i+1,l}$ and the variable which corresponds to $(c'-1) \in [a_{j'}, b_{j'}]$. In the case $c' = a_{j'}$, we again obtain a factor $y_j - \alpha y_{j'}$ with $\alpha = q_3^{\pm 1} \alpha_{c'}/\alpha_c$ due to the fact that f should vanish under larger substitutions.

Let $j > j'$ and let c' be fermionic, $c' \equiv m$ or $c' \equiv m+n$. We have three cases.

If $c+1 < b_j$ then we use the quartic wheel condition involving $x_{i,r}, x_{i+1,l}$ and the variables which correspond to $c+1, c+2 \in [a_j, b_j]$. Moreover, in this case we also have a factor coming from the skew-symmetry with respect to $x_{i+1,l}$ and the variable corresponding to $c+1 \in [a_j, b_j]$. This zero corresponds to the factor of the denominator Π which is a difference of $x_{i+1,l}$ and the variable which corresponds to $c+2 \in [a_j, b_j]$.

If $c+1 = b_j$, we get the factor from the skew-symmetry with respect to $x_{i+1,l}$ and the variable which corresponds to $c+1 \in [a_j, b_j]$.

If $c = b_j$, we get a factor due to the fact that f should vanish under larger substitutions.

This covers all factors in the denominator. It remains to point out that some of the linear factors we discussed coincide. However, all coinciding factors of the form $y_j - \alpha y_{j'}$, $j' < j$, come from using conditions each of which uses a single variable of type j'. Moreover, all these variables of type j' are distinct. Therefore, using the derivatives with respect to these variables, we see that we get $(y_j - \alpha y_{j'})^t$ where t is the number of times this factor appeared in our construction.

Finally we claim the $\operatorname{gr} B_{[\boldsymbol{a},\boldsymbol{b}]}$ is trivial unless $[\boldsymbol{a},\boldsymbol{b}]$ is cyclic. Indeed, fix j and consider the limits of $F \in B \subset Sh$ which sends variables of type j to zero or to infinity. Substitute $x_{i,r} \to \xi x_{i,r}$ in F for variables $x_{i,r}$ of type j. Then we have factors in the denominator Π which do not depend on ξ. These are binomials, where variables either both have type j or both do not have type j. These factors cancel when we take the ratio of the limits $\xi \to 0$ and $\xi \to \infty$. Therefore we can remove these factors. After that we proceed by applying the evaluation as in $\phi_{[\boldsymbol{a},\boldsymbol{b}]}$. Take now the ratio of the limits $\xi \to 0$ and $\xi \to \infty$. Then the result is a possible monomial in $y_j^{\pm 1}$ with a possible coefficient $q_1^a q_3^b$, $a, b \in \mathbb{Z}$. By definition, since $F \in B$, this ratio equals to a product of s_i multiplied by a power of d as in (4.2). Since s is generic, it follows that the length of j-th interval has to be a multiple of $m+n$.

Combining, we see that the dimension of B is at most the number of non-empty N-admissible cyclic sets of intervals which is exactly $\dim \mathcal{R}_{m|n,N}$. □

Remark 4.5. *The same proof shows that if there exists a non-zero $F \in Sh_{m|n,\boldsymbol{N}}$ such that both limits* (4.1) *exist for all $\boldsymbol{k} = (k_1, \dots, k_{m+n})$ satisfying $0 \le k_i \le N_i$ and such that* (4.2) *holds, then $\boldsymbol{N} = (N, \dots, N)$ for some $N \in \mathbb{Z}_{\ge 0}$.*

In other words, the Bethe algebra indeed belongs to $Sh^0_{m|n}$. □

5. Generators and commutativity of $\mathcal{B}_{m|n}(s)$

5.1. **Functions $I^c_{M,N}$.** Before proceeding, we prepare and study some auxiliary rational functions which will be used in the next subsection to construct a family of elements of the subalgebra $\mathcal{B}_{m|n}(s)$.

For variables $y = (y_1, \dots, y_M)$ and $z = (z_1, \dots, z_N)$, we shall use the notation

$$\Delta(y) = \prod_{1 \le a < b \le M} (y_a - y_b)\,, \quad \bar{y} = \prod_{a=1}^{M} y_a\,, \quad \Pi(y,z) = \prod_{a=1}^{M}\prod_{b=1}^{N} (y_a - z_b)\,,$$

and similarly for $\Delta(z), \bar{z}$. We shall use also $[n] = (q^n - q^{-n})/(q - q^{-1})$.

For $c \in \mathbb{Z}$, define rational functions $I^c_{M,N}(y,z)$ by

$$I^c_{M,N}(y,z) = \frac{1}{\Delta(y)} \prod_{a=1}^{M} \Big(\frac{q^c T_{q,y_a} - q^{-c} T^{-1}_{q,y_a}}{q - q^{-1}} \Big) \Big(\frac{\Delta(y)}{\Pi(y,z)} \Big)\,, \tag{5.1}$$

where $(T_{q,y_a} f)(y_1, \dots, y_M) = f(y_1, \dots, q y_a, \dots, y_M)$.

The functions $I^c_{M,N}(y,z)$ are clearly symmetric in $y_1, \dots, y_M$ and in $z_1, \dots, z_N$.

Lemma 5.1. *Functions $I^c_{M,N}(y,z)$ enjoy the following properties:*

$$I^c_{M,N}(y,z) = (-1)^{MN+N} \prod_{i=0}^{M-N-1} [c+i] \cdot I^{1-c}_{N,M}(z,y) \quad \text{if } M \geq N\,, \tag{5.2}$$

$$I^c_{M,N}(y,z) = 0 \quad \text{if } M > N \text{ and } N-M+1 \leq c \leq 0, \tag{5.3}$$

$$\bar{y} \cdot I^1_{N,N}(y,z) = \bar{z} \cdot I^0_{N,N}(y,z)\,. \tag{5.4}$$

Proof. Let $s_\lambda(y)$ denote the Schur polynomial associated with a partition λ:

$$\Delta(y) s_\lambda(y) = \det\Big(y_a^{\lambda_b+M-b}\Big)_{1\leq a,b\leq M}\,.$$

Here $\lambda = (\lambda_1,\dots,\lambda_\ell,0,0,\dots)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$, $\ell = \ell(\lambda)$.

Using the Cauchy identity we can write

$$\begin{aligned}\frac{\Delta(y)}{\Pi(y,z)} &= (-1)^{MN}\bar{z}^{-M}\frac{\Delta(y)}{\prod_{\substack{1\leq a\leq M\\ 1\leq b\leq N}}(1-y_a z_b^{-1})}\\ &= (-1)^{MN}\bar{z}^{-M}\sum_\lambda \Delta(y)s_\lambda(y)s_\lambda(z^{-1})\,,\end{aligned}$$

where $z^{-1} = (z_1^{-1},\dots,z_N^{-1})$, and the sum is over all partitions λ with length $\ell(\lambda) \leq \min(M,N)$. The determinant expression of $\Delta(y)s_\lambda(y)$ readily yields

$$\prod_{a=1}^M \Big(\frac{q^c T_{q,y_a} - q^{-c}T^{-1}_{q,y_a}}{q-q^{-1}}\Big)\Big(\Delta(y)s_\lambda(y)\Big) = \prod_{a=1}^M [\lambda_a + M - a + c]\cdot \Delta(y)s_\lambda(y)\,,$$

from which we obtain

$$I^c_{M,N}(y,z) = (-1)^{MN}\bar{z}^{-M}\sum_\lambda \prod_{a=1}^M [\lambda_a + M - a + c]\, s_\lambda(y)s_\lambda(z^{-1})\,. \tag{5.5}$$

By a similar calculation we have also

$$I^c_{N,M}(z,y) = (-1)^{N}\bar{z}^{-M}\sum_\lambda \prod_{a=1}^N [\lambda_a + M - a + 1 - c]\, s_\lambda(z^{-1})s_\lambda(y)\,. \tag{5.6}$$

Comparing (5.5) and (5.6) we obtain (5.2).

Property (5.3) is an immediate consequence of (5.2).

