

# DEFORMATIONS OF $\mathcal{W}$ ALGEBRAS VIA QUANTUM TOROIDAL ALGEBRAS

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ABSTRACT. We study the uniform description of deformed  $\mathcal{W}$  algebras of type A including the supersymmetric case in terms of the quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$ . In particular, we recover the deformed affine Cartan matrices and the deformed integrals of motion.

We introduce a comodule algebra  $\mathcal{K}$  over  $\mathcal{E}$  which gives a uniform construction of basic deformed  $\mathcal{W}$  currents and screening operators in types B, C, D including twisted and supersymmetric cases. We show that a completion of algebra  $\mathcal{K}$  contains three commutative subalgebras. In particular, it allows us to obtain a commutative family of integrals of motion associated with affine Dynkin diagrams of all non-exceptional types except  $D_{\ell+1}^{(2)}$ . We also obtain in a uniform way deformed finite and affine Cartan matrices in all classical types together with a number of new examples, and discuss the corresponding screening operators.

## 1. INTRODUCTION

Commutative families of operators coming from conformal field theory (CFT), known as local integrals of motion (IM), have attracted a lot of attention in the last quarter of a century. The interest was boosted by a seminal sequence of papers by Bazhanov, Lukyanov, and Zamolodchikov [BLZ1]–[BLZ3] where many related structures were revealed and a number of intriguing conjectures was put forth. One outcome of that study is the celebrated ODE/IM correspondence relating the spectrum of local IM to remarkable differential operators whose monodromy coefficients satisfy Bethe ansatz equations [DT], [MRV], [BLZ4], [BLZ5], see also [BL]. However, many important questions in the area still have no answers and many key statements remain conjectures. The explicit form of the local integrals of motion is unknown even in the simplest cases, see [FF] for the discussion of their existence, and there are no theorems about their spectrum.

From a general philosophy of quantum groups, it is quite natural to consider  $q$ -deformations of local IM in search for clarification of the matters. It turns out that after the  $q$ -deformation, the type A local IM become non-local, but can be written down explicitly [FKSW]. Moreover, one can describe their spectrum via Bethe ansatz, [FJM]. In the present article we address the problem of constructing the  $q$ -deformation of local IM in other types.

The algebraic structure underlying the CFT in question is given by the Virasoro algebra, and more generally,  $\mathcal{W}$  algebras. The  $q$ -deformations of  $\mathcal{W}$  algebras have been provided in [AKOS] for type A, and in [FR1] for simple Lie algebras. It has been recognized in recent years that quantum groups provide a natural framework for studying  $\mathcal{W}$  algebras [MO].

For the deformed  $\mathcal{W}$  algebras of type A, the relevant quantum group is the quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$  (also known as the Ding-Iohara-Miki algebra), see Section 2.1. Algebra  $\mathcal{E}$  has three Fock

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modules depending on a color  $c \in \{1, 2, 3\}$ , see Section 2.3. Consider generating current  $e(z)$  of  $\mathcal{E}$  acting on a tensor product of  $\ell + 1$  Fock modules associated to arbitrary colors  $c_1, \dots, c_{\ell+1}$ . Then  $e(z)$  acts as a sum of  $\ell + 1$  vertex operators, see Section 3.1. When all Fock modules are of the same kind, the current  $e(z)$  turns out to be related to the basic current  $T(z)$  of the deformed  $\mathcal{W}$  algebra of type  $A_\ell$ , as  $e(z) = T(z)Z(z)$ , where  $Z(z)$  is a vertex operator written in a single boson  $\{h_i\}$  given by the action of the Cartan current of  $\mathcal{E}$  and which commutes with  $T(z)$ . For other choices of colors, see for example (3.6), one gets currents which can be viewed as various  $q$ -deformations of  $\mathcal{W}$  algebras associated to Lie superalgebras of types  $\mathfrak{gl}_{m|n}$  with  $m + n = \ell + 1$ . In particular, we introduce the currents  $A_i(z)$ ,  $i = 1, \dots, \ell$ , given by ratios of neighboring terms in this sum of  $\ell + 1$  vertex operators, see (3.3). The current  $A_i(z)$  is bosonic if  $c_i = c_{i+1}$ , and it is fermionic otherwise. Then the table of contractions between the  $A_i(z)$ 's gives the deformed Cartan matrix of finite type (3.8). Moreover, the screening operators which are integrals of  $q$ -primitives of  $A_i(z)$ , see (3.18), (3.19), commute with  $T(z)$ .

Algebra  $\mathcal{E}$  has a family of commutative algebras depending on a parameter  $\mu$  given by the transfer matrices. One can “dress” current  $e(z)$  multiplying by an appropriate combination of Cartan current depending on  $\mu$ , see (3.9). Then generators of the commutative algebras can be computed explicitly and are given by multiple integrals (3.25) of the dressed current  $\mathbf{e}(z)$  with Feigin-Odesskii [FO] kernel, (3.23), see [FJM]. These generators acting on tensor products of Fock spaces are deformations of the local IM associated to  $\mathcal{W}$  of type  $A$ . The spectrum of transfer matrices (or of deformed IM), is computed by Bethe ansatz, see [FJMM1], [FJMM2] for two different derivations. Then one can attempt to obtain the spectrum of the original local IM by taking an appropriate limit, see [FJM].

The dressed current  $\mathbf{e}(z)$  has the form  $e(z)\tilde{Z}_\mu(z)$  where  $\tilde{Z}_\mu(z)$  is a vertex operator written in terms of  $\{h_i\}$ . Then we observe that the ratio  $A_0(z)$  of the last term in the dressed current  $\mathbf{e}(z)$  and the first term shifted by  $\mu$  produces one more screening operator which commutes with the deformed IM. Moreover, the matrix of contractions between all of  $A_i(z)$  is given by the deformed Cartan matrix of affine type as in [KP]. Following the terminology of [N], [KP], we interpret the dressed current  $\mathbf{e}(z)$  as a  $qq$ -character of the first fundamental representation of quantum affine  $\mathfrak{sl}_{\ell+1}$ , see Section 3.3.

Then we develop a similar picture for non-exceptional types other than  $A$ .

We introduce an algebra  $\mathcal{K}$  depending on three parameters  $q_1, q_2, q_3$  such that  $q_1 q_2 q_3 = 1$ , which plays a role of quantum toroidal algebra, see (4.1)-(4.5). Unlike  $\mathcal{E}$ , the algebra  $\mathcal{K}$  is not a Hopf algebra but a left  $\mathcal{E}$  comodule, see (4.8). Algebra  $\mathcal{K}$  has 6 representations in one boson denoted  $\mathcal{F}_c^B$ , see Proposition 4.2, and  $\mathcal{F}_c^{CD}$ ,  $c \in \{1, 2, 3\}$ , see Proposition 4.3, which we call boundary Fock modules of types  $B$  and  $CD$ , respectively. The algebra  $\mathcal{K}$  has a central element  $C$  which determines the level of the module. We have  $C^2 = q_c^{1/2}$  in type  $B$  and  $C^2 = q_c^{-1}$  in type  $CD$ .

The comodule structure allows us to consider a tensor product of a boundary Fock module of color  $c_{\ell+1}$  with  $\ell$  Fock modules of  $\mathcal{E}$  of colors  $c_1, \dots, c_\ell$ . This  $\mathcal{K}$  module is realized in  $\ell + 1$  bosons. The generating current  $E(z)$  of  $\mathcal{K}$  acting on such a tensor product is a sum of  $2\ell + 1$  terms if the boundary module is of type  $B$  and of  $2\ell$  vertex operators if the boundary module is of type  $CD$ . Similarly to type  $A$ , if all colors of  $\mathcal{E}$  Fock modules are the same,  $c_1 = \dots = c_\ell$ , then up to a boson we recover the deformed  $\mathcal{W}$  currents of [FR1]. More precisely, if the boundary module is of type  $B$ , we obtain a deformed  $\mathcal{W}$  current of type  $B$  when  $c_{\ell+1} \neq c_\ell$ , and of type  $\mathfrak{osp}(1, 2\ell)$  when  $c_{\ell+1} = c_\ell$  (the latter is called  $A_{2\ell}^{(2)}$  type in [FR1]). If the boundary module is of type  $CD$ , we recover the deformed  $\mathcal{W}$  current of type  $C$  when  $c_{\ell+1} \neq c_\ell$ , and of type  $D$  when  $c_{\ell+1} = c_\ell$ . If the colors of  $\mathcal{E}$  Fock spaces vary, we get various  $q$ -deformations of  $\mathcal{W}$  currents related to supersymmetric cases. We again introduce

currents  $A_i(z)$  as ratios of neighboring terms (in terms of Dynkin diagram), see (4.15)-(4.18). Then we observe that the contractions give a deformed Cartan matrix of corresponding finite type, and that the screening operators constructed from  $A_i(z)$  commute with our  $E(z)$ .

We define a dressed current  $\mathbf{E}(z)$  and consider integrals of products of  $\mathbf{E}(z)$  with the Feigin-Odesskii kernel similarly to type A, see (4.41). We show that these integrals commute if  $C^2/\mu = q_{c_0}$ ,  $c_0 \in \{1, 2, 3\}$ . Thus we have three families of commutative subalgebras in  $\mathcal{K}$  and three commuting families of operators acting on the representation. Naturally, we call them deformed IM. Note that the simplest deformed IM is always given by the constant term of the deformed  $\mathcal{W}$  current  $\mathbf{E}(z)$ .

We use dressed current  $\mathbf{E}(z)$  to introduce current  $A_0(z)$ , see (4.29)-(4.31), and observe that the contractions of  $A_i(z)$  give deformed affine Cartan matrices. If  $c_0 \neq c_1$  then the affine root turns out to be of type C and if  $c_0 = c_1$  then of type D. In particular, we cover that way all non-exceptional affine Dynkin diagrams except  $D_{\ell+1}^{(2)}$ . Moreover, the screening operator constructed from  $A_0(z)$  commutes with the deformed IM.

It is then natural to consider affine roots of type B. It corresponds to the case when dressing parameter  $\mu$  satisfies  $C^2/\mu = q_{c_0}^{-1/2}$ . In this case we introduce current  $A_0(z)$  in a similar way. Adding one more vertex operator to  $\mathbf{E}(z)$  (that is “adding a one dimensional representation”) we obtain a new current  $\tilde{\mathbf{E}}(z)$ . Then  $\tilde{\mathbf{E}}(z)$  commutes with screening operators constructed from  $A_i(z)$ ,  $i = 1, \dots, \ell$ , and the integral of  $\tilde{\mathbf{E}}(z)$  commutes with the screening operator corresponding to  $A_0(z)$ . In particular, contractions of  $A_i(z)$  lead to deformed affine Cartan matrices whose affine node is of type B.

We do not know how to include the integral of  $\tilde{\mathbf{E}}(z)$  into a family of commuting operators directly. However if the boundary module is of type CD, we can exchange the affine node with the  $\ell$ -th node. It turns out that the integral of  $\tilde{\mathbf{E}}(z)$  coincides with the integral of a different deformed  $\mathcal{W}$  current for which the boundary module is of type B and the affine root is of types C or D, see Remark 4.6. Then the integral of the latter current is a part of the family of integrals of motion as before. Such a recipe does not work for  $D_{\ell+1}^{(2)}$  for which the end nodes are both of type B.

The full list of deformed affine Cartan matrices produced by our construction includes a number of new examples, see Appendix C.

There are several questions arising from our work which we plan to address in the future publications.

- The deformed integrals of motion related to  $D_{\ell+1}^{(2)}$  are not constructed yet. The affine roots of type B need an additional study in general.
- We introduced the commuting algebras in  $\mathcal{K}$  as explicit integrals. We would like to construct these algebras from some version of transfer matrices and obtain their spectrum by a Bethe ansatz method.
- The nature of  $\mathcal{K}$  algebra needs to be clarified. We expect that it should be recognized as a twisted version of the quantum toroidal algebra. Also, it is interesting to understand the relation of  $\mathcal{K}$  to the universal  $\mathcal{W}$  algebras of types B and C, D of [KL].
- There are similar comodule algebras for quantum toroidal algebras  $\mathcal{E}_n$  associated to  $\mathfrak{gl}_n$ . It is important to study the currents they produce and the corresponding integrals of motion.
- The deformations of integrals of motion related to exceptional types seem to be related to other quantum algebras, see Section 5.2 for a discussion of type  $G_2$ .

The plan of the paper is as follows.

In Section 2 we introduce our convention and review generalities about the quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$ .

Section 3 is devoted to deformed  $\mathcal{W}$  algebras and deformed integrals of motion of type A. Section 3.1 reviews the non-affine case. In Section 3.2 we introduce the zeroth root current and recover deformed affine Cartan matrices. We discuss  $qq$ -characters in Section 3.3, screenings in Section 3.4, and the deformed IM in Section 3.5.

In Section 4 we treat the case of other classical types. We introduce the algebra  $\mathcal{K}$  (Section 4.1), along with its left  $\mathcal{E}$  comodule structure (Section 4.2) and the boundary representations  $\mathcal{F}_c^B, \mathcal{F}_c^{CD}$  (Section 4.3). We construct the deformed  $\mathcal{W}$  currents and obtain deformed finite type Cartan matrices in Section 4.4. We discuss the affinization in Section 4.5. Section 4.6 deals with  $qq$ -characters, Section 4.7 with screenings, and Section 4.8 with the deformed commuting integrals.

In Section 5 we make some additional remarks. We discuss integrals of motion of Knizhnik-Zamolodchikov type in Section 5.1 and the situation in type  $G_2$  in Section 5.2.

The text is followed by three appendices. In Appendix A, we give a proof of the commutativity of integrals of motion. Appendix B discusses the existence of operator  $K(u)$ . In Appendix C we give a library of deformed Cartan matrices obtained from  $\mathcal{K}$ .

## 2. THE QUANTUM TOROIDAL ALGEBRA ASSOCIATED TO $\mathfrak{gl}_1$

In this section we recall the quantum toroidal algebra associated to  $\mathfrak{gl}_1$  and its Fock modules.

**2.1. Relations.** Fix  $q_1, q_2, q_3 \in \mathbb{C}^\times$  such that  $q_1 q_2 q_3 = 1$ . We also fix values of  $\ln q_i$  and set  $q_i^a = \exp(a \ln q_i)$  for all  $a \in \mathbb{C}$ . In this paper we assume that our choice of parameters is generic, meaning that  $q_1^a q_2^b = 1$  for  $a, b \in \mathbb{Z}$  if and only if  $a = b = 0$ .

We use the notation

$$s_c = q_c^{1/2}, \quad t_c = s_c - s_c^{-1}, \quad b_c = \frac{t_c}{t_1 t_2 t_3} \quad (c \in \{1, 2, 3\}).$$

We also use

$$g(z, w) = \prod_{i=1}^3 (z - q_i w), \quad \bar{g}(z, w) = \prod_{i=1}^3 (z - q_i^{-1} w) = -g(w, z), \quad \kappa_r = \prod_{i=1}^3 (1 - q_i^r) \quad (r \in \mathbb{Z}).$$

Note that  $\kappa_1 = -t_1 t_2 t_3 = \sum_{i=1}^3 q_i^{-1} - \sum_{i=1}^3 q_i$ .

The quantum toroidal algebra  $\mathcal{E}$  associated to  $\mathfrak{gl}_1$  is an associative algebra with parameters  $q_1, q_2, q_3$  generated by coefficients of the currents

$$e(z) = \sum_{n \in \mathbb{Z}} e_n z^{-n}, \quad f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n}, \quad \psi^\pm(z) = \exp\left(\sum_{r>0} \kappa_r h_{\pm r} z^{\mp r}\right),$$

and an invertible central element<sup>1</sup>  $C$ . The defining relations are as follows.

$$\begin{aligned}
 \psi^\pm(z)\psi^\pm(w) &= \psi^\pm(w)\psi^\pm(z), \\
 \psi^+(z)\psi^-(w) &= \psi^-(w)\psi^+(z)\frac{g(z, Cw)\bar{g}(Cz, w)}{\bar{g}(z, Cw)g(Cz, w)}, \\
 g(z, w)\psi^\pm(C^{(-1\mp 1)/2}z)e(w) &= \bar{g}(z, w)e(w)\psi^\pm(C^{(-1\mp 1)/2}z), \\
 \bar{g}(z, w)\psi^\pm(C^{(-1\pm 1)/2}z)f(w) &= g(z, w)f(w)\psi^\pm(C^{(-1\pm 1)/2}z), \\
 [e(z), f(w)] &= \frac{1}{\kappa_1}\left(\delta\left(\frac{Cw}{z}\right)\psi^+(w) - \delta\left(\frac{Cz}{w}\right)\psi^-(z)\right), \\
 g(z, w)e(z)e(w) &= \bar{g}(z, w)e(w)e(z), \\
 \bar{g}(z, w)f(z)f(w) &= g(z, w)f(w)f(z), \\
 \text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1}[e(z_1), [e(z_2), e(z_3)]] &= 0, \\
 \text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1}[f(z_1), [f(z_2), f(z_3)]] &= 0,
 \end{aligned}$$

where

$$\text{Sym}_{z_1, \dots, z_N} F(z_1, \dots, z_N) = \frac{1}{N!} \sum_{\pi \in \mathfrak{S}_N} F(z_{\pi(1)}, \dots, z_{\pi(N)}).$$

The relations for  $\psi^\pm(z)$  are equivalent to

$$(2.1) \quad [h_r, h_s] = \delta_{r+s, 0} \frac{1}{\kappa_r} \frac{C^r - C^{-r}}{r}.$$

Let  $\mathcal{A}$  be a  $\mathbb{Z}$  graded algebra with a central element  $C$ . The completion of  $\mathcal{A}$  in the positive direction is the algebra  $\tilde{\mathcal{A}}$ , linearly spanned by products of series of the form  $\sum_{i=M}^{\infty} f_i g_i$ , where  $M \in \mathbb{Z}$ ,  $f_i, g_i \in \mathcal{A}$  and  $\deg g_i = i$ .