To see (5.4), choose $M = N$ and $c = 0$ in (5.5). Then only partitions with $\lambda_N > 0$ contribute to the sum. Rewriting λ_a as $\lambda_a + 1$ for $1 \leq a \leq N$ and using $s_{\lambda+(1,\dots,1)}(y) = \bar{y}s_\lambda(y)$, we obtain (5.4). □

Lemma 5.2. *For $0 \leq k \leq M$ and $0 \leq l \leq N$, consider the substitution*

$$(y,z) = (\xi y', y'', \xi z', z'')\,,$$

where

$$\begin{aligned} y' &= (y_1,\dots,y_k)\,, \quad y'' = (y_{k+1},\dots,y_M)\,,\\ z' &= (z_1,\dots,z_l)\,, \quad z'' = (z_{l+1},\dots,z_N)\,.\end{aligned}$$

Then the following asymptotics hold:

$$\begin{aligned}
&I^c_{M,N}(\xi y', y'', \xi z', z'')\\
&= \frac{(-1)^{(M-k)l}\xi^{-kN-(M-k)l}}{(\bar{y'})^{N-l}(\bar{z'})^{M-k}}\Big(I^{c+M-k-N+l}_{k,l}(y',z')I^c_{M-k,N-l}(y'',z'') + O(\xi^{-1})\Big) \quad \text{as } \xi\to\infty\,,\\
&= \frac{(-1)^{k(N-l)}\xi^{-kl}}{(\bar{y''})^{l}(\bar{z''})^{k}}\Big(I^{c}_{k,l}(y',z')I^{c+k-l}_{M-k,N-l}(y'',z'') + O(\xi)\Big) \quad \text{as } \xi\to 0\,.
\end{aligned}$$

Proof. This follows from a direct calculation, noting

$$\begin{aligned}
&\Delta(\xi y', y'') = \xi^{k(k-1)/2}\Delta(y')\Delta(y'')\times\begin{cases}\xi^{k(M-k)}\Big(\big(\bar{y}'\big)^{M-k}+O(\xi^{-1})\Big) & \text{as } \xi\to\infty,\\ (-1)^{k(M-k)}\big(\bar{y}''\big)^{k}+O(\xi) & \text{as } \xi\to 0,\end{cases}\\
&\Pi(\xi y', y'', \xi z', z'') = \xi^{kl}\Pi(y',z')\Pi(y'',z'')\\
&\times\begin{cases}\xi^{k(N-l)+(M-k)l}\Big((-1)^{(M-k)l}\big(\bar{y}'\big)^{N-l}\big(\bar{z}'\big)^{M-k}+O(\xi^{-1})\Big) & \text{as } \xi\to\infty,\\ (-1)^{k(N-l)}\big(\bar{y}''\big)^{l}\big(\bar{z}''\big)^{k}+O(\xi) & \text{as } \xi\to 0.\end{cases}
\end{aligned}$$

□

The following is a corollary of Lemma 5.2.

Lemma 5.3. *Let $M = N$ and $c > 0$. Then the limit*

$$\lim_{\xi\to\infty}\xi^{N(k+l)-kl}I^c_{N,N}(\xi y', y'', \xi z', z'')$$

exists. This limit is non-zero only if $c > k - l$. Similarly the limit

$$\lim_{\xi\to 0}\xi^{kl}I^c_{N,N}(\xi y', y'', \xi z', z'')$$

exists. This limit is non-zero only if $c > -k + l$.

Proof. By (5.3), $I^{c-k+l}_{k,l}(y',z') = 0$ if $k > l$ and $1 \le c \le k - l$. Similarly $I^{c+k-l}_{N-k,N-l}(y'',z'') = 0$ if $k < l$ and $1 \le c \le l - k$. □

Clearing the denominator we define

$$\tilde{I}^c_{M,N}(y,z) = (q-q^{-1})^M\prod_{\substack{1\le a\le M\\ 1\le b\le N}}(qy_a - z_b)(q^{-1}y_a - z_b)\cdot I^c_{M,N}(y,z)\,. \tag{5.7}$$

Then $\tilde{I}^c_{M,N}(y,z)$ is a polynomial in y, z, symmetric in y, symmetric in z, and of total degree MN. It is also a Laurent polynomial in q^c:

$$\begin{aligned}
&\tilde{I}^c_{M,N}(y,z) = \sum_{l=0}^{M}(-1)^l q^{(M-2l)c}\tilde{I}_{M,N,l}(y,z)\,, \qquad (5.8)\\
&\tilde{I}_{M,N,l}(y,z) = q^{(M-l)(M-l-1)/2-l(l-1)/2}\sum_{|I|=l}\prod_{\substack{a\in I\\ b\notin I}}\frac{q^{-1}y_a - qy_b}{y_a-y_b}\prod_{\substack{a\notin I\\ 1\le b\le N}}(q^{-1}y_a - z_b)\prod_{\substack{a\in I\\ 1\le b\le N}}(qy_a - z_b)\,,
\end{aligned}$$

where the summation is over subsets I of $\{1,\dots,M\}$.

In particular we have

$$\tilde{I}^c_{N,N}(y, q^{\pm 1}y) = q^{\pm N(N-1)/2} q^{\pm Nc} \prod_{1\le a,b\le N} (q^{-1}y_a - qy_b)\,. \tag{5.9}$$

Lemma 5.4. *Let $N \ge 2$, $c \in \mathbb{Z}$. Then*

$$I^c_{N,N}(y,z) = 0 \quad \text{if } y_1 = q^{-1}z_1, y_2 = qz_1,$$
$$I^c_{N,N}(y,z) = 0 \quad \text{if } z_1 = q^{-1}y_1, z_2 = qy_1.$$

Proof. The first relation follows from (5.8). Applying the duality (5.2) we obtain the second. □

5.2. **Elements $G^{m|n}_{r,N}(s)$.** In this subsection we introduce rational functions $G^{m|n}_{r,N}(s)$, $G^{*,m|n}_{r,N}(s)$ and show that they belong to the subalgebra $\mathcal{B}_{m|n}(s)$. Moreover, we will show that $G^{m|n}_{r,N}(s)$, $G^{*,m|n}_{r,N}(s)$ generate the Bethe algebra $\mathcal{B}_{m|n}(s)$, see Theorem 5.15 below.

Conjecturally, generating series of the form $\sum_{N=0}^{\infty} \gamma_N G^{m|n}_{r,N}(s) u^{-N}$, $\sum_{N=0}^{\infty} \gamma^*_N G^{*,m|n}_{r,N}(s) u^{-N}$ with some constants γ_N, γ^*_N correspond to the trace of the universal R-matrix of quantum affine algebra $U_q\mathfrak{gl}_{m|n}$ computed on the evaluation module with evaluation parameter u obtained from the r-th skew-symmetric power of the vector and covector representations, with weights determined by s.

We retain the notation in the previous sections, assuming $m \ge 1$, $n \ge 0$, $m \ne n$. In particular, we fix $s_1, \dots, s_{m+n} \in (\mathbb{C}^\times)^{m+n}$ such that $\prod_{i=1}^{m+n} s_i = 1$. We shall write

$$\mathbf{s}_i = \prod_{1\le j\le i-1} s_j\,, \quad \mathbf{s}_1 = 1\,.$$

Fixing N we set $\bar{x}_i = \prod_{a=1}^{N} x_{i,a}$ and

$$t_i(s) = \mathbf{s}_i \frac{\bar{x}_{i-1}}{\bar{x}_i}\,, \quad J^c_i(s) = \left(-\mathbf{s}_i d^N\right)^c \tilde{I}^{c+1}_{N,N}(dx_{i-1}, x_i)\,.$$

Set further

$$P^{m|n}_N = \begin{cases} \dfrac{\prod_{i=1}^{m-1}\prod_{1\le a,b\le N}(x_{i,a} - q_2 x_{i,b})}{\prod_{i=1}^{m+n}\prod_{1\le a,b\le N}(x_{i,a} - x_{i+1,b})} \cdot \bar{x}_m \Delta(x_m)\Delta(x_{m+n}) & \text{if } m,n \ge 1, \\ \displaystyle\prod_{i=1}^{m}\prod_{1\le a,b\le N} \frac{x_{i,a} - q_2 x_{i,b}}{x_{i,a} - x_{i+1,b}} & \text{if } m \ge 2,\ n = 0, \\ \displaystyle\prod_{1\le a<b\le N} \frac{(x_{1,a} - q_2 x_{1,b})(x_{1,a} - q_2^{-1} x_{1,b})}{(x_{1,a} - x_{1,b})^2} & \text{if } m = 1,\ n = 0. \end{cases} \tag{5.10}$$

Definition 5.5. *For $r \ge 0$, we define*

$$G^{m|n}_{r,N}(s) = P^{m|n}_N \sum_{\substack{1\le i_1<\dots<i_p\le m \\ p+c_{m+1}+\dots+c_{m+n}=r}} t_{i_1}(s)\cdots t_{i_p}(s) \prod_{i=m+1}^{m+n} J^{c_i}_i(s)\,, \tag{5.11}$$

The sum is taken over all $i_1, \dots, i_p$, $p \ge 0$, and $c_{m+1}, \dots, c_{m+n} \ge 0$, subject to the indicated condition. □

Terms of $G^{m|n}_{r,N}(s)$ are in one-to-one correspondence with semi-standard Young tableaux of a column of length r in the alphabet $\{1,2,\dots,m+n\}$. Letters are arranged in non-decreasing order, where each $i\in\{1,\dots,m\}$ can appear at most once while each $i\in\{m+1,\dots,m+n\}$ is allowed to repeat c_i times. In other words, the terms of $G^{m|n}_{r,N}(s)$ are in one-to-one correspondence with a basis of the r-th skew-symmetric power of the vector representation $\mathbb{C}^{m|n}$ of $\mathfrak{gl}_{m|n}$.

Clearly, $G^{m|n}_{0,N}=P^{m|n}_N\prod_{i=m+1}^{m+n}J^0_i(s)$.

When $n=0$, we write $P^m_N=P^{m|0}_N$, $G^m_{r,N}(s)=G^{m|0}_N(s)$.

Example 5.6. If $n=0$, then $G^m_{r,N}(s)/P^m_N$ is the r-th elementary symmetric polynomial in the variables $t_1(s),\dots,t_m(s)$. Hence $G^m_{m,N}(s)=G^m_{0,N}(s)\prod_{i=1}^m\mathbf{s}_i$ and $G^m_{r,N}(s)=0$ for $r>m$. □

Example 5.7. Let $m=3,n=2$ and $r=2$. Suppressing s we have

$$\begin{aligned}G^{3|2}_{2,N}/P^{3|2}_N &= t_1t_2J^0_4J^0_5+t_1t_3J^0_4J^0_5+t_2t_3J^0_4J^0_5\\ &\quad+t_1J^1_4J^0_5+t_2J^1_4J^0_5+t_3J^1_4J^0_5+t_1J^0_4J^1_5+t_2J^0_4J^1_5+t_3J^0_4J^1_5\\ &\quad+J^2_4J^0_5+J^1_4J^1_5+J^0_4J^2_5\,.\end{aligned}$$

□

Example 5.8. Let $m=2,n=1$ and $N=1$. Denote $x_{1,1}=x,x_{2,1}=y,x_{3,1}=z$. Then x is bosonic while y,z are fermionic. For $N=1$ we have no wheel conditions.

Then one easily checks that $B_{2|1,1}(s)$ is three dimensional and consists of elements

$$\frac{A\,xyz+B(s_1s_2d^2\,x^2y+s_2d^2\,y^2z+z^2x)+C(s_1s_2d^2\,y^2x+s_1\,x^2z+z^2y)}{(x-y)(y-z)(x-z)},$$

where A,B,C are arbitrary constants. We have $\mathbf{s}_1=1$, $\mathbf{s}_2=s_1,\mathbf{s}_3=s_1s_2$, so $t_1(s)=z/x$, $t_2(s)=s_1x/y$, $t_3(s)=s_1s_2y/z$.

We have a basis given by

$$\begin{aligned}G^{2|1}_{0,1}&=\frac{q^{-1}(1-q_2)^2xyz}{(x-y)(y-z)(z-x)}\,,\\ G^{2|1}_{1,1}&=\frac{q^{-1}(1-q_2)^2}{(x-y)(y-z)(z-x)}\Big(s_1s_2d^2y^2x+s_1x^2z+z^2y-(q_1+q_3^{-1})s_1s_2xyz\Big)\,,\\ G^{2|1}_{2,1}&=\frac{1-q_2}{(x-y)(y-z)(z-x)}\Big(-(q-q^{-1})s_1\big(s_1s_2d^2x^2y+s_2d^2y^2z+z^2x\big)\\ &\quad+(q^2-q^{-2})s_1s_2d\big(s_1s_2d^2y^2x+s_1x^2z+z^2y\big)-(q^3-q^{-3})s_1^2s_2^2d^2xyz\Big)\,.\end{aligned}$$

□

In view of the isomorphism (2.3), we introduce also the following elements for $n\geq 1$:

$$G^{*,m|n}_{r,N}(s)=P^{*,m|n}_N\sum_{\substack{m+1\leq i_1<\dots<i_p\leq m+n\\ p+c_{m+1}+\dots+c_{m+n}=r}}t_{i_1}(s)\cdots t_{i_p}(s)\prod_{i=1}^m J^{*,c_i}_i(s)\,,\tag{5.12}$$

where

$$P_N^{*m|n} = \frac{\prod_{i=m+1}^{m+n-1}\prod_{1\le a,b\le N}(x_{i,a}-q_2^{-1}x_{i,b})}{\prod_{i=1}^{m+n}\prod_{1\le a,b\le N}(x_{i,a}-x_{i+1,b})}\cdot \bar{x}_{m+n}\Delta(x_m)\Delta(x_{m+n})\,,$$
$$J_i^{*c}(s) = \bigl(-\mathbf{s}_i d^{-N}\bigr)^c\,\tilde{I}_{N,N}^{c+1}(d^{-1}x_{i-1},x_i)\,.$$

In terms of the generating series

$$G_N^{m|n}(u;s) = \sum_{r=0}^{\infty}(-u)^{-r}G_{r,N}^{m|n}(s)\,,\quad G_N^{*m|n}(u;s) = \sum_{r=0}^{\infty}(-u)^{-r}G_{r,N}^{*m|n}(s)\,, \tag{5.13}$$

the definition (5.11), (5.12) can be written compactly as

$$G_N^{m|n}(u;s) = P_N^{m|n}\prod_{i=1}^{m}(1-t_i(s)/u)\prod_{i=m+1}^{m+n}J_i(u;s)\,, \tag{5.14}$$

$$G_N^{*,m|n}(u;s) = P_N^{*,m|n}\prod_{i=m+1}^{m+n}(1-t_i(s)/u)\prod_{i=1}^{n}J_i^*(u;s)\,. \tag{5.15}$$

Here we set

$$J_i(u;s) = \sum_{c=0}^{\infty}(-u)^{-c}J_i^c(s) = \sum_{l=0}^{N}(-1)^l\frac{q^{N-2l}}{1-\mathbf{s}_i/(q_2^lq_3^Nu)}\tilde{I}_{N,N,l}(dx_{i-1},x_i)\,,$$
$$J_i^*(u;s) = \sum_{c=0}^{\infty}(-u)^{-c}J_i^{*,c}(s) = \sum_{l=0}^{N}(-1)^l\frac{q^{N-2l}}{1-\mathbf{s}_i/(q_2^lq_1^{-N}u)}\tilde{I}_{N,N,l}(d^{-1}x_{i-1},x_i)\,,$$

with $\tilde{I}_{N,N,l}$ being given in (5.8).

Our goal in this subsection is to show the following.

Proposition 5.9. *For all $r\ge 0$, the elements $G_{r,N}^{m|n}(s)$, $G_{r,N}^{*m|n}(s)$ belong to $\mathcal{B}_{m|n,N}(s)$.*

For the proof we prepare two Lemmas.