We call an  $\mathcal{A}$  module  $V$  admissible if  $V$  is  $\mathbb{Z}$  graded with degrees bounded from above, i.e.,  $V = \bigoplus_{i=-\infty}^N V_i$ , where  $V_i = \{v \in V \mid \deg v = i\}$ , and if  $C$  is diagonalizable. The completion  $\tilde{\mathcal{A}}$  acts on all admissible modules.

Algebra  $\mathcal{E}$  has a  $\mathbb{Z}$  grading given by

$$\deg e_i = \deg f_i = \deg h_i = i, \quad \deg C = 0.$$

In other words, if we formally set  $\deg z = 1$ , then  $e(z), f(z), \psi^\pm(z)$  all have degree zero.

Let  $\mathcal{E} \tilde{\otimes} \mathcal{E}$  be the tensor algebra  $\mathcal{E} \otimes \mathcal{E}$  completed in the positive direction. We use the topological coproduct  $\Delta : \mathcal{E} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{E}$  as in [FJM]:

$$\begin{aligned}
 (2.2) \quad \Delta e(z) &= e(C_2^{-1}z) \otimes \psi^+(C_2^{-1}z) + 1 \otimes e(z), \\
 \Delta f(z) &= f(z) \otimes 1 + \psi^-(C_1^{-1}z) \otimes f(C_1^{-1}z), \\
 (2.3) \quad \Delta \psi^+(z) &= \psi^+(z) \otimes \psi^+(C_1 z), \\
 \Delta \psi^-(z) &= \psi^-(C_2 z) \otimes \psi^-(z),
 \end{aligned}$$

<sup>1</sup>In the standard definition, the quantum toroidal algebra  $\mathcal{E}$  associated to  $\mathfrak{gl}_1$  has two central elements. Here we set the second central element to 1.

where  $C_1 = C \otimes 1$ ,  $C_2 = 1 \otimes C$ .

Note that the coproduct can be extended to the map  $\Delta : \tilde{\mathcal{E}} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{E}$ .

The counit map  $\epsilon : \mathcal{E} \rightarrow \mathbb{C}$  is given by  $\epsilon(e(z)) = \epsilon(f(z)) = 0$ ,  $\epsilon(\psi^\pm(z)) = 1$ ,  $\epsilon(C) = 1$ .

We set also

$$\tilde{f}(z) = S(f(z)) = -\psi^-(z)^{-1} f(Cz)$$

where  $S$  is the antipode. The coproduct reads

$$\Delta \tilde{f}(z) = \tilde{f}(C_2 z) \otimes \psi^-(z)^{-1} + 1 \otimes \tilde{f}(z).$$

Note that the topological Hopf algebra  $\mathcal{E}$  depends on  $q_1, q_2, q_3$  symmetrically, in other words it depends on the unordered set  $\{q_1, q_2, q_3\}$ .

**2.2. Vertex operators.** In this paper we often study vertex operators acting on bosonic Fock spaces. Here we give a brief description of these objects.

A Heisenberg algebra  $\mathcal{H}$  is an associative algebra generated by linearly independent elements  $h_r^{(i)}$ , where  $r \in \mathbb{Z}$ ,  $i = 1, \dots, N$ , with relations  $[h_r^{(i)}, h_s^{(j)}] = \delta_{r+s,0} a_r^{i,j}$ , where  $a_r^{i,j} \in \mathbb{C}$ . Algebra  $\mathcal{H}$  has a  $\mathbb{Z}$  grading such that  $\deg h_r^{(i)} = r$ .

A weight  $\alpha$  is a linear functional  $\alpha : \text{span}_{\mathbb{C}}\{h_0^{(1)}, \dots, h_0^{(N)}\} \rightarrow \mathbb{C}$ . For a weight  $\alpha$ , we define the Fock space  $\mathbb{F}_\alpha$  as an  $\mathcal{H}$  module generated by a vacuum vector  $v_\alpha$  such that  $h_r^{(i)} v_\alpha = 0$  for  $r > 0$ ,  $h_0^{(i)} v_\alpha = \alpha(h_0^{(i)}) v_\alpha$  for  $i = 1, \dots, N$ , and such that  $\mathbb{F}_\alpha$  is freely generated over  $\mathbb{C}[h_r^{(i)}]_{r \in \mathbb{Z}, i=1, \dots, N}$ .

Given a weight  $\alpha$ , let  $e^{Q_\alpha} : \mathbb{F}_\mu \rightarrow \mathbb{F}_{\mu+\alpha}$  be a linear operator such that  $e^{Q_\alpha} v_\mu = v_{\mu+\alpha}$  and such that  $[e^{Q_\alpha}, h_r^{(i)}] = 0$ ,  $r \neq 0$ . We have

$$e^{Q_\alpha} z^{v_0} = z^{-\alpha(v_0)} z^{v_0} e^{Q_\alpha}, \quad v_0 \in \text{span}_{\mathbb{C}}\{h_0^{(1)}, \dots, h_0^{(N)}\}.$$

In this paper, by a vertex operator we mean a formal series  $V(z)$  of the form

$$(2.4) \quad V(z) = b \exp\left(\sum_{r>0} \mathbf{v}_{-r} z^r\right) \exp\left(\sum_{r \geq 0} \mathbf{v}_r z^{-r}\right), \quad \mathbf{v}_r \in \text{span}_{\mathbb{C}}\{h_r^{(1)}, \dots, h_r^{(N)}\} \quad (r \in \mathbb{Z}),$$

where  $b \in \mathbb{C}^\times$ . For any  $v \in \mathbb{F}_\mu$ ,  $V(z)v$  is a well defined Laurent series in  $z$  with values in  $\mathbb{F}_\mu$ .

The product of (2.4) with another vertex operator  $V'(w) = b' \exp(\sum_{r>0} \mathbf{v}'_{-r} w^r) \exp(\sum_{r \geq 0} \mathbf{v}'_r w^{-r})$  has the form

$$V(z)V'(w) = \varphi_{V,V'}(w/z) : V(z)V'(w) :,$$

where  $\varphi_{V,V'}(w/z) \in \mathbb{C}[[w/z]]$  is a formal power series called the contraction of  $V(z)$  and  $V'(w)$ , and the vertex operator

$$: V(z)V'(w) := bb' \exp\left(\sum_{r>0} (\mathbf{v}_{-r} z^r + \mathbf{v}'_{-r} w^r)\right) \exp\left(\sum_{r \geq 0} (\mathbf{v}_r z^{-r} + \mathbf{v}'_r w^{-r})\right)$$

is called the normal ordered product of  $V(z)$  and  $V'(w)$ . Obviously  $: V(z)V'(w) := : V'(w)V(z) :$ .

Our vertex operators will depend on various parameters,  $q_1, q_2, \mu, C$ , etc. One can think of these parameters as variables or as generic complex numbers. We call  $\alpha$  a formal monomial if  $\alpha$  is a product of parameters in rational powers. For example  $\alpha$  can be of the form  $q_1^{a_i} q_2^{b_i}$  with  $a_i, b_i \in \mathbb{Q}$ .

We often study the case when the contraction has the form

$$(2.5) \quad \varphi_{V,V'}(w/z) = \prod_i (1 - \alpha_i w/z)^{-n_i}$$

where  $n_i \in \mathbb{Z}$  and  $\alpha_i$  are formal monomials. Equation (2.5) is equivalent to saying

$$[v_r, v'_s] = \delta_{r+s,0} \frac{\sum_i n_i \alpha_i^r}{r} \quad (r \in \mathbb{Z}_{>0}).$$

We use the notation

$$\mathcal{C}(V(z), V'(w)) = \sum_i n_i \alpha_i$$

to represent (2.5), and by abusing the language we also call the sum in the right hand side the contraction of  $V(z)$  and  $V'(w)$ .

Note that  $\mathcal{C}(V(pz), V'(p'w)) = \mathcal{C}(V(z), V'(w)) \cdot p'/p$ .

We call a contraction rational if  $\sum_i n_i \alpha_i$  has a finite number of summands. We call a contraction elliptic if we can write  $\sum_i n_i \alpha_i = \sum_j m_j \beta_j / (1 - \beta)$  where  $m_j \in \mathbb{Z}$ ,  $\beta_i, \beta$  are formal monomials, and the summation over  $j$  is finite.

We also often study screening currents and screening operators. Screening currents  $S(z)$  have the form

$$(2.6) \quad S(z) = e^{Q_\alpha z^{s_0}} \exp\left(\sum_{r>0} s_{-r} z^r\right) \exp\left(\sum_{r>0} s_r z^{-r}\right), \quad s_r \in \text{span}_{\mathbb{C}}\{h_r^{(1)}, \dots, h_r^{(N)}\} \quad (r \in \mathbb{Z}),$$

where  $\alpha$  is a weight. In particular,  $S(z) = e^{Q_\alpha z^{s_0}} S^{osc}(z)$  where  $S^{osc}(z)$  is a vertex operator with  $\bar{s}_0 = 0$ .

Given a screening current  $S(z) = e^{Q_\alpha z^{s_0}} S^{osc}(z)$  and a vertex operator  $V(w)$ , the normal ordering is given by  $:S(z)V(w): = :V(w)S(z): = e^{Q_\alpha z^{s_0}} :S^{osc}(z)V(w):$ .

The screening operator  $S$  is the constant term of  $zS(z)$ :

$$(2.7) \quad S = \frac{1}{2\pi i} \int S(z) dz.$$

Note that  $S$  is a well defined operator  $\mathbb{F}_\mu \rightarrow \mathbb{F}_{\mu+\alpha}$  if and only if  $\mu(s_0) \in \mathbb{Z}$ .

Let  $V(z), V'(z)$  be vertex operators of the form (2.4), and set

$$A(z) = :V(z)V'(z)^{-1} :.$$

Assume that there are formal monomials  $p_1, p_2, p_3$  such that

$$(2.8) \quad \begin{aligned} \mathcal{C}(A(z), V(w)) &= -(1 - p_1^2)(1 - p_2^2), & \mathcal{C}(V(w), A(z)) &= -(1 - p_1^{-2})(1 - p_2^{-2}), \\ \mathcal{C}(A(z), V'(w)) &= (1 - p_2^{-2})(1 - p_3^{-2}), & \mathcal{C}(V'(w), A(z)) &= (1 - p_2^2)(1 - p_3^2). \end{aligned}$$

For these equations to be consistent, we must have  $(1 - p_2^2)(p_1^2 - p_3^2)(1 - p_1^{-2}p_2^{-2}p_3^{-2}) = 0$ . Excluding the trivial case  $p_2^2 = 1$ , we shall assume that either  $p_1 = p_3$  or  $p_1 p_2 p_3 = 1$ . We also assume that if  $v_0 = av'_0$  for some  $a \in \mathbb{C}$  then  $a = -\log p_1 / \log p_3$ .

Then we can construct a screening operator commuting with  $V(z) + V'(z)$  as follows.

We define the screening current  $S(z)$  of the form (2.6) by

$$(2.9) \quad A(z) = \frac{1 - p_3^2}{1 - p_1^2} p_2^{-2} p_3^{-2} :S(p_2^{-1}z)S(p_2z)^{-1} :.$$

This amounts to the relations

$$\begin{aligned} \mathbf{a}_r &= \mathbf{v}_r - \mathbf{v}'_r = (p_2^r - p_2^{-r})\mathbf{s}_r \quad (r \neq 0), \\ e^{\mathbf{a}_0} &= e^{\mathbf{v}_0 - \mathbf{v}'_0} = \frac{b'}{b} \frac{1 - p_3^2}{1 - p_1^2} p_2^{-2} p_3^{-2} p_2^{-2s_0}. \end{aligned}$$

In addition we choose a weight  $\alpha$  so that

$$(2.10) \quad \alpha(\mathbf{v}_0) = 2 \log p_1, \quad \alpha(\mathbf{v}'_0) = -2 \log p_3.$$

Then we have the following well known lemma.

**Lemma 2.1.** *When the screening operator  $S$  is well defined, we have*

$$[S, V(z) + V'(z)] = 0.$$

*Proof.* We have

$$\begin{aligned} S(z)V(w) &= \frac{1 - p_1^2 p_2 w/z}{1 - p_2 w/z} : S(z)V(w) :, & V(w)S(z) &= \frac{1 - p_1^{-2} p_2^{-1} z/w}{1 - p_2^{-1} z/w} p_1^2 : S(z)V(w) :, \\ S(z)V'(w) &= \frac{1 - p_3^{-2} p_2^{-1} w/z}{1 - p_2^{-1} w/z} : S(z)V'(w) :, & V'(w)S(z) &= \frac{1 - p_3^2 p_2 z/w}{1 - p_2 z/w} p_3^{-2} : S(z)V'(w) :. \end{aligned}$$

It follows that  $[S(z), V(w)] = (1 - p_1^2)\delta(p_2 w/z) : S(z)V(w)$ , where  $\delta(z) = \sum_{i \in \mathbb{Z}} z^i$  is the formal delta function. Integrating over  $z$ , we obtain  $[S, V(w)] = (1 - p_1^2)p_2 w : S(p_2 w)V(w) :$ . Similarly, we obtain  $[S, V'(w)] = (1 - p_3^{-2})p_2^{-1} w : S(p_2^{-1} w)V'(w) :$ .

On the other hand, from the definitions we have

$$: S(p_2 w)V(w) : = \frac{1 - p_3^2}{1 - p_1^2} p_2^{-2} p_3^{-2} : S(p_2^{-1} w)V'(w) :.$$

Hence the lemma follows.  $\square$

We note that in (2.8) one can replace  $A(z)$  with  $A(pz)$  for any formal monomial  $p$ . Then the screening current  $S(z)$  will be also shifted to  $S(pz)$  and  $S$  will be replaced by a constant multiple  $p^{-1}S$ . Therefore Lemma 2.1 will still hold.

When  $p_1 = p_3 \neq p_2$ , the right hand sides of (2.8) become symmetric in  $p_1$  and  $p_2$ . Interchanging the roles of  $p_1$  and  $p_2$ , one can construct another screening operator commuting with  $V(z) + V'(z)$ .

It is easy to check that if one has  $p_1 p_2 p_3 = 1$  in (2.8), then the corresponding screening current is an ordinary fermion [BFM].

**2.3. The Fock modules.** Algebra  $\mathcal{E}$  has three families of Fock representations  $\mathcal{F}_c(u)$ , where  $c \in \{1, 2, 3\}$  and  $u \in \mathbb{C}^\times$ . We call  $c$  the color of the Fock module and  $u$  the evaluation parameter. Quite generally, if the central element  $C$  acts on an  $\mathcal{E}$  module  $M$  by a scalar  $k$ , then we say that  $M$  has level  $k$  and often write  $C = k$ . The Fock module  $\mathcal{F}_c(u)$  has level  $s_c$ .