Let $\mathbf{k}=(k_1,\dots,k_{m+n})$ be such that $0\le k_i\le N$ for all i. Consider each summand in $G_{r,N}^{m|n}(s)$ with the substitution $x_i\to(\xi x_i',x_i'')$, where $x_i'=(x_{i,1},\dots,x_{i,k_i})$ and $x_i''=(x_{i,k_i+1},\dots,x_{i,N})$:

$$\bigl(P_N^{m|n}t_{i_1}(s)\cdots t_{i_p}(s)\prod_{i=m+1}^{m+n}J_i^{c_i}(s)\bigr)_\xi^{\mathbf{k}}\,. \tag{5.16}$$

Our first task is to estimate its behavior as $\xi\to\infty$ or $\xi\to 0$.

Lemma 5.10. *(i) The limit of each term* (5.16) *exists for both $\xi\to\infty$ and $\xi\to 0$.*
(ii) The limit $\xi\to\infty$ is non-zero only if the following are satisfied:

$$k_{i-1}-k_i\in\{0,1\}\quad \text{if } i\in\{i_1,\dots,i_p\}\,,$$
$$k_{i-1}-k_i\in\{0,-1\}\quad \text{if } i\in\{1,\dots,m\}\backslash\{i_1,\dots,i_p\}\,,$$
$$c_i\ge k_{i-1}-k_i\quad \text{if } i\in\{m+1,\dots,m+n\}.$$

(iii) The limit $\xi \to 0$ is non-zero only if the following are satisfied:

$$\begin{aligned} &k_{i-1} - k_i \in \{0, -1\} \quad \text{if } i \in \{i_1, \dots, i_p\}\,, \\ &k_{i-1} - k_i \in \{0, 1\} \quad \text{if } i \in \{1, \dots, m\} \backslash \{i_1, \dots, i_p\}\,, \\ &c_i \geq -k_{i-1} + k_i \quad \text{if } i \in \{m+1, \dots, m+n\}. \end{aligned}$$

Proof. A direct calculation shows that

$$(P_N^{m|n})_\xi^{\mathbf{k}} = \begin{cases} O(\xi^{\mu_\infty}) & \text{as } \xi \to \infty, \\ O(\xi^{\mu_0}) & \text{as } \xi \to 0, \end{cases}$$

where

$$\begin{aligned} \mu_\infty &= -N \sum_{i=m+1}^{m+n} \big(k_{i-1} + k_i\big) - \sum_{i=1}^{m-1} k_i^2 + \sum_{i=1}^{m+n} k_{i-1}k_i - \frac{1}{2}k_m(k_m - 1) - \frac{1}{2}k_{m+n}(k_{m+n} + 1)\,, \\ \mu_0 &= \sum_{i=1}^{m-1} k_i^2 - \sum_{i=1}^{m+n} k_{i-1}k_i + \frac{1}{2}k_m(k_m + 1) + \frac{1}{2}k_{m+n}(k_{m+n} - 1)\,. \end{aligned}$$

Clearly $\big(t_i\big)_\xi^{\mathbf{k}} = \xi^{k_{i-1} - k_i} t_i$. From Lemma 5.2 we have also

$$\big(J_i^{c_i}\big)_\xi^{\mathbf{k}} = \begin{cases} O(\xi^{N(k_{i-1}+k_i) - k_{i-1}k_i}) & \text{as } \xi \to \infty, \\ O(\xi^{k_{i-1}k_i}) & \text{as } \xi \to 0. \end{cases}$$

Combining these, we find that

$$\Big(P_N^{m|n} t_{i_1}(s) \cdots t_{i_p}(s) \prod_{i=m+1}^{m+n} J_i^{c_i}(s)\Big)_\xi^{\mathbf{k}} = \begin{cases} O(\xi^{\nu_\infty}) & \text{as } \xi \to \infty, \\ O(\xi^{\nu_0}) & \text{as } \xi \to 0, \end{cases}$$

with

$$\begin{aligned} \nu_\infty &= \mu_\infty + \sum_{t=1}^{p} \big(k_{i_t - 1} - k_{i_t}\big) + \sum_{i=m+1}^{m+n} \big(N(k_{i-1} + k_i) - k_{i-1}k_i\big) \\ &= -\frac{1}{2}\sum_{t=1}^{p} (k_{i_t - 1} - k_{i_t})(k_{i_t - 1} - k_{i_t} - 1) - \frac{1}{2} \sum_{\substack{1 \leq i \leq m \\ i \neq i_1, \dots, i_p}} (k_{i-1} - k_i)(k_{i-1} - k_i + 1)\,, \\ \nu_0 &= \mu_0 + \sum_{t=1}^{p} \big(k_{i_t - 1} - k_{i_t}\big) + \sum_{i=m+1}^{m+n} k_{i-1}k_i \\ &= \frac{1}{2}\sum_{t=1}^{p} (k_{i_t - 1} - k_{i_t})(k_{i_t - 1} - k_{i_t} + 1) + \frac{1}{2} \sum_{\substack{1 \leq i \leq m \\ i \neq i_1, \dots, i_p}} (k_{i-1} - k_i)(k_{i-1} - k_i - 1)\,. \end{aligned}$$

For any k_i we have $\nu_\infty \leq 0$, which implies that the limit $\xi \to \infty$ exists. The equality holds if and only if $k_{i-1} - k_i \in \{0, 1\}$ for $i = i_1, \dots, i_p$ and $k_{i-1} - k_i \in \{0, -1\}$ for $1 \leq i \leq m$, $i \neq i_1, \dots, i_p$. Furthermore, by Lemma 5.3, we must have $c_i \geq k_{i-1} - k_i$ for $m+1 \leq i \leq m+n$ in order that the limit be non-zero. This proves (ii).

Assertion (iii) is shown similarly. □

Next we compute the leading coefficients.

Lemma 5.11. *We have*

$$\big(P_N^{m|n}\big)^{\mathbf{k}}_{\xi} = \begin{cases} \xi^{\mu_\infty}\Big(X_\infty + O(\xi^{-1})\Big) & \text{as } \xi \to \infty, \\ \xi^{\mu_0}\Big(X_0 + O(\xi)\Big) & \text{as } \xi \to 0, \end{cases}$$

where μ_∞, μ_0 are as in the proof of Lemma 5.10 and

$$\frac{X_\infty}{X_0} = (-1)^{k_m(N-k_m)+k_{m+n}(N-k_{m+n})} \prod_{i=1}^{m} \Big(\frac{\bar{x}_{i-1}}{\bar{x}_i}\Big)^{k_i-k_{i-1}} \prod_{i=m+1}^{m+n} \frac{\bar{x}_{i-1}^{k_i}\bar{x}_i^{k_{i-1}}}{\big(\bar{x}'_{i-1}\bar{x}'_i\big)^N}\,.$$

Proof. This is a direct computation. Assuming $n \geq 1$, set

$$P' = \frac{\prod_{i=1}^{m-1}\prod_{1\leq a,b\leq k_i}(x_{i,a}-q_2x_{i,b})}{\prod_{i=1}^{m+n}\prod_{\substack{1\leq a\leq k_i\\ 1\leq b\leq k_{i+1}}}(x_{i,a}-x_{i+1,b})} \cdot \bar{x}'_m\,\Delta(x'_m)\Delta(x'_{m+n})\,,$$

$$P'' = \frac{\prod_{i=1}^{m-1}\prod_{k_i+1\leq a,b\leq N}(x_{i,a}-q_2x_{i,b})}{\prod_{i=1}^{m+n}\prod_{\substack{k_i+1\leq a\leq N\\ k_{i+1}+1\leq b\leq N}}(x_{i,a}-x_{i+1,b})} \cdot \bar{x}''_m\,\Delta(x''_m)\Delta(x''_{m+n})\,.$$

Then the factors X_∞, X_0 are given respectively as follows.