The Fock modules are irreducible with respect to the Heisenberg subalgebra of  $\mathcal{E}$  generated by  $\psi^\pm(z)$ . Thus we have the identification of vector spaces  $\mathcal{F}_c(u) = \mathbb{C}[h_{-r}]_{r>0} v_c$ , where  $v_c$  is the vacuum vector such that  $h_r v_c = 0$  ( $r > 0$ ) and  $C v_c = s_c v_c$ . Then the generators  $e(z)$  and  $\tilde{f}(z)$  are given by vertex operators

$$e(z) \mapsto b_c : V_c(z; u) :, \quad \tilde{f}(z) \mapsto b_c : V_c(z; u)^{-1} :,$$

where  $b_c = -t_c/\kappa_1$  and

$$(2.11) \quad V_c(z; u) = u \exp\left(\sum_{r>0} \frac{\kappa_r h_{-r}}{1 - q_c^r} z^r\right) \exp\left(\sum_{r>0} \frac{\kappa_r h_r}{1 - q_c^r} q_c^{r/2} z^{-r}\right).$$

Note that

$$(2.12) \quad V_c(s_c^{-1}z; u)\psi^+(z) = \psi^-(s_c^{-1}z)V_c(s_c z; u).$$

For vertex operators (2.11), the contractions are rational. The non-trivial ones are

$$(2.13) \quad \begin{aligned} \mathcal{C}(V_c(z; u), V_c(w; v)) &= -\frac{\kappa_1}{1 - q_c}, \\ \mathcal{C}(\psi^+(q_c^{-1/2}z), V_c(w; u)) &= \mathcal{C}(V_c(z; u), \psi^-(w)) = -\kappa_1, \\ \mathcal{C}(\psi^+(q_c^{-1/2}z), \psi^-(w)) &= -(1 - q_c)\kappa_1. \end{aligned}$$

### 3. $\mathcal{W}$ ALGEBRAS AND INTEGRALS OF MOTION OF TYPE $\mathbf{A}$

The deformed  $\mathcal{W}$  algebras are introduced in [FR1] starting from a deformed non-affine Cartan matrix, or in a more general context, a Dynkin quiver in [KP]. In these papers, supersymmetric cases were not considered.

It turns out that the deformed  $\mathcal{W}$  currents of type  $\mathbf{A}$  are essentially the images of the current  $e(z)$  of the quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$  acting on tensor products of Fock modules. In this section we discuss this connection, and explain how the deformed affine Cartan matrix and integrals of motion are recovered from the data of Fock modules.

**3.1. The current  $e(z)$  and root currents  $A_i(z)$ .** Fix a tensor product of  $\ell + 1$  Fock modules

$$(3.1) \quad \mathcal{F}_{c_1}(u_1) \otimes \cdots \otimes \mathcal{F}_{c_{\ell+1}}(u_{\ell+1}), \quad c_1, \dots, c_{\ell+1} \in \{1, 2, 3\},$$

which has level  $C = \prod_{j=1}^{\ell+1} s_{c_j}$ . By coproduct formulas (2.2), (2.3), the current  $e(z)$  acts as a sum of vertex operators in  $\ell + 1$  bosons

$$\begin{aligned} e(z) &= b_{c_1}\Lambda_1(z) + \cdots + b_{c_{\ell+1}}\Lambda_{\ell+1}(z), \\ \Lambda_i(z) &= 1 \otimes \cdots \otimes 1 \otimes V_{c_i}(\overset{i}{a_i^{\mathbf{A}}}z; u_i) \otimes \psi^+(s_{c_{i+1}}^{-1}a_{i+1}^{\mathbf{A}}z) \otimes \cdots \otimes \psi^+(s_{c_{\ell+1}}^{-1}a_{\ell+1}^{\mathbf{A}}z), \end{aligned}$$

where

$$(3.2) \quad a_i^{\mathbf{A}} = \prod_{j=i+1}^{\ell+1} s_{c_j}^{-1}.$$

Note that the current  $e(z)$  with evaluation parameters  $au_i$ , where  $a \in \mathbb{C}^\times$ , is obtained from  $e(z)$  with evaluation parameters  $u_i$  by scalar multiplication by  $a$ .

To each neighboring pair of Fock spaces

$$\cdots \otimes \mathcal{F}_{c_i}(\overset{i}{u_i}) \otimes \mathcal{F}_{c_{i+1}}(\overset{i+1}{u_{i+1}}) \otimes \cdots, \quad i = 1, \dots, \ell,$$

we associate a current  $A_i(z)$ . Namely, for  $i = 1, \dots, \ell$ , let  $A_i(z)$  be given by a normally ordered ratio:

$$(3.3) \quad A_i(z) = : \frac{\Lambda_i((a_i^A)^{-1}z)}{\Lambda_{i+1}((a_i^A)^{-1}z)} :,$$

where  $a_i^A$  is given in (3.2). We call  $A_i(z)$  a root current.

From (2.13) we have the following contractions:

$$\mathcal{C}(\Lambda_i(z), \Lambda_j(w)) = \begin{cases} -\kappa_1 & (i < j), \\ -\kappa_1/(1 - q_{c_i}) & (i = j), \\ 0 & (i > j), \end{cases}$$

where  $i, j = 1, \dots, \ell + 1$ .

Denote the contractions between root currents by

$$(3.4) \quad B_{i,j} = \mathcal{C}(A_i(z), A_j(w)) \quad (i, j = 1, \dots, \ell).$$

The only non-trivial ones are

$$(3.5) \quad B_{i-1,i} = B_{i,i-1} = \frac{t_1 t_2 t_3}{t_{c_i}}, \quad B_{j,j} = -\frac{t_1 t_2 t_3}{t_{c_j} t_{c_{j+1}}} (s_{c_j} s_{c_{j+1}} - s_{c_j}^{-1} s_{c_{j+1}}^{-1}),$$

where  $i = 2, \dots, \ell$ ,  $j = 1, \dots, \ell$ .

In other words, only neighboring  $A_i(z)$  have non-trivial contractions which can be described by  $2 \times 2$  matrices corresponding to the choice of three Fock spaces, see type A matrices in Appendix C.2.

Note that all these contractions are rational and do not depend on evaluation parameters  $u_i$ .

The matrix of contractions  $B = (B_{i,j})_{i,j=1,\dots,\ell}$  should be compared with the Cartan matrix of type A. We say that  $A_i(z)$  is a bosonic current of type  $c$  if it corresponds to a pair  $\mathcal{F}_c \otimes \mathcal{F}_c$ , and a fermionic current of type  $c$  if it corresponds to a pair  $\mathcal{F}_{c_1} \otimes \mathcal{F}_{c_2}$ , where  $\{c_1, c_2, c\} = \{1, 2, 3\}$ .

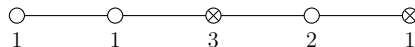
**Example 3.1.** Consider the tensor product

$$(3.6) \quad \mathcal{F}_1(u_1) \otimes \mathcal{F}_1(u_2) \otimes \mathcal{F}_1(u_3) \otimes \mathcal{F}_2(u_4) \otimes \mathcal{F}_2(u_5) \otimes \mathcal{F}_3(u_6), \quad C = s_1^3 s_2^2 s_3.$$

The matrix  $B$  is given by

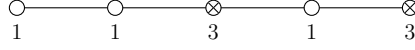
$$(3.7) \quad B = \begin{pmatrix} -t_2 t_3 (s_1 + s_1^{-1}) & t_2 t_3 & 0 & 0 & 0 \\ t_2 t_3 & -t_2 t_3 (s_1 + s_1^{-1}) & t_2 t_3 & 0 & 0 \\ 0 & t_2 t_3 & t_3^2 & t_1 t_3 & 0 \\ 0 & 0 & t_3 t_t & -t_1 t_3 (s_2 + s_2^{-1}) & t_1 t_3 \\ 0 & 0 & 0 & t_1 t_3 & t_1^2 \end{pmatrix}.$$

We picture this matrix as a Dynkin diagram where a circle represents a bosonic node and a crossed circle a fermionic node. We also attach a marking by 1, 2, 3 to each vertex, indicating its type.



□

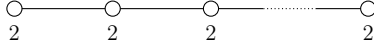
The diagram in the example looks like a diagram of  $\mathfrak{sl}_{4|2}$  but it is different because of the markings. In the diagram of  $\mathfrak{sl}_{4|2}$  all bosonic nodes have the same marking and all fermionic nodes have the same marking too:



And if all colors are the same, e.g.  $c_i = 2$ ,  $i = 1, \dots, \ell + 1$ , then

$$(3.8) \quad B_{i,j} = -t_1 t_3 ((s_2 + s_2^{-1})\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}).$$

Up to an overall multiple  $-t_1 t_3$ , this coincides with the matrix  $B(q, t)$  of type  $A_\ell$  in [FR1], eq.(2.4) with the identification  $s_2 = qt^{-1}$ .



However, in contrast to [FR1], we have one extra boson. Indeed, by construction our root currents commute with the diagonal Heisenberg algebra

$$[A_i(z), \Delta^{(\ell)} h_r] = 0 \quad (i = 1, \dots, \ell, r \neq 0),$$

where  $\Delta^{(\ell)}$  signifies the iterated coproduct with  $\Delta^{(1)} = \Delta$ . This extra boson allows us to have rational commutation relations between  $\Lambda_i(z)$ 's, and will play a role in the affinization to be discussed below.

**3.2. Root current  $A_0(z)$ .** Fix an arbitrary  $\mu \in \mathbb{C}^\times$  with  $|\mu| < 1$ . We define the dressed version of the current  $e(z)$  of the algebra  $\mathcal{E}$  by

$$(3.9) \quad \mathbf{e}(z) = e(z)\psi_\mu^+(C^{-1}z)^{-1}, \quad \psi_\mu^+(z) = \prod_{s=0}^{\infty} \psi^+(\mu^{-s}z) = \exp\left(\sum_{r>0} \frac{\kappa_r}{1-\mu^r} h_r z^{-r}\right).$$

The coefficients of the dressed current  $\mathbf{e}(z)$  are elements of  $\tilde{\mathcal{E}}$ . We warn the reader that  $\mathbf{e}(z)$  at  $\mu = 0$  does not reduce to  $e(z)$  but rather to  $e(z)\psi^+(C^{-1}z)^{-1}$ . This somewhat unusual convention is due to historical reasons.

While the current  $e(z)$  has rational commutation relations, the dressed current  $\mathbf{e}(z)$  satisfies the elliptic commutation relations:

$$\mathbf{e}(z)\mathbf{e}(w) = \mathbf{e}(w)\mathbf{e}(z) \prod_{i=1}^3 \frac{\Theta_\mu(q_i w/z)}{\Theta_\mu(q_i^{-1} w/z)}.$$

Here we use symbols for infinite products and theta functions

$$(z_1, \dots, z_r; p)_\infty = \prod_{i=1}^r \prod_{k=0}^{\infty} (1 - z_i p^k), \quad \Theta_p(z) = (z, pz^{-1}, p; p)_\infty.$$

These relations have to be understood as equality of matrix elements. Namely, if  $M$  is an admissible  $\mathcal{E}$  module then the matrix coefficients of both sides converge to meromorphic functions which are analytic continuation of each other.

There are two separate motivations for the definition of the dressed current. One is the form of the integrals of motion, see Section 3.5. In this section we discuss a different reason: the appearance of the current  $A_0(z)$  corresponding to the affine node and ultimately of the screening operator corresponding to the affine node, see Section 3.4. In type A these two points lead to the same definition of the dressed current.

The dressed current  $e(z)$  has the form

$$e(z) = b_{c_1} \mathbf{\Lambda}_1(z) + \cdots + b_{c_{\ell+1}} \mathbf{\Lambda}_{\ell+1}(z),$$

where  $\mathbf{\Lambda}_i(z) = \Lambda_i(z) \Delta^{(\ell)} \psi_\mu^+(C^{-1}z)^{-1}$ .

For  $i = 1, \dots, \ell$ , the ratios of the terms in dressed and undressed currents are the same:

$$: \frac{\mathbf{\Lambda}_i(z)}{\mathbf{\Lambda}_{i+1}(z)} : = : \frac{\Lambda_i(z)}{\Lambda_{i+1}(z)} : = A_i(a_i^A z), \quad i = 1, \dots, \ell.$$

However the contractions of individual terms change to elliptic ones:

$$(3.10) \quad \mathcal{C}(\mathbf{\Lambda}_i(z), \mathbf{\Lambda}_j(w)) = \mathcal{C}(\Lambda_i(z), \Lambda_j(w)) + \frac{\kappa_1}{1 - \mu}.$$

This allows us to define a new current  $A_0(z)$  which has rational contractions with all root currents  $A_i(z)$ ,  $i = 1, \dots, \ell$ , and is independent of them. Namely, define the zeroth root current by

$$(3.11) \quad A_0(z) = : \frac{\mathbf{\Lambda}_{\ell+1}(z)}{\mathbf{\Lambda}_1(\mu z)} : = : \frac{\Lambda_{\ell+1}(z)}{\Lambda_1(\mu z)} \Delta^{(\ell)} \psi^+(C^{-1}\mu z) : .$$

The contractions involving  $A_0(z)$  are as follows. Retaining notation (3.4) we have  $B_{i,0} = B_{0,i} = 0$  if  $i \neq 0, 1, \ell$ . The non-trivial ones are:

$$B_{0,0} = -\frac{t_1 t_2 t_3}{t_{c_{\ell+1}} t_{c_1}} (s_{c_{\ell+1}} s_{c_1} - s_{c_{\ell+1}}^{-1} s_{c_1}^{-1}),$$

$$B_{0,\ell} = B_{\ell,0} = \frac{t_1 t_2 t_3}{t_{c_{\ell+1}}}, \quad B_{0,1} = \frac{t_1 t_2 t_3}{t_{c_1}} C \mu^{-1}, \quad B_{1,0} = \frac{t_1 t_2 t_3}{t_{c_1}} C^{-1} \mu,$$

where  $C = \prod_{j=1}^{\ell+1} s_{c_j}$  is the total level and  $\ell > 1$ .

For  $\ell = 1$ ,  $B_{0,1}$  and  $B_{1,0}$  take the form:

$$B_{0,1} = -\kappa_1 \left( \frac{1}{t_{c_2}} + \frac{C \mu^{-1}}{t_{c_1}} \right), \quad B_{1,0} = -\kappa_1 \left( \frac{1}{t_{c_2}} + \frac{C^{-1} \mu}{t_{c_1}} \right).$$

Thus we have a  $U_q \widehat{\mathfrak{sl}}_2$  type deformed Cartan matrix corresponding to  $c_1 = c_2 = 2$  and a  $U_q \widehat{\mathfrak{gl}}_{1|1}$  type deformed Cartan matrix corresponding to  $c_1 = 1, c_2 = 2$ :

$$-t_1 t_3 \begin{pmatrix} s_2 + s_2^{-1} & -1 - C \mu^{-1} \\ -1 - C^{-1} \mu & s_2 + s_2^{-1} \end{pmatrix}, \quad t_3 \begin{pmatrix} t_3 & t_1 + t_2 C \mu^{-1} \\ t_1 + t_2 C^{-1} \mu & t_3 \end{pmatrix}.$$

We call the extended matrix of contractions  $\hat{B} = (B_{i,j})_{i,j=0,1,\dots,\ell}$  the symmetrized deformed affine Cartan matrix.

**Example 3.2.** We continue to consider the tensor product (3.6). The matrix  $\hat{B}$  is given by

$$(3.12) \quad \hat{B} = \begin{pmatrix} t_2^2 & t_2 t_3 C \mu^{-1} & 0 & 0 & 0 & t_1 t_2 \\ t_2 t_3 C^{-1} \mu & -t_2 t_3 (s_1 + s_1^{-1}) & t_2 t_3 & 0 & 0 & 0 \\ 0 & t_2 t_3 & -t_2 t_3 (s_1 + s_1^{-1}) & t_2 t_3 & 0 & 0 \\ 0 & 0 & t_2 t_3 & t_3^2 & t_1 t_3 & 0 \\ 0 & 0 & 0 & t_1 t_3 & -t_1 t_3 (s_2 + s_2^{-1}) & t_1 t_3 \\ t_1 t_2 & 0 & 0 & 0 & t_1 t_3 & t_1^2 \end{pmatrix}.$$

With the root  $A_0(z)$  the Dynkin diagram becomes the following.

$$(3.13) \quad \begin{array}{c} \textcircled{\otimes} \\ \diagup \quad \diagdown \\ \textcircled{\circ} \quad \textcircled{\circ} \quad \textcircled{\otimes} \quad \textcircled{\circ} \quad \textcircled{\otimes} \\ \text{1} \quad \text{1} \quad \text{3} \quad \text{2} \quad \text{1} \end{array}$$

The affine node is fermionic and we label it by 2, since it corresponds to the ratio of the last and first terms obtained from  $\mathcal{F}_3(u_6)$  and  $\mathcal{F}_1(u_1)$ . However, note that there is an arbitrary parameter  $\mu$  which does not appear in the diagram.  $\square$

One can easily describe the restrictions on the markings of the Dynkin diagram which can appear. There is a single global condition: if there are  $a_c$  fermionic nodes of type  $c$ ,  $c = 1, 2, 3$ , then  $a_1 \equiv a_2 \equiv a_3$  modulo 2. In addition, there are several local conditions. For example, neighboring bosonic nodes have to have the same marking, neighboring bosonic and fermionic nodes cannot have the same marking.

In particular, (3.13) is not a diagram of affine  $\mathfrak{sl}_{4|2}$ , as in any affine Dynkin diagram of type  $\mathfrak{sl}_{m|n}$  the number of simple odd roots is even (which is true in our setting if all bosonic nodes have the same type).

If all colors are the same, e.g.  $c_i = 2$ ,  $i = 1, \dots, \ell + 1$ , then the matrix  $\hat{B}$  is, up to a scalar multiple, the deformed affine Cartan matrix of type **A** in [KP].

The following lemma shows that in all cases the currents  $A_i(z)$ ,  $i = 0, 1, \dots, \ell$ , are independent generating currents of the Heisenberg algebra provided  $\mu \neq 1, C^2$ .

**Lemma 3.3.** *We have*

$$\det \hat{B} = \kappa_1^{\ell+1} \prod_{j=1}^{\ell+1} t_{c_j}^{-1} \times (1 - \mu)(C^{-1} - C\mu^{-1}).$$

*Proof.* It is easy to see that  $\det \hat{B}$  as a function of  $x = C^{-1}\mu$  is a linear function of  $x + x^{-1}$ . One can check that if  $x = C^{-1}$  then the vector  $(y_0, y_1, \dots, y_\ell)^T$  with  $y_0 = C$  and  $y_i = s_{c_1} \cdots s_{c_i}$  ( $1 \leq i \leq \ell$ ) is in the kernel of  $\hat{B}$ . It follows that  $\det \hat{B} = a(C + C^{-1} - x - x^{-1})$  with some  $a$ . The coefficient  $a$  can be determined from the behavior as  $x \rightarrow \infty$

$$\det \hat{B} = (-1)^\ell \hat{B}_{1,0} \hat{B}_{0,\ell} \prod_{j=2}^{\ell} \hat{B}_{j,j-1} + O(1) = -x \prod_{j=1}^{\ell+1} \frac{\kappa_1}{t_{c_j}} + O(1).$$

□

**3.3. Currents  $Y_i(z)$  and  $qq$ -characters.** In this section we describe current  $e(z)$  as a  $qq$ -character in the spirit of [N], [KP].

In what follows we write  $\hat{B} = \hat{D}\hat{C}$ ,  $\hat{C} = (C_{i,j})_{i=0}^\ell$ , choosing a diagonal matrix  $\hat{D} = \text{diag}(d_0, \dots, d_\ell)$  in such a way that

$$\begin{aligned} d_i &= -\frac{t_1 t_2 t_3}{t_c}, & C_{i,i} &= s_c + s_c^{-1}, & \text{if } A_i(z) \text{ is bosonic of type } c, \\ d_i &= t_c, & C_{i,i} &= s_c - s_c^{-1}, & \text{if } A_i(z) \text{ is fermionic of type } c. \end{aligned}$$

The currents  $A_i(z)$  correspond to roots. In order to understand the combinatorics of deformed  $\mathcal{W}$ -currents, it is convenient to introduce currents  $Y_i(z)$ ,  $i = 0, \dots, \ell$ , which correspond to the fundamental weights.

Define the modes  $\mathbf{a}_{i,r}$  by setting

$$A_i(z) = e^{\mathbf{a}_{i,0}} : e^{\sum_{r \neq 0} \mathbf{a}_{i,r} z^{-r}} : .$$

The zero mode  $e^{\mathbf{a}_{i,0}}$  is a variable, which takes the value  $u_i/u_{i+1}$  in (3.3) and (3.11), where  $u_0 = u_{\ell+1}$ .

Write elements of non-symmetrized deformed affine Cartan matrix as  $C_{i,j} = \sum_k m_{i,j}^{(k)} - \sum_s n_{i,j}^{(s)}$ , where  $m_{i,j}^{(k)}$ ,  $n_{i,j}^{(s)}$  are monomials of the form  $s_1^a s_2^b \mu^c$ ,  $a, b, c \in \mathbb{Z}$ , and define  $Y_i(z)$  by the set of equations

$$(3.14) \quad A_j(z) = : \prod_i \prod_k Y_i(m_{i,j}^{(k)} z) \prod_s \left( Y_i(n_{i,j}^{(s)} z) \right)^{-1} :, \quad j = 0, \dots, \ell,$$

where  $Y_i(z)$  are of the form

$$Y_i(z) = : e^{\sum_{r \neq 0} y_{i,r} z^{-r}} : \times \begin{cases} e^{y_{i,0}} & \text{if } A_i(z) \text{ is bosonic,} \\ e^{Q_{y_i} z^{y_{i,0}}} & \text{if } A_i(z) \text{ is fermionic.} \end{cases}$$

In the right hand side of (3.14),  $Y_i(z)$  for fermionic  $A_i(z)$  appears only as a ratio  $: Y_i(az)Y_i(bz)^{-1} :$ , so that  $e^{Q_{y_i}}$  cancels out. We shall specify the latter when we discuss the screening currents, see Section 3.4 below.

Due to Lemma 3.3, such  $Y_i(z)$ 's exist and are unique up to an overall shift of zero modes  $y_{i,0}$ . Moreover, we have

$$(3.15) \quad \mathcal{C}(A_j(w), Y_i(z)) = \pm \mathcal{C}(Y_i(z), A_j(w)) = d_i \delta_{i,j},$$

where the sign is  $+$  for bosonic nodes and  $-$  for fermionic nodes.

We explain this definition in an Example 3.4 below.

Let us set up some language convenient for discussing combinatorics. In what follows we set  $\mu = C s_1^{-\gamma}$ ,  $\gamma \in \mathbb{C}$ , and rewrite monomials  $s_1^i s_2^j \mu^k$ ,  $i, j, k \in \mathbb{Z}$ , in the form  $s_1^a s_2^b$ ,  $a, b \in \mathbb{C}$ . We adopt the shorthand notation  $\mathbf{l}_{a,b} = Y_l(s_1^a s_2^b z)$ ,  $l = 0, 1, \dots, \ell$ .

Let  $\mathcal{A}$  be the commutative ring in formal free variables  $\mathbf{l}_{a,b}^{\pm 1}$ ,  $l = 0, 1, \dots, \ell$ ,  $a, b \in \mathbb{C}$ .

Let  $\pi_0 : \mathcal{A} \rightarrow \mathcal{A}$  be the ring homomorphism sending  $\mathbf{0}_{a,b} \mapsto 1$ ,  $\mathbf{l}_{a,b} \mapsto \mathbf{l}_{a,b}$ ,  $l \neq 0$ .

For a formal monomial  $\alpha = s_1^a s_2^b$  define a ring homomorphism  $\tau_\alpha : \mathcal{A} \rightarrow \mathcal{A}$  shifting indices by  $(a, b)$ . For example, we have  $\tau_{s_1^a s_2^b} \mathbf{1}_{c,d} = \mathbf{1}_{a+c, b+d}$ .

The combinatorial study of the  $qq$ -characters and  $q$ -characters is based on the concept of dominant monomial, see [FR2]. The characters are obtained by a combinatorial algorithm “expanding the dominant monomials”, see [FM], [KP].

If current  $A_l(z)$  is bosonic, then a monomial  $m \in \mathcal{A}$  is called  $\mathbf{l}$ -dominant if  $m$  does not contain negative powers of  $\mathbf{l}_{a,b}$ . If current  $A_l(z)$  is fermionic, then we suggest to call a monomial  $m \in \mathcal{A}$   $\mathbf{l}$ -dominant if  $m$  is a product of  $\tilde{\mathbf{l}}_{a,b}$  with  $\tilde{l} \neq l$  and monomials of the form  $\mathbf{l}_{a,b\tau_{qc}}(\mathbf{l}_{a,b}^{-1})$ , where in the Cartan matrix we have either  $\hat{C}_{l,l-1} = t_c$  or  $\hat{C}_{l,l+1} = t_c$ .

**Example 3.4.** . We continue to work with (3.7). In this case we have

$$d_0 = t_2, \quad d_1 = d_2 = -t_2 t_3, \quad d_3 = t_3, \quad d_4 = -t_1 t_3, \quad d_5 = t_1,$$

and

$$\hat{C} = \begin{pmatrix} t_2 & t_3 C \mu^{-1} & 0 & 0 & 0 & t_1 \\ -C^{-1} \mu & s_1 + s_1^{-1} & -1 & 0 & 0 & 0 \\ 0 & -1 & s_1 + s_1^{-1} & -1 & 0 & 0 \\ 0 & 0 & t_2 & t_3 & t_1 & 0 \\ 0 & 0 & 0 & -1 & s_2 + s_2^{-1} & -1 \\ t_2 & 0 & 0 & 0 & t_3 & t_1 \end{pmatrix}.$$

Reading off the columns of  $\hat{C}$ , we obtain  $A_i(z)$  currents in terms of the  $Y_j(z)$ 's:

$$\begin{aligned} A_0(z) &= \mathbf{0}_{0,1} \mathbf{0}_{0,-1}^{-1} \mathbf{1}_{-\gamma,0}^{-1} \mathbf{5}_{0,1} \mathbf{5}_{0,-1}^{-1}, & A_1(z) &= \mathbf{0}_{\gamma-1,-1} \mathbf{0}_{\gamma+1,+1}^{-1} \mathbf{1}_{1,0} \mathbf{1}_{-1,0} \mathbf{2}_{0,0}^{-1}, & A_2(z) &= \mathbf{1}_{0,0}^{-1} \mathbf{2}_{1,0} \mathbf{2}_{-1,0} \mathbf{3}_{0,1} \mathbf{3}_{0,-1}^{-1}, \\ A_3(z) &= \mathbf{2}_{0,0}^{-1} \mathbf{3}_{-1,-1} \mathbf{3}_{1,1}^{-1} \mathbf{4}_{0,0}^{-1}, & A_4(z) &= \mathbf{3}_{1,0} \mathbf{3}_{-1,0}^{-1} \mathbf{4}_{0,1} \mathbf{4}_{0,-1} \mathbf{5}_{-1,-1} \mathbf{5}_{1,1}^{-1}, & A_5(z) &= \mathbf{0}_{1,0} \mathbf{0}_{-1,0}^{-1} \mathbf{4}_{0,0}^{-1} \mathbf{5}_{1,0} \mathbf{5}_{-1,0}^{-1}, \end{aligned}$$

where  $s_1^\gamma = C \mu^{-1}$ .

Set  $\delta_i = \ln q_i$ ,  $\delta_1 + \delta_2 + \delta_3 = 0$ . Then zero modes are given by

$$\begin{aligned} \mathbf{a}_{0,0} &= \delta_2 y_{0,0} - y_{1,0} + \delta_2 y_{5,0}, & \mathbf{a}_{1,0} &= \delta_3 y_{0,0} + 2y_{1,0} - y_{2,0}, & \mathbf{a}_{2,0} &= -y_{1,0} + 2y_{2,0} + \delta_2 y_{3,0}, \\ \mathbf{a}_{3,0} &= -y_{2,0} + \delta_3 y_{3,0} - y_{4,0}, & \mathbf{a}_{4,0} &= \delta_1 y_{3,0} + 2y_{4,0} + \delta_3 y_{5,0}, & \mathbf{a}_{5,0} &= \delta_1 y_{0,0} - y_{4,0} + \delta_1 y_{5,0}. \end{aligned}$$

These conditions uniquely determine  $y_{i,0}$ 's up to a common additive shift.

The current  $\mathbf{e}(z)$  can be written:

$$\begin{aligned} \mathbf{e}(a_0^{-1} z) &= \mathbf{\Lambda}_1(a_0^{-1} z) \left( b_{c_1} 1 + b_{c_2} A_1^{-1}(s_1 z) + b_{c_3} A_1^{-1}(s_1 z) A_2^{-1}(s_1^2 z) + b_{c_4} A_1^{-1}(s_1 z) A_2^{-1}(s_1^2 z) A_3^{-1}(s_1^3 z) \right. \\ &\quad \left. + b_{c_5} A_1^{-1}(s_1 z) A_2^{-1}(s_1^2 z) A_3^{-1}(s_1^3 z) A_4^{-1}(s_1^3 s_2 z) + b_{c_6} A_1^{-1}(s_1 z) A_2^{-1}(s_1^2 z) A_3^{-1}(s_1^3 z) A_4^{-1}(s_1^3 s_2 z) A_5^{-1}(s_1^3 s_2^2 z) \right), \end{aligned}$$

which we symbolically depict as follows.

$$\mathbf{\Lambda}_1 \xrightarrow{A_1^{-1}} \mathbf{\Lambda}_2 \xrightarrow{A_2^{-1}} \mathbf{\Lambda}_3 \xrightarrow{A_3^{-1}} \dots \xrightarrow{A_{\ell-1}^{-1}} \mathbf{\Lambda}_\ell \xrightarrow{A_\ell^{-1}} \mathbf{\Lambda}_{\ell+1}$$

Ignoring constants  $b_{c_i}$  we write it using  $Y_i(z)$  in the form:

$$(3.16) \quad \chi = \mathbf{0}_{\gamma,-1} \mathbf{0}_{\gamma+2,1}^{-1} \mathbf{1}_{0,0} + \mathbf{1}_{2,0}^{-1} \mathbf{2}_{1,0} + \mathbf{2}_{3,0}^{-1} \mathbf{3}_{2,-1} \mathbf{3}_{2,1}^{-1} + \mathbf{3}_{4,1} \mathbf{3}_{2,1}^{-1} \mathbf{4}_{3,0} + \mathbf{4}_{3,2}^{-1} \mathbf{5}_{4,2} \mathbf{5}_{2,0}^{-1} + \mathbf{0}_{2,2} \mathbf{0}_{4,2}^{-1} \mathbf{5}_{2,2} \mathbf{5}_{2,0}^{-1}.$$

Following [N], we call the expression  $\chi$  the  $qq$ -character of the vector representation corresponding to the Dynkin diagram (3.13).

According to the general rule, for  $l = 1, 2, 4$  (that is, for bosonic nodes) a monomial is  $\mathbf{l}$ -dominant if it is a product of monomials  $\mathbf{l}_{a,b}$  and  $\tilde{\mathbf{l}}_{a,b}^{\pm 1}$  with  $a, b \in \mathbb{C}$  and  $\tilde{l} \neq l$ . A monomial is  $\mathbf{0}$ -dominant if it

is a product of monomials of the form  $\mathbf{0}_{a,b}\mathbf{0}_{a+2,b}^{-1}$ ,  $\mathbf{0}_{a,b}\mathbf{0}_{a-2,b-2}^{-1}$ , and  $\mathbf{l}_{a,b}^{\pm 1}$  with  $l \neq 0$ ;  $\mathbf{3}$ -dominant if it is a product of monomials of the form  $\mathbf{3}_{a,b}\mathbf{3}_{a+2,b}^{-1}$ ,  $\mathbf{3}_{a,b}\mathbf{3}_{a,b+2}^{-1}$ , and  $\mathbf{l}_{a,b}^{\pm 1}$  with  $l \neq 3$ ;  $\mathbf{5}$ -dominant if it is a product of monomials of the form  $\mathbf{5}_{a,b}\mathbf{5}_{a,b+2}^{-1}$ ,  $\mathbf{5}_{a,b}\mathbf{5}_{a-2,b-2}^{-1}$ , and  $\mathbf{l}_{a,b}^{\pm 1}$  with  $l \neq 5$ .

In (3.16) the  $l$ -th term is  $\mathbf{l}$ -dominant for  $1 \leq l \leq 5$ , and the last term is  $\mathbf{0}$ -dominant. The  $(l+1)$ st term is obtained by multiplying the  $l$ -th term by the inverse of current  $A_l(z)$  with an appropriate shift, see (3.3). Moreover, the last term and the first term are connected as follows

$$(3.17) \quad A_0^{-1}(s_1^3 s_2^2 s_3 z) \mathbf{0}_{2,2} \mathbf{0}_{4,2}^{-1} \mathbf{5}_{2,2} \mathbf{5}_{2,0}^{-1} = \mathbf{0}_{2,0} \mathbf{0}_{4,2}^{-1} \mathbf{1}_{\gamma+2,1} = \tau_\mu(\mathbf{0}_{\gamma,-1} \mathbf{0}_{\gamma+2,1}^{-1} \mathbf{1}_{0,0}).$$

□

While our  $qq$ -character is closely connected to that of [KP], they are different in several ways. First, our first monomial is  $\mathbf{l}$ -dominant for  $l = 1, \dots, \ell$ , but not  $\mathbf{0}$ -dominant. Second, our  $qq$ -character is of finite type, namely, applying  $\pi_0$  we obtain the  $qq$ -character corresponding to the non-affine Cartan matrix. In particular, our  $qq$ -character contains finitely many terms. Third, the fermionic nodes are not considered in [KP]<sup>2</sup>. Fourth, the variables  $\mathbf{0}_{a,b}$  do play an important role as they correspond to the dressing of the current  $e(z)$  and the finite type  $qq$ -character “closes up” in the sense of (3.17).

The  $qq$ -characters of [KP] commute with all screenings operators including the one corresponding to the zeroth node and correspond to modules of quantum toroidal algebra  $\mathcal{E}_{\ell+1}$  associated to  $\mathfrak{gl}_{\ell+1}$ . If we start with our  $qq$ -character and formally require such commutativity, we would have to add infinitely many terms which correspond to the vector representation of  $\mathcal{E}_{\ell+1}$ , while [KP] starts with a dominant monomial which produces a  $qq$ -character corresponding to a Fock module of  $\mathcal{E}_{\ell+1}$ .