$$\begin{aligned}\frac{X_\infty}{P'P''} &= (-q_2)^{\sum_{i=1}^{m-1}k_i(N-k_i)}(-1)^{\sum_{i=1}^{m+n}k_i(N-k_{i-1})} \\ &\quad\times \frac{\prod_{i=1}^{m-1}\big(\bar{x}'_i\big)^{2(N-k_i)}}{\prod_{i=1}^{m+n}\big(\bar{x}'_i\big)^{N-k_{i-1}}\big(\bar{x}'_{i-1}\big)^{N-k_i}}\big(\bar{x}'_m\big)^{N-k_m}\big(\bar{x}'_{m+n}\big)^{N-k_{m+n}}, \\ \frac{X_0}{P'P''} &= (-q_2)^{\sum_{i=1}^{m-1}k_i(N-k_i)}(-1)^{\sum_{i=1}^{m+n}k_{i-1}(N-k_i)+k_m(N-k_m)+k_{m+n}(N-k_{m+n})} \\ &\quad\times \frac{\prod_{i=1}^{m-1}\big(\bar{x}''_i\big)^{2k_i}}{\prod_{i=1}^{m+n}\big(\bar{x}''_i\big)^{k_{i-1}}\big(\bar{x}''_{i-1}\big)^{k_i}}\big(\bar{x}''_m\big)^{k_m}\big(\bar{x}''_{m+n}\big)^{k_{m+n}}.\end{aligned}$$

Taking their ratio we obtain the assertion of the Lemma.

In the case $n = 0$ the result can be checked easily. □

Proof of Proposition 5.9. It is enough to consider $G^{m|n}_{r,N}(s)$.

First we need to check that the elements $G^{m|n}_{r,N}(s)$ belong to the shuffle algebra $Sh^0_{m|n,N}$.

For $1 \leq i \leq m-1$ we have $P_N^{m|n} = 0$ if $x_{i,a} = q_2x_{i,b}$. By Lemma 5.4, for $m+1 \leq i \leq m+n$ we have $J_i^{c_i}(s) = 0$ if $q_3^{-1}x_{i-1,a} = q_1x_{i-1,b} = x_{i,c}$, or if $q_3x_{i,a} = q_1^{-1}x_{i,b} = x_{i-1,c}$. Therefore $G^{m|n}_{r,N}(s)$ satisfies the wheel conditions.

Next we show the asymptotic conditions. We may assume that $k_{i-1}-k_i \in \{-1,0,1\}$ for $1 \leq i \leq m$; otherwise $(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{\infty}$, $(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{0}$ are both zero. For $\epsilon \in \{-1,0,1\}$ set

$$A_\epsilon = \{i \mid 1 \leq i \leq m\,,\ k_{i-1}-k_i = \epsilon\}\,,$$

so that $\{1,\dots,m\} = A_+ \sqcup A_0 \sqcup A_-$ and $|A_+| - |A_-| = \sum_{i=1}^m (k_{i-1}-k_i) = k_{m+n}-k_m$.

By Lemma 5.2, Lemma 5.11 and Lemma 5.10, we obtain

(5.17)

$$(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{\infty} = Y_\infty \sum t_{i_1}(s)\cdots t_{i_p}(s) \prod_{i=m+1}^{m+n} \big(-\mathbf{s}_i d^N\big)^{c_i} \tilde{I}^{c_i+1-k_{i-1}+k_i}_{k_{i-1},k_i}(dx'_{i-1},x'_i)\tilde{I}^{c_i+1}_{N-k_{i-1},N-k_i}(dx''_{i-1},x''_i)\,,$$

$$Y_\infty = X_\infty \cdot \prod_{i=m+1}^{m+n} (-1)^{(N-k_{i-1})k_i} d^{k_{i-1}(N-k_i)} (\bar{x}'_{i-1})^{N-k_i} (\bar{x}'_i)^{N-k_{i-1}}\,,$$

where the sum is taken over $i_1,\dots,i_p$, $p\geq 0$, $c_{m+1},\dots,c_{m+n}\geq 0$, such that

$$\begin{aligned} &A_+ \subset \{i_1,\dots,i_p\}\,, \quad A_-\cap\{i_1,\dots,i_p\}=\emptyset\,,\\ &c_i \geq k_{i-1}-k_i \quad \text{for } m+1\leq i\leq m+n\,,\\ &p+\sum_{i=m+1}^{m+n} c_i = r\,. \end{aligned}$$

Similarly we have

$$(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{0} = Y_0 \sum t_{i'_1}(s)\cdots t_{i'_{p'}}(s) \prod_{i=m+1}^{m+n} \big(-\mathbf{s}_i d^N\big)^{c'_i} \tilde{I}^{c'_i+1}_{k_{i-1},k_i}(dx'_{i-1},x'_i)\tilde{I}^{c'_i+1+k_{i-1}-k_i}_{N-k_{i-1},N-k_i}(dx''_{i-1},x''_i)\,, \tag{5.18}$$

$$Y_0 = X_0 \cdot \prod_{i=m+1}^{m+n} (-1)^{k_{i-1}(N-k_i)} d^{(N-k_{i-1})k_i} (\bar{x}''_{i-1})^{k_i} (\bar{x}''_i)^{k_{i-1}}\,,$$

where the sum is taken over $i'_1,\dots,i'_{p'}$, $p'\geq 0$, $c'_{m+1},\dots,c'_{m+n}\geq 0$, such that

$$\begin{aligned} &A_- \subset \{i'_1,\dots,i'_t\}\,, \quad A_+\cap\{i'_1,\dots,i'_t\}=\emptyset\,,\\ &c'_i \geq -k_{i-1}+k_i \quad \text{for } m+1\leq i\leq m+n\,,\\ &p'+\sum_{i=m+1}^{m+n} c'_i = r\,. \end{aligned}$$

If we set $\{i_1,\dots,i_p\}=A_+\sqcup\{j_1,\dots,j_u\}$ and $c_i=c'_i+k_{i-1}-k_i$, then (5.17) becomes

$$\begin{aligned} (G^{mn}_{r,N}(s))^{\mathbf{k}}_{\infty} = Y_\infty &\prod_{i\in A_+} t_i(s)\cdot \prod_{i=m+1}^{m+n} \big(-\mathbf{s}_i d^N\big)^{k_{i-1}-k_i}\\ &\times \sum t_{j_1}(s)\cdots t_{j_u}(s) \prod_{i=m+1}^{m+n} \big(-\mathbf{s}_i d^N\big)^{c'_i} \tilde{I}^{c'_i+1}_{k_{i-1},k_i}(dx'_{i-1},x'_i)\tilde{I}^{c'_i+1+k_{i-1}-k_i}_{N-k_{i-1},N-k_i}(dx''_{i-1},x''_i)\,. \end{aligned}$$

Similarly, by setting $\{i'_1,\dots,i'_{p'}\}=A_-\sqcup\{j_1,\dots,j_u\}$, (5.18) yields

$$\begin{aligned} (G^{m|n}_{r,N}(s))^{\mathbf{k}}_{0} = Y_0 &\prod_{i\in A_-} t_i(s)\\ &\times \sum t_{j_1}(s)\cdots t_{j_u}(s) \prod_{i=m+1}^{m+n} \big(-\mathbf{s}_i d^N\big)^{c'_i} \tilde{I}^{c'_i+1}_{k_{i-1},k_i}(dx'_{i-1},x'_i)\tilde{I}^{c'_i+1+k_{i-1}-k_i}_{N-k_{i-1},N-k_i}(dx''_{i-1},x''_i)\,. \end{aligned}$$