The usual  $q$ -character of the evaluation vector representation of  $U_q \widehat{\mathfrak{sl}}_\ell$  in the sense of [FR1] is recovered after applying  $\pi_0$  in the case of  $\otimes_{i=1}^{\ell+1} \mathcal{F}_2(u_i)$ , when all Fock spaces are of the same sort. In this case we have

$$b_2^{-1} e(z) = \mathbf{0}_{0,\gamma+1}^{-1} \mathbf{1}_{0,0} + \mathbf{2}_{0,1} \mathbf{1}_{0,2}^{-1} + \mathbf{3}_{0,2} \mathbf{2}_{0,3}^{-1} + \dots + \mathbf{\ell}_{0,\ell+1}^{-1} \mathbf{0}_{0,\ell}$$

where  $s_2^\gamma = C\mu^{-1}$ .

### 3.4. Screenings.

We discuss the screening operators.

First, we clearly have  $[A_i(z), \mathbf{\Lambda}_j(w)] = 0$  whenever  $j \not\equiv i, i+1 \pmod{\ell+1}$ .

Moreover, for  $i = 0, 1, \dots, \ell$ , the non-trivial contractions of  $A_i(z)$  have the form (2.8) where

$$A(z) = \frac{b_{c_i}}{b_{c_{i+1}}} A_i(z), \quad V(z) = b_{c_i} \mathbf{\Lambda}_i((a_i^A)^{-1} z), \quad V'(z) = b_{c_{i+1}} \mathbf{\Lambda}_{i+1}((a_i^A)^{-1} \mu^{\delta_{0,i}} z).$$

Here we set  $\mathbf{\Lambda}_{\ell+1}(z) = \mathbf{\Lambda}_0(z)$ . If the node is bosonic,  $c_i = c_{i+1}$ , then  $p_3 = p_1 = s_c$ ,  $p_2 = s_b$ , where  $\{c_i, b, c\} = \{1, 2, 3\}$ . If the node is fermionic, then  $p_1 = s_{c_{i+1}}$ ,  $p_2 = s_d$ ,  $p_3 = s_{c_i}$ , where  $\{c_i, c_{i+1}, d\} = \{1, 2, 3\}$ .

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<sup>2</sup>Fermionic root currents  $A_i(z)$  appeared in [BFM].

Accordingly we define two screening currents  $S_i^\pm(z) = e^{Q_{s_i} z^{s_{i,0}^\pm}} S_i^{\pm,osc}(z)$  for each bosonic node and one screening current  $S_i^f(z) = e^{Q_{s_i} z^{s_{i,0}^f}} S_i^{f,osc}(z)$  for each fermionic node, see (2.9), (2.10):

$$(3.18) \quad c_i = c_{i+1} : \quad A_i(z) = s_{c_i} s_{c_{i+1}} : \frac{S_i^+(s_b^{-1}z)}{S_i^+(s_b z)} : = s_{c_i} s_{c_{i+1}} : \frac{S_i^-(s_c^{-1}z)}{S_i^-(s_c z)} :,$$

$$(3.19) \quad c_i \neq c_{i+1} : \quad A_i(z) = s_{c_i} s_{c_{i+1}} : \frac{S_i^f(s_d^{-1}z)}{S_i^f(s_d z)} :,$$

where  $(c_i, b, c) = cycl(1, 2, 3)$  and  $(c_i, c_{i+1}, d) = \{1, 2, 3\}$ . We choose  $Q_{s_i}$  by demanding  $[y_{j,0}, Q_{s_i}] = g_i \delta_{i,j}$ , where

$$(3.20) \quad g_i = \begin{cases} \log q_c & \text{for } S_i^+, \\ \log q_b & \text{for } S_i^-, \\ -1 & \text{for } S_i^f. \end{cases}$$

In the fermionic case (3.19), we set in addition  $[s_{i,0}, Q_{y_j}] = -\delta_{i,j}$  so that we have

$$S_i^f(z) Y_i(w) = \frac{1}{z-w} : S_i^f(z) Y_i(w) : , \quad Y_i(w) S_i^f(z) = \frac{1}{w-z} : Y_i(w) S_i^f(z) : .$$

Introduce the corresponding screening operators  $S_i^\pm, S_i^f$  by (2.7) when well-defined.

Let  $S_i$  stand for either  $S_i^\pm$  when  $A_i(z)$  is a bosonic current, or  $S_i^f$  when  $A_i(z)$  is a fermionic current. It follow from Lemma 2.1 that the currents  $e(z)$  and  $\mathbf{e}(z)$  both commute with all  $S_i$  with  $i \neq 0$ , and that  $\mathbf{e}(z)$  commutes with  $S_0$  up to a  $\mu$ -difference:

$$(3.21) \quad [S_i, e(z)] = [S_i, \mathbf{e}(z)] = 0, \quad i = 1, \dots, \ell,$$

$$(3.22) \quad [S_0, \mathbf{e}(z)] = b_{c_1} [S_0, \mathbf{\Lambda}_1(z) - \mathbf{\Lambda}_1(\mu z)].$$

The relation (3.22) implies the commutativity of  $S_0$  with integrals of motion, see Theorem 3.7 below. In view of these relations, we call  $\mathbf{e}(z)$  the deformed  $\mathcal{W}$ -current of type  $\mathbf{A}$ .

**Example 3.5.** We again illustrate the construction of screenings on the example of (3.6), see also Example 3.4. In this case

$$\begin{aligned} \frac{\mathbf{\Lambda}_6(z)}{\mathbf{\Lambda}_1(\mu z)} &= s_2^{-1} \frac{S_0^f(s_2^{-1}z)}{S_0^f(s_2 z)}, & \frac{\mathbf{\Lambda}_1(z)}{\mathbf{\Lambda}_2(z)} &= q_1 \frac{S_1^\pm(s_{(5\pm 1)/2}^{-1} a_1^A z)}{S_1^\pm(s_{(5\pm 1)/2} a_1^A z)}, \\ \frac{\mathbf{\Lambda}_2(z)}{\mathbf{\Lambda}_3(z)} &= q_1 \frac{S_2^\pm(s_{(5\pm 1)/2}^{-1} a_2^A z)}{S_2^\pm(s_{(5\pm 1)/2} a_2^A z)}, & \frac{\mathbf{\Lambda}_3(z)}{\mathbf{\Lambda}_4(z)} &= s_3^{-1} \frac{S_3^f(s_3^{-1} a_3^A z)}{S_3^f(s_3 a_3^A z)}, \\ \frac{\mathbf{\Lambda}_4(z)}{\mathbf{\Lambda}_5(z)} &= q_2 \frac{S_4^\pm(s_{2\pm 1}^{-1} a_4^A z)}{S_4^\pm(s_{2\pm 1} a_4^A z)}, & \frac{\mathbf{\Lambda}_5(z)}{\mathbf{\Lambda}_6(z)} &= s_1^{-1} \frac{S_5^f(s_1^{-1} a_5^A z)}{S_5^f(s_1 a_5^A z)}. \end{aligned}$$

In particular the zero modes of screening operators satisfy

$$\begin{aligned} \frac{u_6}{u_1} &= s_2^{-1} q_2^{-s_{0,0}^f}, \quad \frac{u_1}{u_2} = q_1 q_{(5\pm 1)/2}^{-s_{1,0}^\pm}, \quad \frac{u_2}{u_3} = q_1 q_{(5\pm 1)/2}^{-s_{2,0}^\pm}, \\ \frac{u_3}{u_4} &= s_3^{-1} q_3^{-s_{3,0}^f}, \quad \frac{u_4}{u_5} = q_2 q_{2\pm 1}^{-s_{4,0}^\pm}, \quad \frac{u_5}{u_6} = s_1^{-1} q_1^{-s_{5,0}^f}. \end{aligned}$$

Recall that we need  $s_{i,0}$  to act as an integer in order to have well defined screening operators. It dictates some conditions on  $u_i$ .

It is convenient to rename the zeroth screening current by setting  $S_0^f(z) = z^{\Delta-1}\bar{S}_0^f(z)$ ,  $s_{0,0}^f = \bar{s}_{0,0}^f + \Delta - 1$ , where  $q_2^{\Delta-1} = C^2$ . Then the screening operator reads

$$S_0^f = \int z^{\Delta-1}\bar{S}_0^f(z) dz$$

and  $\bar{s}_{0,0}^f$  is determined from

$$1 = q_2^{-\bar{s}_{0,0}^f} q_{(5\pm 1)/2}^{-s_{1,0}^\pm - s_{2,0}^\pm} q_3^{-s_{3,0}^f} q_{2\pm 1}^{-s_{4,0}^\pm} q_1^{-s_{5,0}^f}.$$

□

**3.5. Integrals of motion.** One of our primary interests is in constructing commuting families of operators. In the case of type **A** the source of such families is the standard transfer matrix construction, see [FJM]. Here we recall the answer, which appeared first in [FKSW].

Define the Feigin-Odesskii kernel function [FO]:

$$(3.23) \quad \omega_2(z) = \frac{\Theta_\mu(z)\Theta_\mu(q_2^{-1}z)}{\Theta_\mu(q_1z)\Theta_\mu(q_3z)}.$$

The function  $\omega_2(z)$  satisfies a series of identities parametrized by  $m, n \in \mathbb{Z}_{\geq 1}$ , see [FKSW]:

$$(3.24) \quad \text{Sym}_{z_1, \dots, z_{m+n}} \prod_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \omega_2(z_j/z_i)^{-1} = \text{Sym}_{z_1, \dots, z_{m+n}} \prod_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \omega_2(z_i/z_j)^{-1}.$$

The following theorem describes the integrals of motion.

**Theorem 3.6.** [FKSW], [FJM] *The following elements  $\{\mathbf{I}_n\}_{n=1}^\infty$  are mutually commutative:*

$$(3.25) \quad \mathbf{I}_n = \int \cdots \int \mathbf{e}(z_1) \cdots \mathbf{e}(z_n) \cdot \prod_{j < k} \omega_2(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{2\pi i z_j},$$

where the integral is taken on the unit circle  $|z_j| = 1$ ,  $j = 1, \dots, n$  (or a common circle of any radius, due to homogeneity) in the region  $|q_1|, |q_3| > 1$  and extended by analytic continuation everywhere else.

*Proof.* In [FKSW] the theorem is proved directly (on tensor products of Fock spaces) with the use of the identity (3.24).

In [FJM], it is shown that  $\mathbf{I}_n$ , up to a constant, are Taylor coefficients of the transfer matrix corresponding to the Fock space  $\mathcal{F}_2(u)$ , where  $\mu$  is the twist parameter. Then the commutativity follows from the standard argument with R-matrix. □

We note that in the construction of integrals of motion one can replace  $\omega_2(z)$  with functions

$$\omega_1(z) = \frac{\Theta_\mu(z)\Theta_\mu(q_1^{-1}z)}{\Theta_\mu(q_2z)\Theta_\mu(q_3z)}, \quad \omega_3(z) = \frac{\Theta_\mu(z)\Theta_\mu(q_3^{-1}z)}{\Theta_\mu(q_1z)\Theta_\mu(q_2z)}.$$

It is known that while individual integrals  $\mathbf{I}_n$  with  $n \geq 2$  depend on this choice, the algebra generated by all  $\mathbf{I}_n$  does not.

Let us verify the commutativity with screenings.

**Theorem 3.7.** *The integrals of motion commute with all screening operators,*

$$[S_i, \mathbf{I}_n] = 0, \quad i = 0, 1, \dots, \ell, \quad n \geq 1.$$

*Proof.* For  $i \neq 0$ , this follows readily from (3.21). We check the case  $i = 0$  using (3.22). By replacing  $\omega_2(x)$  by  $\omega_3(x)$  or  $\omega_1(x)$  if necessary, it suffices to consider the two cases,  $c_1 = c_{\ell+1} = 2$ , or  $c_1 = 1, c_{\ell+1} = 3$ . We assume  $|q_1|, |q_3| > 1$ .

First consider the case  $c_1 = c_{\ell+1} = 2, S_0 = S_0^-$ . Then

$$[S_0, \mathbf{I}_1(z)] = \text{const.} \mathcal{A}^{(1)}(z), \quad \mathcal{A}^{(1)}(z) = z : S_0(s_1^{-1} \mu^{-1} z) \mathbf{I}_1(z) : .$$

Noting the symmetry of the integrand and the commutativity  $S_0(z) \mathbf{e}(w) = \mathbf{e}(w) S_0(z)$  as meromorphic functions, we obtain

$$\begin{aligned} [S_0, \mathbf{I}_n] &\propto \sum_{i=1}^n \int \cdots \int \mathbf{e}(z_1) \cdots (\mathcal{A}^{(1)}(z_i) - \mathcal{A}^{(1)}(\mu z_i)) \cdots \mathbf{e}(z_n) \cdot \prod_{j < k} \omega_2(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \\ &\propto \int \cdots \int (\mathcal{A}^{(1)}(z_1) - \mathcal{A}^{(1)}(\mu z_1)) \mathbf{e}(z_2) \cdots \mathbf{e}(z_n) \cdot \prod_{j < k} \omega_2(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{2\pi i z_j}. \end{aligned}$$

Let us examine the relevant poles of  $\mathcal{A}^{(1)}(z_1) \mathbf{e}(z_j)$ . For symmetry reasons we consider  $j = 2$ . We have

$$\omega_2(z_2/z_1) \mathcal{A}^{(1)}(z_1) \mathbf{I}_1(z_2) = F(z_2/z_1) F(\mu^{-1} z_1/z_2) f_l^{(1)}(z_2/z_1) \times : \mathcal{A}^{(1)}(z_1) \mathbf{I}_1(z_2) :,$$

where

$$\begin{aligned} F(x) &= \frac{(\mu x, \mu q_2 x; \mu)_\infty}{(\mu q_1^{-1} x, \mu q_3^{-1} x; \mu)_\infty}, \\ f_l^{(1)}(x) &= \begin{cases} \frac{1 - q_2 x}{1 - q_1^{-1} x} & (l = 1), \\ \frac{1 - \mu q_2^{-1} x}{1 - \mu q_1 x} & (l = \ell + 1), \\ 1 & (l \neq 1, \ell + 1). \end{cases} \end{aligned}$$

All integrals are taken over the unit circle. If  $|\mu q_1| > 1$ , one easily checks that the  $z_1$  integral vanishes. Otherwise one has to pick the residues at  $z_1 = q_1^{-1} z_2, \mu q_1 z_2$  coming from  $f_l^{(1)}(z_2/z_1)$ . Using again the relation  $q_2 : S_0(s_1^{-1} z) \mathbf{I}_1(\mu z) : = : S_0(s_1 z) \mathbf{I}_{\ell+1}(z) :$ , we find that

$$[S_0, \mathbf{I}_n] \propto \int \cdots \int (\mathcal{A}^{(2)}(z_2) - \mathcal{A}^{(2)}(\mu q_1 z_2)) \mathbf{e}(z_3) \cdots \mathbf{e}(z_n) \cdot \prod_{k=3}^n \omega_2^{(2)}(z_k/z_2) \cdot \prod_{3 \leq j < k} \omega_2(z_k/z_j) \prod_{j=2}^n \frac{dz_j}{2\pi i z_j},$$

where

$$\mathcal{A}^{(2)}(z) = z : S_0(s_1^{-1} q_1^{-1} \mu^{-1} z) \mathbf{I}_1(q_1^{-1} z) \mathbf{I}_1(z) :, \quad \omega_2^{(2)}(x) = \omega_2(q_1 x) \omega_2(x).$$

This current has properties similar to  $\mathcal{A}^{(1)}(z)$  except that  $\mu$  is changed to  $\mu q_1$ , namely

$$\omega_2^{(2)}(z_2/z_1) \mathcal{A}^{(2)}(z_2) \mathbf{I}_1(z_3) = F(z_3/z_2) F(q_1^{-1} \mu^{-1} z_2/z_3) f_l^{(2)}(z_3/z_2) \times : \mathcal{A}^{(2)}(z_2) \mathbf{I}_1(z_3) :,$$

where  $f_{\ell+1}^{(2)}(x) = f_{\ell+1}^{(1)}(q_1 x)$  and  $f_l^{(2)}(x) = f_l^{(1)}(x)$  ( $l \neq \ell + 1$ ). We repeat this process several times, until either there are no other integration variables left, or until  $|\mu q_1^m| > 1$  with some  $m$  and the shifting  $z \rightarrow \mu q_1^m z$  is no longer obstructed.

In the case  $c_1 = 1$  and  $c_{\ell+1} = 3$ , the analog of  $f_l^{(1)}(x)$  becomes

$$\tilde{f}_l^{(1)}(x) = \begin{cases} \frac{1-q_1x}{1-q_2^{-1}x} & (l = 1), \\ \frac{1-\mu q_3^{-1}x}{1-\mu q_2x} & (l = \ell + 1), \\ 1 & (l \neq 1, \ell + 1). \end{cases}$$

There are no poles which obstruct the shift  $z \rightarrow \mu z$ .  $\square$

#### 4. ALGEBRA $\mathcal{K}$

We define an algebra  $\mathcal{K}$  which produces the deformed  $\mathcal{W}$  currents of types B, C, D in the same way as the algebra  $\mathcal{E}$  produces the deformed  $\mathcal{W}$  current of type A. We think of algebra  $\mathcal{K}$  as a “twisted” [O] analog of algebra  $\mathcal{E}$  or as a “coideal” subalgebra [L].

4.1. **Algebra  $\mathcal{K}$ .** We define an algebra  $\mathcal{K}$  with generating currents

$$E(z) = \sum_{n \in \mathbb{Z}} E_n z^{-n}, \quad K^\pm(z) = \exp\left(\sum_{\pm r > 0} H_r z^{-r}\right),$$

and an invertible central element  $C$ . We set

$$K(z) = K^-(z)K^+(C^2 z).$$

The defining relations are as follows.

$$(4.1) \quad g(z, w)E(z)E(w) + g(w, z)E(w)E(z) = \frac{1}{\kappa_1} \left( g(z, w)\delta\left(C^2 \frac{z}{w}\right)K(z) + g(w, z)\delta\left(C^2 \frac{w}{z}\right)K(w) \right),$$

$$(4.2) \quad K^\pm(z)K^\pm(w) = K^\pm(w)K^\pm(z),$$

$$(4.3) \quad g(z, w)g(z, C^2 w)K^+(z)K^-(w) = \bar{g}(z, w)\bar{g}(z, C^2 w)K^-(w)K^+(z),$$

$$(4.4) \quad g(z, w)K^\pm(z)E(w) = \bar{g}(z, w)E(w)K^\pm(z),$$

$$(4.5) \quad \text{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [E(z_1), [E(z_2), E(z_3)]] = \text{Sym}_{z_1, z_2, z_3} X(z_1, z_2, z_3) \kappa_1^{-1} \delta(C^2 z_1/z_3) K^-(z_1) E(z_2) K^+(z_3),$$

where

$$\text{Sym}_{z_1, \dots, z_N} f(z_1, \dots, z_N) = \frac{1}{N!} \sum_{\pi \in \mathfrak{S}_N} f(z_{\pi(1)}, \dots, z_{\pi(N)}),$$

$$X(z_1, z_2, z_3) = \frac{(z_1 + z_2)(z_3^2 - z_1 z_2)}{z_1 z_2 z_3} G(z_2/z_3) + \frac{(z_2 + z_3)(z_1^2 - z_2 z_3)}{z_1 z_2 z_3} G(z_1/z_2) + \frac{(z_3 + z_1)(z_2^2 - z_3 z_1)}{z_1 z_2 z_3},$$

and  $G(w/z)$  stands for the power series expansion of  $\bar{g}(z, w)/g(z, w)$  in  $w/z$ . We note that as a rational function

$$X(z_1, z_2, z_3) = \frac{\kappa_1 g(z_1, z_3)}{g(z_3, z_2)g(z_2, z_1)} (z_3 + z_2)(z_1 + z_2)(z_2^2 - z_3 z_1) \frac{z_2}{z_3 z_1}.$$

The relations for  $K^\pm(z)$  are equivalent to

$$[H_r, H_s] = -\delta_{r+s, 0} \kappa_r \frac{1 + C^{2r}}{r}.$$

Note the difference to (2.1).

In the presence of quadratic relations, the Serre relation (4.5) can be reformulated in terms of correlation functions, see Lemma 4.9 below.

Note that the Fourier coefficients of  $K(z)$  are infinite series. Therefore the relation (4.1) requires some justification. We proceed as follows.

We define the homogeneous grading of generators by  $\deg E_n = n$ ,  $n \in \mathbb{Z}$ ,  $\deg H_{\pm r} = \pm r$ ,  $r \in \mathbb{Z}_{>0}$ . Consider the free algebra  $\mathcal{A}$  generated by  $E(z), H(z)$ . Let  $\tilde{\mathcal{A}}$  be the completion of  $\mathcal{A}$  with respect to the grading in the positive direction. The elements of  $\tilde{\mathcal{A}}$  are series of the form  $\sum_{i > -N} f_i g_i$ , where  $f_i, g_i \in \mathcal{A}$  and  $\deg g_i = i$ . Then we consider  $\mathcal{K}$  to be a graded quotient algebra of  $\tilde{\mathcal{A}}$ .

We call a  $\mathcal{K}$  module  $V$  admissible if  $V$  is a graded module and  $V = \bigoplus_{i < N} V_i$ , where  $V_i = \{v \in V, \deg v = i\}$ , and if in addition  $C$  is diagonalizable. If  $V$  is an admissible module, then for any  $v \in V$ , the series  $K^+(z)v$  is a polynomial in  $z^{-1}$  with values in  $V$  and therefore  $K(z)v$  is well defined.

Finally, we note that there are obvious automorphisms of  $\mathcal{K}$ :

$$(4.6) \quad \iota : \mathcal{K} \rightarrow \mathcal{K}, \quad E(z) \mapsto -E(z), \quad K^\pm(z) \mapsto K^\pm(z),$$

$$(4.7) \quad \tau_a : \mathcal{K} \rightarrow \mathcal{K}, \quad E(z) \mapsto E(az), \quad K^\pm(z) \mapsto K^\pm(az) \quad (a \in \mathbb{C}^\times).$$

**4.2. Left comodule structure.** The algebra  $\mathcal{K}$  does not seem to have a natural coproduct. Instead it is a comodule over the quantum toroidal algebra  $\mathcal{E}$ .

Consider the tensor product algebra of  $\mathcal{E}$  and  $\mathcal{K}$ . We denote by  $\mathcal{E} \tilde{\otimes} \mathcal{K}$  the completion of the tensor algebra  $\mathcal{E} \otimes \mathcal{K}$  with respect to the homogeneous grading in the positive direction.

**Proposition 4.1.** *The following map  $\Delta : \mathcal{K} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{K}$  endows  $\mathcal{K}$  with a structure of a left  $\mathcal{E}$ -comodule:*

$$(4.8) \quad \begin{aligned} \Delta E(z) &= e(C_2^{-1}z) \otimes K^+(z) + 1 \otimes E(z) + \tilde{f}(C_2z) \otimes K^-(z), \\ \Delta K^+(z) &= \psi^+(C_1^{-1}C_2^{-1}z) \otimes K^+(z), \\ \Delta K^-(z) &= \psi^-(C_2z)^{-1} \otimes K^-(z), \\ \Delta C &= C \otimes C, \end{aligned}$$

where  $C_1 = C \otimes 1$ ,  $C_2 = 1 \otimes C$ . □

*Proof.* Checking that  $\Delta$  preserves the relations (4.1)–(4.5), especially the last one, demands a straightforward but long calculation. We use the identity

$$\begin{aligned} & \left( \frac{z_1}{z_2} - \frac{z_2}{z_1} \right) (g_{1,3}g_{2,3} - g_{3,1}g_{3,2}) + \left( \frac{z_2}{z_3} - \frac{z_3}{z_2} \right) g_{3,1}(g_{2,3} + g_{3,2}) + \left( \frac{z_1}{z_3} - \frac{z_3}{z_1} \right) (g_{1,3} + g_{3,1})g_{2,3} \\ &= \kappa_1 \left( 1 - \frac{z_3^2}{z_1 z_2} \right) (z_1 + z_3)(z_2 + z_3) z_3 g_{1,2}, \end{aligned}$$

where  $g_{i,j} = g(z_i, z_j)$ .

Coassociativity  $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$  mapping  $\mathcal{K} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{E} \tilde{\otimes} \mathcal{K}$  is a short direct calculation. The counit property  $(\epsilon \otimes \text{id}) \circ \Delta = \text{id}$  mapping  $\mathcal{K} \rightarrow \mathcal{K}$  is obvious. □

**4.3. Boundary Fock modules.** We describe several representations of algebra  $\mathcal{K}$ , given in one free boson.

For a complex number  $k \in \mathbb{C}^\times$ , let  $\mathcal{H}_k$  be the Heisenberg algebra generated by  $\{H_r\}_{r \neq 0}$  with the relations  $[H_r, H_s] = -\delta_{r+s,0} \kappa_r (1 + k^{2r})/r$ .

Keeping in mind the Dynkin types **B** and **C, D**, for  $c \in \{1, 2, 3\}$  we set

$$\mathcal{H}_c^{\mathbf{B}} = \mathcal{H}_{s_c^{1/2}}, \quad \mathcal{H}_c^{\mathbf{CD}} = \mathcal{H}_{s_c^{-1}},$$

and denote by  $\mathcal{F}_c^{\mathbf{B}}, \mathcal{F}_c^{\mathbf{CD}}$  the corresponding Fock modules of the Heisenberg algebra.

We have three  $\mathcal{K}$  modules of type **B**. Set

$$k_c^{\mathbf{B}} = \frac{(1 + s_c)(s_d - s_b)}{\kappa_1}, \quad (c, d, b) = \text{cycl}(1, 2, 3).$$

Define a vertex operator  $\tilde{K}_c^{\pm}(z)$  by

$$\tilde{K}_c^{\pm}(z) = \exp\left(\sum_{\pm r > 0} \frac{1}{1 + s_c^{-r}} H_r z^{-r}\right).$$

We have  $\tilde{K}_c^{\pm}(z)\tilde{K}_c^{\pm}(s_c z) = K^{\pm}(z)$ . Set  $\tilde{K}_c(z) = \tilde{K}_c^{-}(z)\tilde{K}_c^{+}(s_c z)$ .

**Proposition 4.2.** *For  $c \in \{1, 2, 3\}$ , the map  $\mathcal{K} \rightarrow \mathcal{H}_c^{\mathbf{B}}$  sending*

$$E(z) \mapsto k_c^{\mathbf{B}} \tilde{K}_c(z), \quad K^{\pm}(z) \mapsto K^{\pm}(z), \quad C \mapsto s_c^{1/2},$$

*endows  $\mathcal{F}_c^{\mathbf{B}}$  with a structure of a  $\mathcal{K}$  module of level  $s_c^{1/2}$ .*

*Proof.* The proposition is proved by a direct computation. □

We also have three  $\mathcal{K}$  modules of type **CD**.

**Proposition 4.3.** *For  $c \in \{1, 2, 3\}$ , the map  $\mathcal{K} \rightarrow \mathcal{H}_c^{\mathbf{CD}}$  sending*

$$E(z) \mapsto 0, \quad K^{\pm}(z) \mapsto K^{\pm}(z), \quad C \mapsto s_c^{-1},$$

*endows  $\mathcal{F}_c^{\mathbf{CD}}$  with a structure of a  $\mathcal{K}$  module of level  $s_c^{-1}$ .*

*Proof.* The proposition is proved by a direct computation. □

Using the comodule map  $\Delta$  we obtain the following corollary.

**Corollary 4.4.** *There exists an algebra homomorphism  $\mathcal{K} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{H}_c^{\mathbf{B}}$  such that*

$$\begin{aligned} E(z) &\mapsto e(s_c^{-1/2}z) \otimes K^+(z) + k_c^{\mathbf{B}} \otimes \tilde{K}_c(z) + \tilde{f}(s_c^{1/2}z) \otimes K^-(z), \\ K^+(z) &\mapsto \psi^+(C_1^{-1}s_c^{-1/2}z) \otimes K^+(z), \quad K^-(z) \mapsto \psi^-(s_c^{1/2}z)^{-1} \otimes K^-(z), \\ C &\mapsto s_c^{1/2}C_1. \end{aligned}$$

*Similarly, there exists an algebra homomorphism  $\mathcal{K} \rightarrow \mathcal{E} \tilde{\otimes} \mathcal{H}_c^{\mathbf{CD}}$  such that*

$$\begin{aligned} E(z) &\mapsto e(s_c z) \otimes K^+(z) + \tilde{f}(s_c^{-1}z) \otimes K^-(z), \\ K^+(z) &\mapsto \psi^+(C_1^{-1}s_c z) \otimes K^+(z), \quad K^-(z) \mapsto \psi^-(s_c^{-1}z)^{-1} \otimes K^-(z), \\ C &\mapsto s_c^{-1}C_1. \end{aligned}$$

□

It seems likely that the homomorphisms in Corollary 4.4 are injective. If so, it would give us an inclusion of algebra  $\mathcal{K}$  into algebra  $\mathcal{E}$  extended by an extra Heisenberg algebra.

Note that twisting the boundary modules by automorphism (4.6) we obtain a new set of boundary modules. Twisting by automorphisms (4.7) leads to isomorphic modules.

**4.4. Root currents  $A_i(z)$  of types B, C, D.** Fix a sequence of colors  $c_1, \dots, c_{\ell+1} \in \{1, 2, 3\}$ , and consider a  $\mathcal{K}$  module defined as a tensor product of  $\ell$  Fock modules of  $\mathcal{E}$  with a boundary Fock module  $\mathcal{F}_c^{\mathbf{B}}$  or  $\mathcal{F}_c^{\mathbf{CD}}$ :

$$(4.9) \quad \mathcal{F}_{c_1}(u_1) \otimes \cdots \otimes \mathcal{F}_{c_\ell}(u_\ell) \otimes \mathcal{F}_{c_{\ell+1}}^{\mathbf{B}},$$

$$(4.10) \quad \mathcal{F}_{c_1}(u_1) \otimes \cdots \otimes \mathcal{F}_{c_\ell}(u_\ell) \otimes \mathcal{F}_{c_{\ell+1}}^{\mathbf{CD}}.$$

We say that the tensor product has

$$\begin{aligned} \text{type B} & \quad \text{for (4.9),} \\ \text{type C} & \quad \text{for (4.10) with } c_\ell \neq c_{\ell+1}, \\ \text{type D} & \quad \text{for (4.10) with } c_\ell = c_{\ell+1}. \end{aligned}$$

Let  $C_{\ell+1}$  denote the level of the boundary Fock module:

$$(4.11) \quad C_{\ell+1} = \begin{cases} s_{c_{\ell+1}}^{1/2} & \text{for type B,} \\ s_{c_{\ell+1}}^{-1} & \text{for type C, D.} \end{cases}$$

The total level is

$$(4.12) \quad C = C_{\ell+1} \prod_{i=1}^{\ell} s_{c_i}.$$

By the comodule formula (4.8) and the coproduct formulas (2.2), (2.3), current  $E(z)$  acts as a sum of vertex operators in  $\ell + 1$  bosons of the form

$$E(z) = \sum_{i=1}^{\ell} b_{c_i} \Lambda_i(z) + k_{c_{\ell+1}}^{\mathbf{B}} \Lambda_0(z) + \sum_{i=1}^{\ell} b_{c_i} \Lambda_{\bar{i}}(z) \quad \text{for type B,}$$

$$E(z) = \sum_{i=1}^{\ell} b_{c_i} \Lambda_i(z) + \sum_{i=1}^{\ell} b_{c_i} \Lambda_{\bar{i}}(z) \quad \text{for type C, D.}$$

In these formulas,

$$\Lambda_i(z) = 1 \otimes \cdots \otimes 1 \otimes V_{c_i}^{\overset{i}{\sim}}(a_i z; u_i) \otimes \psi^+(s_{c_{i+1}}^{-1} a_{i+1} z) \otimes \cdots \otimes \psi^+(s_\ell^{-1} a_\ell z) \otimes K^+(z) \quad (i = 1, \dots, \ell),$$

$$\Lambda_0(z) = 1 \otimes \cdots \otimes 1 \otimes \tilde{K}_{c_{\ell+1}}(z),$$

$$\Lambda_{\bar{i}}(z) = 1 \otimes \cdots \otimes 1 \otimes V_{c_i}^{\overset{i}{\sim}}(a_i^{-1} z; u_i) \otimes \psi^-(a_{i+1}^{-1} z)^{-1} \otimes \cdots \otimes \psi^-(a_\ell^{-1} z)^{-1} \otimes K^-(z) \quad (i = 1, \dots, \ell),$$

where  $a_i$ 's are given by

$$(4.13) \quad a_i = C_{\ell+1}^{-1} \prod_{j=i+1}^{\ell} s_{c_j}^{-1}.$$

We have the following contractions:

$$\begin{aligned}
(4.14) \quad \mathcal{C}(\Lambda_i(z), \Lambda_j(w)) &= \begin{cases} -\kappa_1 & (i \prec j, i \neq \bar{j}) \\ 0 & (i \succ j, i \neq \bar{j}) \end{cases}, \\
\mathcal{C}(\Lambda_i(z), \Lambda_i(w)) &= \begin{cases} -\frac{\kappa_1}{1-q_{c_i}} & (i \neq 0) \\ -\frac{\kappa_1}{1+s_{c_{\ell+1}}} & (i = 0) \end{cases}, \\
\mathcal{C}(\Lambda_i(z), \Lambda_{\bar{i}}(w)) &= -\kappa_1 + \frac{\kappa_1 q_{c_i}}{1-q_{c_i}} a_i^{-2} \quad (1 \leq i \leq \ell), \\
\mathcal{C}(\Lambda_{\bar{i}}(z), \Lambda_i(w)) &= \frac{\kappa_1}{1-q_{c_i}} a_i^2 \quad (1 \leq i \leq \ell).
\end{aligned}$$

Here the indices are ordered as  $1 \prec \dots \prec \ell \prec 0 \prec \bar{\ell} \prec \dots \prec \bar{1}$ , and we set  $c_{\bar{i}} = c_i$ .