It is simple to check that the second lines of these equations have the same ranges of summation and therefore coincide. Noting

$$\frac{\prod_{i\in A_+} t_i(s)}{\prod_{i\in A_-} t_i(s)} = \prod_{i=1}^{m} t_i(s)^{k_{i-1}-k_i}$$

and using Lemma 5.11, we obtain

$$\begin{aligned}\frac{(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{\infty}}{(G^{m|n}_{r,N}(s))^{\mathbf{k}}_{0}} &= \frac{Y_\infty}{Y_0}\cdot\frac{\prod_{i\in A_+} t_i(s)}{\prod_{i\in A_-} t_i(s)}\prod_{i=m+1}^{m+n}\bigl(-\mathbf{s}_i d^N\bigr)^{k_{i-1}-k_i}\\ &= \frac{X_\infty}{X_0}\prod_{i=m+1}^{m+n}(-d)^{N(k_{i-1}-k_i)}\prod_{i=m+1}^{m+n}\frac{(\bar{x}'_{i-1}\bar{x}'_i)^N}{\bar{x}_{i-1}^{k_i}\bar{x}_i^{k_{i-1}}}\cdot\prod_{i=1}^{m} t_i(s)^{k_{i-1}-k_i}\prod_{i=m+1}^{m+n}\bigl(-\mathbf{s}_i d^N\bigr)^{k_{i-1}-k_i}\\ &= \prod_{i=1}^{m+n} s_i^{k_i}\cdot d^{2N(k_m-k_{m+n})}\,.\end{aligned}$$

This completes the proof of Proposition 5.9. □

5.3. **Main theorem.** In this section we prove Theorem 5.13 and Theorem 5.15 below, which are the main results of this paper. We shall make explicit the dependence of various objects on parameters q_1, q_2, q_3 by writing $\mathcal{B}_{m|n}(q_1,q_2,q_3;s)$, $P_N^{m|n}(q_1,q_2,q_3)$, $G^{m|n}_{r,N}(q_1,q_2,q_3;s)$ and so forth. We recall that $\mathcal{R}_{m|n} = \mathbb{C}[T_{i,\nu} \mid 1\le i\le m+n\,,\ \nu\in\mathbb{Z}_{>0}]$ with $\deg T_{i,\nu}=\nu$.

We start with the special case $m=1, n=0$.

Theorem 5.12. [FHHSY] *The subalgebra $\mathcal{B}_1(q_1,q_2,q_3)$ of $Sh_1(q_1,q_2,q_3)$ is commutative, and is isomorphic to the polynomial algebra $\mathcal{R}_1$. For each choice of $i\in\{1,2,3\}$, the following set of elements generate $\mathcal{B}_1(q_1,q_2,q_3)$:*

$$\epsilon_N(x;q_i) = \prod_{1\le a<b\le N}\frac{(x_a-q_ix_b)(x_a-q_i^{-1}x_b)}{(x_a-x_b)^2}\,,\quad N\ge 0\,.$$

□

Consider the case $n=0$ with m general.

Theorem 5.13. *Let $m\ge 1$, and assume that s is generic.*

(i) The elements $\{G^m_{r,N}(q_1,q_2,q_3;s) \mid 0\le r\le m-1,\ N\ge 0\}$ generate $\mathcal{B}_m(q_1,q_2,q_3;s)$.

(ii) The restriction of π_m^{m-1} to $\mathcal{B}_m(q_1,q_2,q_3;s)$ gives an isomorphism of $\mathbb{Z}_{\ge 0}$-graded algebras

$$\pi_m^{m-1}:\mathcal{B}_m(q_1,q_2,q_3;s)\longrightarrow \mathcal{B}_{m-1}(\tilde{q}_1,q_2,\tilde{q}_3;\tilde{s})\otimes\mathcal{B}_1(q_1q_3^{-m+1},q_2,q_3^m)\,,$$

where $\tilde{q}_i$ are as in (3.7), and $\tilde{s}=(\tilde{s}_1,\dots,\tilde{s}_{m-2})$ is given by

$$\tilde{s}_1 = (s_2\cdots s_{m-1})^{-1}\,,\quad \tilde{s}_i = s_i\quad \text{for } 2\le i\le m-2\,.$$

(iii) Algebra $\mathcal{B}_m(q_1,q_2,q_3;s)$ is commutative, and is isomorphic to the polynomial algebra $\mathcal{R}_m$.

Proof. If $m = 1$, then $G^1_{0,N} = \epsilon^{(2)}_N$, where $\epsilon^{(2)}_N(x) = \epsilon_N(x; q_2)$. Hence the assertions are true by Theorem 5.12. Let $m \geq 2$, and assume that (i)–(iii) hold for all $m' < m$. For brevity we write $G_{r,N} = G^m_{r,N}(q_1, q_2, q_3; s)$, $\tilde{G}_{r,N} = G^{m-1}_{r,N}(\tilde{q}_1, q_2, \tilde{q}_3; \tilde{s})$, $\mathcal{B}_m = \mathcal{B}_m(q_1, q_2, q_3; s)$, $\mathcal{B}_{m-1} = \mathcal{B}_{m-1}(\tilde{q}_1, q_2, \tilde{q}_3; \tilde{s})$, and $\mathcal{B}_1 = \mathcal{B}_1(q_1 q_3^{-m+1}, q_2, q_3^m)$.

Denote by $\mathcal{B}'_m$ the subalgebra of $\mathcal{B}_m$ generated by the elements $\{G_{r,N} \mid 0 \leq r, N \geq 0\}$. (We recall that $G_{m,N}$ is proportional to $G_{0,N}$ and $G_{r,N} = 0$ for $r > m$.) We use induction on N to prove the following statement:

$$(S_N): \qquad \tilde{G}_{r,N} \otimes 1,\ 1 \otimes \epsilon^{(2)}_N \quad \in \pi^{m-1}_m\big(\mathcal{B}'_m\big) \quad \text{for } r, N \geq 0\,.$$

If $N = 0$, then there is nothing to prove. Assume $(S_{N'})$ for $N' < N$. Then by assertion (i) for $m-1$ and $(S_{N'})$, we have

$$\mathcal{B}_{m-1,N'} \otimes 1 = \langle \tilde{G}_{r,K} \otimes 1 \mid 0 \leq r,\ K \leq N' \rangle \subset \pi^{m-1}_m\big(\mathcal{B}'_m\big) \quad \text{for } N' < N.$$

Likewise we have

$$1 \otimes \mathcal{B}_{1,N'} = \langle 1 \otimes \epsilon^{(2)}_K \mid\ K \leq N' \rangle \subset \pi^{m-1}_m\big(\mathcal{B}'_m\big) \quad \text{for } N' < N.$$

This implies that

$$\mathcal{B}_{m-1,L} \otimes \mathcal{B}_{1,N-L} \subset \pi^{m-1}_m\big(\mathcal{B}'_m\big) \quad \text{for } 1 \leq L \leq N-1.$$

For an element $X \in \mathcal{B}_{m-1} \otimes \mathcal{B}_1$, let $X_{L,N-L} \in \mathcal{B}_{m-1,L} \otimes \mathcal{B}_{1,N-L}$ stand for its tensor component. In view of the discussion above, we have

$$\pi^{m-1}_m\big(G_{r,N}\big) \equiv \pi^{m-1}_m\big(G_{r,N}\big)_{N,0} + \pi^{m-1}_m\big(G_{r,N}\big)_{0,N} \mod \pi^{m-1}_m\big(\mathcal{B}'_m\big)\,.$$

On the other hand, a simple computation shows that

$$\pi^{m-1}_m\big(G_N(u)\big)_{N,0} = C(1 - q_1^N/u) \cdot \tilde{G}_N(q_1^{N/(m-1)} u) \otimes 1\,,$$

$$\pi^{m-1}_m\big(G_N(u)\big)_{0,N} = C'(1 - q_3^{(m-1)N}/u) \prod_{i=1}^{m-1} (1 - \mathbf{s}_i q_3^{-N}/u) \cdot 1 \otimes \epsilon^{(2)}_N\,.$$

Here C, C' are non-zero constants. Setting $u = q_1^N$ we find $1 \otimes \epsilon^{(2)}_N \in \pi^{m-1}_m\big(\mathcal{B}'_{m,N}\big)$. In turn, this implies that all coefficients of the polynomial $\tilde{G}_N(u) \otimes 1$ belong to $\pi^{m-1}_m\big(\mathcal{B}'_{m,N}\big)$. Thus we have proved (S_N).

It follows that the restriction of π^{m-1}_m to $\mathcal{B}'_m$ is surjective. In particular we have $\dim \mathcal{B}_{m,N} \geq \dim \mathcal{B}'_{m,N} \geq \dim \mathcal{R}_{m,N}$. Combining this with the upper estimate in Proposition 4.4, we conclude that (i),(ii) hold for m.