As before, to each neighboring pair of Fock spaces we associate a current  $A_i(z)$ . Namely, for  $i = 1, \dots, \ell - 1$ , we define  $A_i(z)$  similarly as in (3.3),

$$(4.15) \quad A_i(z) = : \frac{\Lambda_i(a_i^{-1}z)}{\Lambda_{i+1}(a_i^{-1}z)} : = : \frac{\Lambda_{\bar{i+1}}(a_i z)}{\Lambda_{\bar{i}}(a_i z)} : .$$

The second equality is due to the identity (2.12).

In addition we define a current  $A_\ell(z)$  for each type as follows.

$$(4.16) \quad A_\ell(z) = : \frac{\Lambda_\ell(a_\ell^{-1}z)}{\Lambda_0(a_\ell^{-1}z)} : = : \frac{\Lambda_0(a_\ell z)}{\Lambda_{\bar{\ell}}(a_\ell z)} : \quad \text{for type B,}$$

$$(4.17) \quad A_\ell(z) = : \frac{\Lambda_\ell(z)}{\Lambda_{\bar{\ell}}(z)} : \quad \text{for type C,}$$

$$(4.18) \quad A_\ell(z) = : \frac{\Lambda_{\ell-1}(z)}{\Lambda_{\bar{\ell}}(z)} : = : \frac{\Lambda_\ell(z)}{\Lambda_{\bar{\ell}-1}(z)} : \quad \text{for type D.}$$

We study the contractions of the root currents  $A_i(z)$ . Denote as before  $B_{i,j} = \mathcal{C}(A_i(z), A_j(w))$ .

The contractions of  $B_{i,j}$  with  $i, j \neq \ell$  are clearly the same as in type A and are given in (3.5). The new feature is the contractions  $B_{i,\ell}, B_{\ell,i}$ .

First, we have

$$B_{i,\ell} = B_{\ell,i} = 0, \quad (i < \ell - 2).$$

Then for type B we have

$$(4.19) \quad B_{\ell-2,\ell} = B_{\ell,\ell-2} = 0, \quad B_{\ell-1,\ell} = B_{\ell,\ell-1} = \frac{t_1 t_2 t_3}{t_{c_\ell}},$$

$$(4.20) \quad B_{\ell,\ell} = -\frac{t_1 t_2 t_3}{t_{c_\ell} t_{c_{\ell+1}}} (s_{c_{\ell+1}}^{1/2} - s_{c_{\ell+1}}^{-1/2}) (s_{c_\ell} s_{c_{\ell+1}}^{1/2} + s_{c_\ell}^{-1} s_{c_{\ell+1}}^{-1/2}).$$

For type C we have

$$(4.21) \quad B_{\ell-2,\ell} = B_{\ell,\ell-2} = 0, \quad B_{\ell-1,\ell} = B_{\ell,\ell-1} = \frac{t_1 t_2 t_3}{t_{c_\ell}} (s_{c_{\ell+1}} + s_{c_{\ell+1}}^{-1}),$$

$$(4.22) \quad B_{\ell,\ell} = -\frac{t_1 t_2 t_3}{t_{c_\ell}} (s_{c_{\ell+1}} + s_{c_{\ell+1}}^{-1}) (s_{c_\ell} s_{c_{\ell+1}}^{-1} + s_{c_\ell}^{-1} s_{c_{\ell+1}}).$$

Finally, for type D we have

$$(4.23) \quad B_{\ell-2,\ell} = B_{\ell,\ell-2} = \frac{t_1 t_2 t_3}{t_{c_{\ell-1}}}, \quad B_{\ell-1,\ell} = B_{\ell,\ell-1} = \frac{t_1 t_2 t_3}{t_{c_{\ell-1}} t_{c_\ell}} (s_{c_{\ell-1}} s_{c_\ell}^{-1} - s_{c_{\ell-1}}^{-1} s_{c_\ell}),$$

$$(4.24) \quad B_{\ell,\ell} = -\frac{t_1 t_2 t_3}{t_{c_{\ell-1}} t_{c_\ell}} (s_{c_{\ell-1}} s_{c_\ell} - s_{c_{\ell-1}}^{-1} s_{c_\ell}^{-1}).$$

Note that, in particular,  $B_{ij} = B_{ji}$  in all cases.

We illustrate these contractions in the library of Appendix C. Here we consider the simplest case when all Fock spaces are of the same color to describe the connection to Cartan matrices of types B, C, D.

Let  $c_i = 2$ ,  $i = 1, \dots, \ell$ . Then the only nonzero  $B_{i,j}$  with both  $i \neq \ell$  and  $j \neq \ell$  are  $B_{i,i} = -t_1 t_3 (s_2 + s_2^{-1})$  and  $B_{i,i-1} = B_{i-1,i} = t_1 t_3$ .

Consider now the case of type B. We have  $C^2 = q_2^\ell s_{c_{\ell+1}}$ .

Let  $c_{\ell+1} = 1$ . Then  $B_{\ell,\ell-1} = B_{\ell-1,\ell} = t_1 t_3$ , and  $B_{\ell,\ell} = -t_3 (s_1^{1/2} - s_1^{-1/2})(s_2 s_1^{1/2} + s_2^{-1} s_1^{-1/2})$ . The other entries involving index  $\ell$  are zero.

Thus the matrix  $-B t_3^{-1} (s_1^{1/2} - s_1^{-1/2})^{-1}$  coincides with type B matrix  $B(q, t)$  (2.4) of [FR1] under the identification  $q = s_1^{1/2}$ ,  $t = s_2^{-1}$ . In particular, in the limit  $s_1 = s_2 = s_3 = 1$  it recovers the symmetrized Cartan matrix of type B.

Moreover, all the terms in  $E(z)$  are obtained from the first one by multiplications by  $A_i(z)^{-1}$  as in the natural representation of  $B_\ell$  shown below (we do not show the arguments of currents or constants in front of vertex operators here).

$$\Lambda_1 \xrightarrow{A_1^{-1}} \Lambda_2 \xrightarrow{A_2^{-1}} \dots \xrightarrow{A_{\ell-1}^{-1}} \Lambda_\ell \xrightarrow{A_\ell^{-1}} \Lambda_0 \xrightarrow{A_\ell^{-1}} \Lambda_{\bar{\ell}} \xrightarrow{A_{\bar{\ell}-1}^{-1}} \dots \xrightarrow{A_2^{-1}} \Lambda_{\bar{2}} \xrightarrow{A_1^{-1}} \Lambda_{\bar{1}}$$

Therefore we depict this case by the following Dynkin diagram.

$$(4.25) \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \dots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ 2 & & 2 & & & & 2 & & 2 & & 1 \end{array}$$

The case  $c_{\ell+1} = 3$  is similar.

However, the deformed Cartan matrix in the case of  $c_{\ell+1} = 2$  is essentially different; it corresponds to the case studied in [BrL], see also Section 7 of [FR1]. We have  $B_{\ell,\ell-1} = B_{\ell-1,\ell} = t_1 t_3$ , and  $B_{\ell,\ell} = -t_1 t_3 (s_2 - 1 + s_2^{-1})$ . In this case  $-B t_3^{-1} t_1^{-1}$  in the limit  $s_1 = s_2 = s_3 = 1$  is equal to symmetrized Cartan matrix of type  $\mathfrak{osp}(1, 2\ell)$ . We associate to this case the following diagram.

$$(4.26) \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \dots & \text{---} & \circ & \text{---} & \circ & \text{---} & \bullet \\ 2 & & 2 & & & & 2 & & 2 & & 2 \end{array}$$

Note that the symmetrized Cartan matrix of type  $\mathfrak{so}(2\ell + 1)$  is twice the symmetrized Cartan matrix of type  $\mathfrak{osp}(1, 2\ell)$ . Moreover, the weight diagrams of the  $\mathfrak{so}(2\ell + 1)$  modules and  $\mathfrak{osp}(1, 2\ell)$  modules are the same and the nilpotent subalgebras in both cases are the same. Therefore, for the purposes of this paper either diagram fits. However, we prefer not to use the affine node to distinguish between these two finite types which was suggested by [FR1]. Instead the affine node distinguishes between affine versions of the deformed Cartan matrices, see Section 4.5.

Let us now switch to the types C and D. We have  $C^2 = q_2^\ell q_{c_{\ell+1}}^{-1}$ .



4.5. **Root current  $A_0(z)$  of types B, C, D.** Define the dressed current  $\mathbf{E}(z)$  depending on  $\mu \in \mathbb{C}^\times$ ,  $|\mu| < 1$ ,

$$\mathbf{E}(z) = E(z)K_\mu^+(z)^{-1}, \quad K_\mu^+(z) = \prod_{s=0}^{\infty} K^+(\mu^{-s}z).$$

The Fourier coefficients of  $\mathbf{E}(z)$  are elements of the algebra  $\tilde{\mathcal{K}}$ , the algebra  $\mathcal{K}$  completed with respect to homogeneous grading in the positive direction.

Similarly to type A, our motivation for the definition of the dressing current is twofold : we would like to have integrals of motion and we also would like to have the current  $A_0(z)$  which produces a screening operator corresponding to the affine node. It turns out that the second requirement restricts  $\mu$  to specific values, in stark contrast to type A where  $\mu$  is arbitrary; see Example 4.5 and Theorems 4.7, 4.8 below. There are 6 possible choices of  $\mu$ :

$$C^2/\mu = s_{c_0}^{-1} \quad \text{or} \quad C^2/\mu = s_{c_0}^2, \quad c_0 \in \{1, 2, 3\},$$

which match the values of  $C^2$  in 6 boundary modules we have defined in Section 4.3. We have been able to find integrals of motion only for the last 3 of them (types C and D below), see Section 4.8.

We make the choice for  $\mu$  and label our choices by types B, C, D as we did for the  $\ell$ -th node. Namely, we have the following choices for the affine node:

$$\begin{aligned} \text{type B :} & \quad \mu = C^2 s_{c_0}, \\ \text{type C :} & \quad \mu = C^2 s_{c_0}^{-2}, \quad c_0 \neq c_1, \\ \text{type D :} & \quad \mu = C^2 s_{c_0}^{-2}, \quad c_0 = c_1, \end{aligned}$$

$C$  being the level given by (4.12). We stress that the choice of  $c_0$  is independent of other  $c_i$ , that is, it is independent of the choice of the  $\mathcal{K}$  representation. The type of the zeroth node is not to be confused with the type of tensor product modules (4.9), (4.10).

To that we associate the affine Dynkin diagram, where the affine node has the prescribed type and the color is given the same way as for the  $\ell$ -th node, see C.1. Namely, we depict the affine node as follows.

|                     |            |                                 |
|---------------------|------------|---------------------------------|
| B, $c_0 = c_1$ :    | ●<br>$c_0$ |                                 |
| B, $c_0 \neq c_1$ : | ○<br>$b$   | $\{c_0, c_1, b\} = \{1, 2, 3\}$ |
| C, $c_0 \neq c_1$ : | ○<br>$b$   | $\{c_0, c_1, b\} = \{1, 2, 3\}$ |
| D, $c_0 = c_2$ :    | ○<br>$c_0$ |                                 |
| D, $c_0 \neq c_2$ : | ⊗<br>$b$   | $\{c_0, c_2, b\} = \{1, 2, 3\}$ |

Current  $\mathbf{E}(z)$  has the form

$$\begin{aligned} \mathbf{E}(z) &= \sum_{i=1}^{\ell} b_{c_i} \mathbf{\Lambda}_i(z) + k_{c_{\ell+1}}^{\mathbf{B}} \mathbf{\Lambda}_0(z) + \sum_{i=1}^{\ell} b_{c_i} \mathbf{\Lambda}_{\bar{i}}(z) && \text{for type B,} \\ \mathbf{E}(z) &= \sum_{i=1}^{\ell} b_{c_i} \mathbf{\Lambda}_i(z) + \sum_{i=1}^{\ell} b_{c_i} \mathbf{\Lambda}_{\bar{i}}(z) && \text{for types C, D,} \end{aligned}$$

where  $\mathbf{\Lambda}_i(z) = \Lambda_i(z) \Delta^{(\ell)} K_{\mu}^+(z)^{-1}$ . For  $\mathbf{\Lambda}_i(z)$  the contraction rule (3.10) applies.

Define the current  $A_0(z)$  of each type as follows.

$$(4.29) \quad : A_0(s_{c_0}^{-1/2} z) A_0(s_{c_0}^{1/2} z) : = : \frac{\mathbf{\Lambda}_{\bar{1}}(\mu^{-1/2} z)}{\mathbf{\Lambda}_1(\mu^{1/2} z)} : \quad \text{for type B,}$$

$$(4.30) \quad A_0(z) = : \frac{\mathbf{\Lambda}_{\bar{1}}(\mu^{-1/2} z)}{\mathbf{\Lambda}_1(\mu^{1/2} z)} : \quad \text{for type C,}$$

$$(4.31) \quad A_0(z) = : \frac{\mathbf{\Lambda}_{\bar{2}}(\mu^{-1/2} z)}{\mathbf{\Lambda}_1(\mu^{1/2} z)} : = : \frac{\mathbf{\Lambda}_{\bar{1}}(\mu^{-1/2} z)}{\mathbf{\Lambda}_2(\mu^{1/2} z)} : \quad \text{for type D.}$$

These definitions are to be compared with (4.16)–(4.18). Note that, for type B, (4.16) implies

$$: A_{\ell}(s_{c_{\ell+1}}^{-1/2} z) A_{\ell}(s_{c_{\ell+1}}^{1/2} z) : = : \frac{\Lambda_{\ell}(z)}{\Lambda_{\bar{\ell}}(z)} : .$$

The corresponding contractions  $B_{i,j}$  are given by the same rule as (4.19)–(4.23) if we interchange  $A_i(z)$  with  $A_{\ell-i}(z)$  and  $c_j$  with  $c_{\ell+1-j}$ .

Namely we have

$$(4.32) \quad B_{0,1} = B_{1,0} = \frac{t_1 t_2 t_3}{t_{c_1}}, \quad B_{0,0} = -\frac{t_1 t_2 t_3}{t_{c_0} t_{c_1}} (s_{c_0}^{1/2} - s_{c_0}^{-1/2}) (s_{c_0}^{1/2} s_{c_1} + s_{c_0}^{-1/2} s_{c_1}^{-1}) \quad \text{for type B;}$$

$$(4.33) \quad B_{0,1} = B_{1,0} = \frac{t_1 t_2 t_3}{t_{c_1}} (s_{c_0} + s_{c_0}^{-1}), \quad B_{0,0} = -\frac{t_1 t_2 t_3}{t_{c_1}} (s_{c_0} + s_{c_0}^{-1}) (s_{c_0} s_{c_1}^{-1} + s_{c_0}^{-1} s_{c_1}) \quad \text{for type C;}$$

$$(4.34) \quad B_{0,2} = B_{2,0} = \frac{t_1 t_2 t_3}{t_{c_2}}, \quad B_{0,1} = B_{1,0} = \frac{t_1 t_2 t_3}{t_{c_1} t_{c_2}} (s_{c_1}^{-1} s_{c_2} - s_{c_1} s_{c_2}^{-1}),$$

$$B_{0,0} = -\frac{t_1 t_2 t_3}{t_{c_1} t_{c_2}} (s_{c_1} s_{c_2} - s_{c_1}^{-1} s_{c_2}^{-1}) \quad \text{for type D.}$$

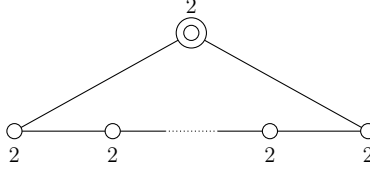
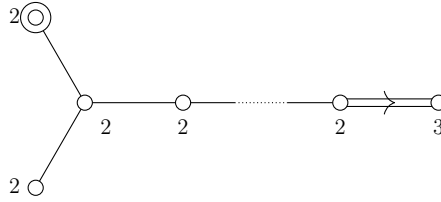
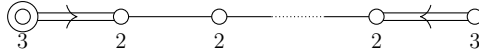
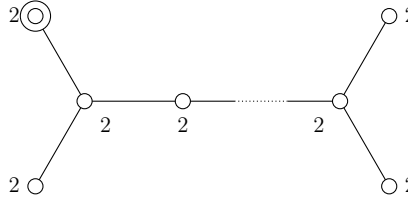
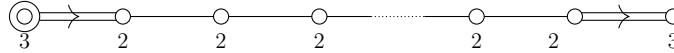
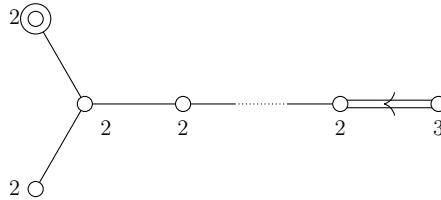
For  $\ell > 3$ , all other  $B_{i,j}$  involving index 0 are zero. The full list of  $B_{i,j}$  for small values of  $\ell$  is given in Appendix C.