Finally assertion (iii) for m is a corollary of (ii) together with assertion (iii) for $m-1$.

This completes the proof of Theorem. □

Remark 5.14. *Statements (i), (iii) are due to* [FT] *except for the following point. In the proof of Lemma 2.7,* [FT]*, it is stated that if $F, G \in \mathcal{B}_m(s)$ then $F * G - G * F$ becomes zero after clearing the denominator and letting $x_{i,a} \to y_i$. While true, this is a non-trivial statement that requires a proof. In order to circumvent this issue we adopt here a different approach using fusion homomorphisms.* □

Finally we consider the case $m \geq n+1$ and $n \geq 1$.

Theorem 5.15. *Let $m \geq n+1$, $n \geq 1$, and assume that s is generic.*

(i) The elements $\{G^{m|n}_{r,N}(q_1,q_2,q_3;s), G^{*,m|n}_{r,N}(q_1,q_2,q_3;s) \mid 0 \leq r,\ N \geq 0\}$ generate $\mathcal{B}_{m|n}(q_1,q_2,q_3;s)$.

(ii) The restriction of $\pi^{m|n-1}_{m|n}$ to $\mathcal{B}_{m|n}(q_1,q_2,q_3;s)$ gives an isomorphism

$$\pi^{m|n-1}_{m|n} : \mathcal{B}_{m|n}(q_1,q_2,q_3;s) \longrightarrow \mathcal{B}_{m|n-1}(\bar{q}_1,q_2,\bar{q}_3;s) \otimes \mathcal{B}_1(q_2q_3^{-m+n}, q_2^{-1}, q_3^{m-n}),$$

where $\bar{q}_i$ are as in (3.1).

(iii) Algebra $\mathcal{B}_{m|n}(q_1,q_2,q_3;s)$ is commutative, and is isomorphic to the polynomial algebra $\mathcal{R}_{m|n}$.

Proof. The proof is quite similar to that of Theorem 5.13. We use the latter as a base of induction and the following formulas:

$$\pi^{m|n-1}_{m|n}\big(G_N(u)\big)_{N,0} = C_1 \cdot \frac{1}{1-\mathbf{s}_{m+n}q_1^N/u} \cdot \bar{G}_N(q_1^{-N/(m-n+1)}u) \otimes 1,$$

$$\pi^{m|n-1}_{m|n}\big(G_N(u)\big)_{0,N} = C_2 \frac{\prod_{i=1}^m (1-\mathbf{s}_i/(q_3^N u))}{\prod_{i=m+1}^{m+n-1}(1-\mathbf{s}_i/(q_3^N u))} \sum_{l=0}^N \frac{C_{N,l}}{1-\mathbf{s}_{m+n}/(q_2^l q_3^N u)} 1 \otimes \big(\epsilon^{(1)}_{N-l} * \epsilon^{(3)}_l\big),$$

$$\pi^{m|n-1}_{m|n}\big(G^*_N(u)\big)_{N,0} = C_3\ (1-\mathbf{s}_{m+n}q_1^{-N}/u) \cdot \bar{G}^*_N(q_1^{N/(m-n+1)}u) \otimes 1,$$

$$\pi^{m|n-1}_{m|n}\big(G^*_N(u)\big)_{0,N} = C_4\ \frac{\prod_{i=m+1}^{m+n-1}(1-\mathbf{s}_i q_3^N/u)\cdot(1-\mathbf{s}_{m+n}q_3^{(m-n+1)N}/u)}{\prod_{i=1}^m(1-\mathbf{s}_i q_3^N/u)} \cdot 1 \otimes \epsilon^{(2)}_N .$$

Here $\epsilon^{(1)}_N(z) = \epsilon_N(z;q_2q_3^{-m+n})$, $\epsilon^{(3)}_N(z) = \epsilon_N(z;q_3^{m-n})$, and $C_1,\dots,C_4$, $C_{N,l}$ are non-zero constants. □

Appendix A. Trace construction

In this Appendix we discuss the relation between the subalgebra $\mathcal{B}_{m|n}(s)$ of $Sh_{m|n}$ and the commutative subalgebra of the quantum toroidal algebra $\mathcal{E}_{m|n}$ of type $\mathfrak{gl}_{m|n}$.

Algebra $\mathcal{E}_{m|n}$ has generators $E_{i,k}, F_{i,k}$, $K^{\pm1}_{i,0}$, $H_{i,r}$, where $i \in \mathbb{Z}/(m+n)\mathbb{Z}$, $k \in \mathbb{Z}$, $r \in \mathbb{Z}\backslash\{0\}$, and an invertible central element C. The defining relations are presented in terms of the generating series $E_i(z) = \sum_k E_{i,k}z^{-k}$, $F_i(z) = \sum_k F_{i,k}z^{-k}$, $K^\pm_i(z) = K^{\pm1}_{i,0}\exp\big(\pm(q-q^{-1})\sum_{\pm r>0} H_{i,r}z^{-r}\big)$. For the details we refer the reader to [BM1]. We note only that the quadratic relations for the currents $E_i(z), F_i(z)$ can be written as

$$\omega_{i,j}(z,w)E_i(z)E_j(w) = (-1)^{|i||j|}\omega_{j,i}(w,z)E_j(w)E_i(z),$$
$$(-1)^{|i||j|}\omega_{j,i}(w,z)F_i(z)F_j(w) = \omega_{i,j}(z,w)F_j(w)F_i(z),$$

where $\omega_{i,j}(z,w)$ are the structure functions of $Sh_{m|n}$. We use also the Hopf algebra structure of $\mathcal{E}_{m|n}$ given by the Drinfeld coproduct, see [BM1], Section 2.4.

A.1. **The case $\mathcal{E}_m$.** First we restrict to $\mathcal{E}_m = \mathcal{E}_{m|0}$, and collect known facts following [FT].

Let $\mathcal{E}^+_m$ (resp. $\mathcal{E}^-_m$) be the subalgebra of $\mathcal{E}_m$ generated by $\{E_{i,k}\}$ (resp. $\{F_{i,k}\}$). There is an anti-isomorphism of algebras $\mathcal{E}^+_m \to \mathcal{E}^-_m$ sending $E_{i,k}$ to $F_{i,k}$. These subalgebras have realization as shuffle algebras due to [N2]. Namely we have an isomorphism of algebras

$$\Psi^\pm : \mathcal{E}^\pm_m \longrightarrow Sh^\pm_m$$

sending $E_{i,k}$ (resp. $F_{i,k}$) to the element $x_i^k \in Sh_{m,\mathbf{1}_i}$, where $Sh_m^- = Sh_m$, and $Sh_m^+ = (Sh_m)^{op}$ is the opposite algebra defined by the structure functions $\omega_{i,j}^{op}(z,w) = \omega_{j,i}(w,z)$. Note that the zeroes of $\omega_{i,j}^{op}(z,w)$ are obtained from those of $\omega_{i,j}(z,w)$ by the interchange $q_1 \leftrightarrow q_3^{-1}$, which leaves the wheel conditions unchanged.

Let $\mathcal{B}_m^+$ (resp. $\mathcal{B}_m^-$) be the Hopf subalgebra of $\mathcal{E}_m$ generated by $E_{i,k}, K_{i,0}^{\pm 1}, H_{i,-r}$ and C (resp. $F_{i,k}$, $K_{i,0}^{\pm 1}$, $H_{i,r}$ and C), $i \in \mathbb{Z}/m\mathbb{Z}$, $k \in \mathbb{Z}$, $r > 0$. There is a non-degenerate Hopf pairing

$$\langle\ ,\ \rangle : \mathcal{B}^+ \times \mathcal{B}^- \longrightarrow \mathbb{C}$$

given in [FT], Theorem 1.7. The isomorophism Ψ^+ can be described explicitly in terms of this pairing. To this aim, introduce the notation

$$F_{i_1,\dots,i_k}(z_1,\dots,z_k) = \prod_{1\le a<b\le k} \omega_{i_b,i_a}(z_b,z_a)\cdot F_{i_1}(z_1)\cdots F_{i_k}(z_k)\,.$$

The quadratic relations for $F_i(z)$ imply the symmetry

$$F_{i_{\sigma(1)},\dots,i_{\sigma(k)}}(z_{\sigma(1)},\dots,z_{\sigma(k)}) = F_{i_1,\dots,i_k}(z_1,\dots,z_k) \quad \text{for } \sigma \in \mathfrak{S}_k\,.$$

We have then (see [FT], Lemma 3.9)

$$(q-q^{-1})^N \Psi^+(X)(x_1,\dots,x_m) = \langle X, F_{1^{N_1},\dots,m^{N_m}}(x_1,\dots,x_m)\rangle \quad \text{for } X \in \mathcal{E}^+_{m,N_1,\dots,N_m}\,, \tag{A.1}$$

where $N = \sum_{i=1}^m N_i$, $i^{N_i} = i,\dots,i$ (N_i times) and $x_i = (x_{i,1},\dots,x_{i,N_i})$, $1 \le i \le m$.

A standard way to construct commutative subalgebras in $\mathcal{E}_m$ is to consider transfer matrices, namely appropriate traces of the universal R-matrix $\mathcal{R}$. More specifically, consider the Fock modules $\mathcal{F}_r$, $r = 0,1,\dots,m$, of $\mathcal{E}_m$. These modules are graded, $\mathcal{F}_r = \bigoplus_{n=-\infty}^0 \mathcal{F}_{r,n}$, the top degree subspace $\mathcal{F}_{r,0}$ being the r-th fundamental representation of $U_q\mathfrak{gl}_m$. Fixing parameters $u_i \in \mathbb{C}^\times$, $1 \le i \le m-1$, consider the weighted trace

$$(\mathrm{id}\otimes\phi_r)\mathcal{R}\,, \quad \phi_r(Y) = \mathrm{Tr}_{\mathcal{F}_r}\Big(P_0 \prod_{i=1}^{m-1} u_i^{-\bar\Lambda_i}\ Y\Big). \tag{A.2}$$

Here $P_0 : \mathcal{F}_r \to \mathcal{F}_{r,0}$ stands for the projection, and $\bar\Lambda_i$ are the fundamental weights of $\mathfrak{gl}_m$. The universal R-matrix $\mathcal{R}$ is an element of the completed tensor product $\mathcal{B}_m^+ \otimes \mathcal{B}_m^-$. Due to the Yang-Baxter equation, the elements $(\mathrm{id}\otimes\phi_r)\mathcal{R}$, $r = 0,\dots,m$, mutually commute.

The functional $\phi_r : \mathcal{B}^- \to \mathbb{C}$ annihilates $H_{i,r}$, $r > 0$. Moreover the dependence of $(\mathrm{id}\otimes\phi_r)\mathcal{R}$ on $K_{i,0}$ is such that it is a linear combination of elements of the form $Xf(s_1,\dots,s_m)$, where $X \in \mathcal{E}_m^+$ has weight 0 and f is a function in $s_i = u_i K_{i,0}^{-1}$. Since s_i's commute with X, (A.2) is essentially an element of $\mathcal{E}_m^+$ in which s_i's enter as parameters.

On the other hand, since $\mathcal{R}$ is the canonical element we have

$$\langle(\mathrm{id}\otimes\phi_r)\mathcal{R}, Y\rangle = \phi_r(Y) \quad \text{for } Y \in \mathcal{B}_m^-. \tag{A.3}$$

Using (A.3) and (A.1) one can rewrite (A.2) as commutative elements in the shuffle algebra Sh_m.

Thus we conclude:

Proposition A.1. *The collection of elements*

$$\prod_{i=1}^m \prod_{1\le a<b\le N} \omega_{i,i}(x_{i,b},x_{i,a}) \prod_{1\le i<j\le m} \prod_{1\le a,b\le N} \omega_{j,i}(x_{j,b},x_{i,a})\cdot \phi_r\Big(\overset{\curvearrowright}{\prod_{1\le i\le m}} \overset{\curvearrowright}{\prod_{1\le a\le N}} F_i(x_{i,a})\Big),$$

where $0 \le r \le m$, $N \ge 0$, constitutes a commutative family in Sh_m.

Using the explicit realization of $F_i(z)$ by vertex operators, one checks that these elements coincide with $G^m_{r,N}(s)$ up to a scalar multiple, with the identification $s_i = u_i K^{-1}_{i,0}$.

A.2. **The case $\mathcal{E}_{2|1}$.** In the previous section, we explained the scheme for constructing commutative elements in Sh_m. We expect that the same method carries over to the case $Sh_{m|n}$ with $m \ne n$. In place of $\mathcal{F}_r$ we use admissible modules $\mathcal{F}_\Lambda$ which have a realization in terms of free fields [BM1]. A new feature is that, for i beyond the equator , the currents $E_i(z), F_i(z)$ are no longer a single vertex operator but a sum of them.

As an example, consider the case $m = 2, n = 1$. (In the following we quote relevant formulas from [BM1]. For the unexplained notation we refer the reader to Section 3 thereof.) The F_i currents have the form

$$F_1(z) = \Gamma^-_1(z), \quad F_2(z) =: \Gamma^-_2(z)\partial_z\big[C^-_2(d^2z)\big] :, \quad F_0(z) = d : \Gamma^-_0(z)C^+_2(dz) :,$$

where $\Gamma^-_i(z)$, $C^\pm_i(z)$ are certain vertex operators, and ∂_z signifies the q-difference operator

$$\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}\,.$$

Notice that $F_2(z)$ is a sum of two vertex operators.

In the simplest case $\Lambda = 0$, the top degree subspace $\mathcal{F}_{\Lambda,0}$ is one dimensional with basis $v_0 = 1$. The weighted trace is simply the action of $\prod^{\curvearrowleft}_{1\le i\le 3}\prod^{\curvearrowleft}_{1\le a\le N} F_i(x_{i,a})$. We find

$$P^{2|1}_N \tilde I^1_{N,N}(dx_2, x_3) = G^{2|1}_{0,N}\,.$$

The appearance of the functions $I^c_{M,N}$ in section 5.1 is due to the presence of ∂_z in $F_2(z)$.

For $\Lambda = \bar\Lambda_1$, $\mathcal{F}_{\bar\Lambda_1,0}$ is three dimensional with basis $v_1 = e^{\bar\Lambda_1}$, $v_2 = e^{-\bar\alpha_1}e^{\bar\Lambda_1}$, and $v_3 = c_{2,-1}e^{-\bar\alpha_1-\bar\alpha_2-c_2}e^{\bar\Lambda_1}$. Taking trace we obtain

$$P^{2|1}_N\Big(\frac{\bar x_3}{\bar x_1}\tilde I^1_{N,N}(dx_2, x_3) + s_1\frac{\bar x_1}{\bar x_2}\tilde I^1_{N,N}(dx_2, x_3) - s_1s_2d^N\frac{\bar x_2}{\bar x_3}\tilde I^2_{N,N}(dx_2, x_3)\Big) = G^{2|1}_{1,N}(s)\,.$$

Here, computing the action on v_3 we use further the identity

$$\frac{1}{\Delta(y)}\prod_{a=1}^N \frac{qT_{y_a,q} - q^{-1}T^{-1}_{q,y_a}}{q - q^{-1}}\Big[\frac{\Delta(y)}{\Pi(y,z)}\Big(1 - \sum_{a,b}(y_a - z_a)(y_b^{-1} - z_b^{-1})\Big)\Big] = \frac{\bar y}{\bar z}I^2_{N,N}(y,z)\,. \tag{A.4}$$

Further examples suggest the formula for $G^{m|n}_{r,N}(s)$ in Section 5.2. We expect that these functions coincide with weighted traces of products of $F_i(z)$ given by free fields. To verify this in general, we would need a host of identities of the type given above. For that reason we have chosen to prove the commutativity of $G^{m|n}_{r,N}(s)$ directly.

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