We think of the matrix  $\hat{B} = (B_{i,j})_{i,j=0}^{\ell}$  as the affinization of matrix  $B$ . In case all colors except 0 and  $\ell$  are equal, e.g.  $c_1 = \dots = c_{\ell} = 2$ , the corresponding Dynkin diagrams are those of non-exceptional affine type, where the colors  $c_0$  and  $c_{\ell}$  determine the ends of the diagram, see Table 1 below. For comparison we include type A in the table.

| type                | Fock space  | $C^2$                | $C^2/\mu$    | $\det \hat{C}$  |
|---------------------|---|----------------------|--------------|---|
| $A_\ell^{(1)}$      | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_2$      | $q_2^{\ell+1}$       | arbitrary    | $(1 - \mu)(s_2^{-\ell-1} - \mu^{-1}s_2^{\ell+1})$                     |
| $B_\ell^{(1)}$      | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_3^B$    | $q_2^\ell q_3^{1/2}$ | $q_2$        | $(s_2^2 - s_2^{-2})(s_2^{\ell-1}s_3^{1/2} - s_2^{-\ell+1}s_3^{-1/2})$ |
| $C_\ell^{(1)}$      | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_3^{CD}$ | $q_2^\ell q_3^{-1}$  | $q_3$        | $(s_2 - s_2^{-1})(s_2^\ell s_3^{-2} - s_2^{-\ell} s_3^2)$             |
| $D_\ell^{(1)}$      | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_2^{CD}$ | $q_2^{\ell-1}$       | $q_2$        | $(s_2 + s_2^{-1})(s_2^2 - s_2^{-2})(s_2^{\ell-2} - s_2^{-\ell+2})$    |
| $A_{2\ell}^{(2)}$   | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_3^B$    | $q_2^\ell q_3^{1/2}$ | $q_3$        | $(s_2 - s_2^{-1})(s_2^\ell s_3^{-1/2} - s_2^{-\ell} s_3^{1/2})$       |
| $A_{2\ell-1}^{(2)}$ | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_3^{CD}$ | $q_2^\ell q_3^{-1}$  | $q_2$        | $(s_2^2 - s_2^{-2})(s_2^{\ell-1}s_3^{-1} - s_2^{-\ell+1}s_3)$         |
| $D_{\ell+1}^{(2)}$  | $\mathcal{F}_2^{\otimes \ell} \otimes \mathcal{F}_3^B$    | $q_2^\ell q_3^{1/2}$ | $q_3^{-1/2}$ | $(s_2 - s_2^{-1})(s_2^\ell s_3 - s_2^{-\ell} s_3^{-1})$               |

TABLE 1.

We draw the corresponding Dynkin diagrams, where labels are as in the finite case and the double circle denotes the affine node which corresponds to the dressing rather than a pair of modules.

$A_\ell^{(1)}$  $B_\ell^{(1)}$  $C_\ell^{(1)}$  $D_\ell^{(1)}$  $A_{2\ell}^{(2)}$  $A_{2\ell-1}^{(2)}$  $D_{\ell+1}^{(2)}$ 

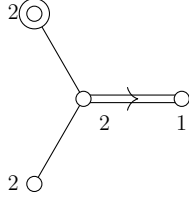
4.6. **The  $qq$ -characters.** As in type A we write  $\hat{B} = \hat{D}\hat{C}$ , choosing  $\hat{D} = \text{diag}(d_0, \dots, d_\ell)$  in such a way that  $C_{i,i}$  becomes one of the following, see Appendix C:

$$s_c + s_c^{-1}, \quad s_b s_c^{-1} + s_b^{-1} s_c, \quad s_b s_c^{1/2} + s_b^{-1/2} s_c, \quad s_c^{3/2} + s_c^{-3/2}, \quad s_c - s_c^{-1}.$$

We then introduce the currents  $Y_i(z)$  following the rule (3.15). Then the current  $\mathbf{E}(z)$  follows the  $qq$ -character.

We give the following example illustrating various phenomena.

**Example 4.5.** Consider the following affine Dynkin diagram.



We label the nodes in the standard way: double circled affine node is the zeroth node, the middle point is node 2 and the shorter node is node 3.

This diagram corresponds to the  $\mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_1^{\mathbf{B}}$  of level  $C = s_2^3 s_1^{1/2}$  with the dressing parameter  $\mu = C^2 s_2^{-2} = s_1 s_2^4$ .

We have the following deformed Cartan matrix (see Appendix C).

$$\hat{C} = \begin{pmatrix} s_2 + s_2^{-1} & 0 & -1 & 0 \\ 0 & s_2 + s_2^{-1} & -1 & 0 \\ -1 & -1 & s_2 + s_2^{-1} & -1 \\ 0 & 0 & -s_1^{1/2} - s_1^{-1/2} & s_1^{1/2} s_2 + s_1^{-1/2} s_2^{-1} \end{pmatrix}.$$

We use notation  $Y_l(s_1^{a/2} s_2^{b/2} z) = \mathbf{l}_{a,b}$ ,  $l = 0, 1, 2, 3$ . Then we have

$$\begin{aligned} A_0(z) &= \mathbf{0}_{0,-2} \mathbf{0}_{0,2} \mathbf{2}_{0,0}^{-1}, & A_1(z) &= \mathbf{1}_{0,-2} \mathbf{1}_{0,2} \mathbf{2}_{0,0}^{-1}, \\ A_2(z) &= \mathbf{0}_{0,0}^{-1} \mathbf{1}_{0,0}^{-1} \mathbf{2}_{0,-2} \mathbf{2}_{0,2} \mathbf{3}_{1,0}^{-1} \mathbf{3}_{-1,0}^{-1}, & A_3(z) &= \mathbf{2}_{0,0}^{-1} \mathbf{3}_{1,2} \mathbf{3}_{-1,-2}. \end{aligned}$$

Then the dressed current  $\mathbf{E}(z)$  takes the form (ignoring the constants and an overall shift) of the  $qq$ -character with 7 monomials:

$$\chi = \mathbf{0}_{0,4}^{-1} \mathbf{1}_{0,0} + \mathbf{0}_{0,4}^{-1} \mathbf{1}_{0,4}^{-1} \mathbf{2}_{0,2} + \mathbf{2}_{0,6}^{-1} \mathbf{3}_{-1,4} \mathbf{3}_{1,4} + \mathbf{3}_{1,4} \mathbf{3}_{1,8}^{-1} + \mathbf{2}_{2,6} \mathbf{3}_{3,8}^{-1} \mathbf{3}_{1,8}^{-1} + \mathbf{0}_{2,8} \mathbf{1}_{2,8} \mathbf{2}_{2,10}^{-1} + \mathbf{0}_{2,8} \mathbf{1}_{2,12}^{-1}.$$

Note the following features of this  $qq$ -character. We follow the terminology of Section 3.3.

Let  $\mathbb{C}P$  be the group ring of the weight lattice generated by fundamental weights  $\omega_l$ ,  $l = 1, \dots, \ell$ . Let  $\rho : \mathcal{A} \rightarrow \mathbb{C}P$  be the ring homomorphism sending  $\mathbf{l}_{a,b} \mapsto \omega_l$ ,  $l = 1, \dots, \ell$ ,  $\mathbf{0}_{a,b} \mapsto 1$ . Then  $\rho(\chi)$  is the character of vector representation of  $\mathbf{B}_3$ .

The initial monomial has  $\mathbf{1}_{0,0}$ , the final monomial has  $\mathbf{1}_{2,12}^{-1}$ . The shift  $s_1 s_2^6$  is  $C^2$ .

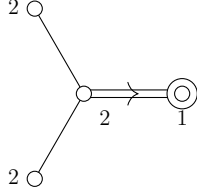
The ratios between neighboring monomials are  $A_1(s_2 z)$ ,  $A_2(s_2^2 z)$ ,  $A_3(s_2^3 z)$ ,  $A_3(s_1 s_2^3 z)$ ,  $A_2(s_1 s_2^4 z)$ , and  $A_1(s_1 s_2^5 z)$  respectively. The first and the 6th monomial are  $\mathbf{1}$ -dominant, the second and the 5th are  $\mathbf{2}$ -dominant, all these “generate”  $q$ -characters of 2 dimensional  $U_q \widehat{\mathfrak{sl}}_2$ -modules. The third monomial is  $\mathbf{3}$ -dominant and “generates” the  $q$ -character of a 3 dimensional  $U_q \widehat{\mathfrak{sl}}_2$ -module. This is the reason  $\mathbf{E}(z)$  commutes with screening operators  $S_i$ ,  $i = 1, \dots, \ell$ , see Section 4.7.

The 6th and the 7th monomial are  $\mathbf{0}$ -dominant. They “generate” monomials which are  $\mu$ -shifts of the first and the second monomial:

$$\begin{aligned} A_0^{-1}(s_1 s_2^5) \mathbf{0}_{2,8} \mathbf{1}_{2,8} \mathbf{2}_{2,10}^{-1} &= \mathbf{0}_{2,12}^{-1} \mathbf{1}_{2,8} = \tau_\mu(\mathbf{0}_{0,4}^{-1} \mathbf{1}_{0,0}), \\ A_0^{-1}(s_1 s_2^5) \mathbf{0}_{2,8} \mathbf{1}_{2,12}^{-1} &= \mathbf{0}_{2,12}^{-1} \mathbf{1}_{2,12}^{-1} \mathbf{2}_{2,10} = \tau_\mu(\mathbf{0}_{0,4}^{-1} \mathbf{1}_{0,4}^{-1} \mathbf{2}_{0,2}). \end{aligned}$$

This is the reason for our choice of  $\mu$  and why the screening  $S_0$  commutes with the integral of  $\mathbf{E}(z)$ .

Now we consider the same Dynkin diagram and the same deformed Cartan matrix but treat node 3 as affine:



Then we consider the  $D_3$  current corresponding to  $\mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_2 \otimes \mathcal{F}_2^{CD}$  with the dressing of type B. The new level  $C^2 = s_2^4$  is different but  $\mu = C^2 s_1 = s_2^4 s_1$  is the same.

Then the  $\mathbf{E}(z)$  current corresponds to the following  $qq$ -character with 6 summands:

$$\bar{\chi} = \mathbf{2}_{0,0} \mathbf{3}_{-1,2}^{-1} \mathbf{3}_{1,2}^{-1} + \mathbf{2}_{0,4}^{-1} \mathbf{1}_{0,2} \mathbf{0}_{0,2} + \mathbf{1}_{0,6}^{-1} \mathbf{0}_{0,2} + \mathbf{1}_{0,2} \mathbf{0}_{0,6}^{-1} + \mathbf{2}_{0,4} \mathbf{1}_{0,6}^{-1} \mathbf{0}_{0,6}^{-1} + \mathbf{2}_{0,8}^{-1} \mathbf{3}_{1,6} \mathbf{3}_{-1,6}.$$

Note also that  $D_3 = A_3$ . Here we consider the deformed  $\mathcal{W}$  current corresponding to the second fundamental module.

The first and the fifth monomials are 2-dominant, the second and the sixth are 2-antidominant. Moreover the ratio of the first to the second is  $A_2(s_2 z)$  and the fifth to the sixth is  $A_2(s_2^3 z)$ . Similarly, in direction 1 the ratio of the second to the third is  $A_1(s_2)$  and the fourth to the fifth is also  $A_1(s_2^2 z)$ . Finally, in the direction 0, the ratio of the second to the fourth equals to the ratio of the third to the fifth and equal to  $A_0(s_2^2 z)$ . It means that  $\bar{\chi}$  is “closed” in the directions 0, 1, and 2 (and that  $\bar{\chi}$  commutes with screenings  $S_0, S_1, S_2$ ).

In the direction of 3, we do not have such a property. However, one can add a monomial and consider

$$(4.35) \quad \tilde{\chi} = \bar{\chi} + \mathbf{3}_{1,6} \mathbf{3}_{1,10}^{-1}.$$

With respect to the  $D_3$  algebra it correspond to adding a trivial 1 dimensional representation (in particular, the new current still commutes with screenings  $S_0, S_1, S_2$ ).

On the other hand, the last monomial  $\mathbf{2}_{0,8}^{-1} \mathbf{3}_{1,5} \mathbf{3}_{1,7}$ , the new monomial  $\mathbf{3}_{1,6} \mathbf{3}_{3,10}^{-1}$  and the shifted monomial  $\tau_{\bar{\mu}}(\mathbf{2}_{0,0} \mathbf{3}_{-1,2}^{-1} \mathbf{3}_{1,2}^{-1})$  together correspond to the 3-dimensional evaluation  $U_q \widehat{\mathfrak{sl}}_2$  module in the third direction (in particular, the new current also commutes with the screening  $S_3$ , see Theorem 4.8).

Finally we note that the two currents corresponding to  $qq$ -characters  $\chi$  and  $\bar{\chi}$  are closely related. Namely each monomial of  $\chi$  is a shift of a monomial in  $\tilde{\chi}$ . Namely, if we apply  $\tau_{\bar{\mu}}$  to the last three monomials of  $\chi$  we obtain  $\tau_{s_2}(\tilde{\chi})$ . In particular, the integrals (constant terms) of  $\tilde{\chi}$  and  $\chi$  coincide.  $\square$

**Remark 4.6.** The phenomena described in the example is rather general. Namely, if one can choose the affine node in several different ways then the corresponding deformed  $\mathcal{W}$  currents are all obtained from each other by shifting some of the monomials. In particular, the integrals (constant terms) of all these currents coincide.

In this paper we study commutative families of operators which include the integral of a deformed  $\mathcal{W}$  current. Thus we expect that the families for different choices of the affine node all commute.

While it seems to be difficult to check directly, this fact would follow if the integral of the deformed  $\mathcal{W}$  current had simple spectrum for generic evaluation parameters  $u_i$ . We expect this is the case and we intend to return to this question when we study the spectrum of IM.

The family of commuting operators correspond to the choices of the affine node of type other than  $B$ . Such a choice exists in all cases except when both zeroth and  $\ell$ -th nodes are of  $B$  type (e.g. in type  $D_{\ell+1}^{(2)}$ ).  $\square$

**4.7. Screenings.** The screening operators are defined in the same way as for type  $A$ , see (3.18), (3.19).

We call the matrix  $\hat{B}$  stable if  $B_{0,\ell} = B_{\ell,0} = 0$ . Most of the deformed affine Cartan matrices of types  $B, C, D$  are stable except for a few low rank cases, see Appendix C.

In what follows we often assume  $\hat{B}$  is stable to simplify the considerations. We expect that in all such cases this assumption can be dropped.

Let  $\hat{B}$  be stable. For  $i = 1, \dots, \ell - 1$ , the screenings are defined by (3.18), (3.19). The screening currents for the  $\ell$ -th node are given as follows.

For type  $B$ ,

$$\begin{aligned} c_\ell = c_{\ell+1} : \quad A_\ell(z) &= s_{c_\ell}^2 s_c : \frac{S_\ell^+(s_b^{-1}z)}{S_\ell^+(s_b z)} : && \left( (c_\ell, b, c) = \text{cycl}(1, 2, 3) \right), \\ c_\ell \neq c_{\ell+1} : \quad A_\ell(z) &= s_{c_\ell}^2 s_{c_{\ell+1}} : \frac{S_\ell^+(s_b^{-1}z)}{S_\ell^+(s_b z)} : = s_{c_\ell}^2 s_{c_{\ell+1}} : \frac{S_\ell^-(s_{c_{\ell+1}}^{-1/2}z)}{S_\ell^-(s_{c_{\ell+1}}^{1/2}z)} : && \left( \{c_\ell, c_{\ell+1}, b\} = \{1, 2, 3\} \right). \end{aligned}$$

For type  $C$ ,

$$c_\ell \neq c_{\ell+1} : \quad A_\ell(z) = q_{c_\ell} q_{c_{\ell+1}}^{-1} : \frac{S_\ell^+(s_b^{-1}z)}{S_\ell^+(s_b z)} : = q_{c_\ell} q_{c_{\ell+1}}^{-1} : \frac{S_\ell^-(s_{c_{\ell+1}}^{-2}z)}{S_\ell^-(s_{c_{\ell+1}}^2z)} : \quad \left( \{c_\ell, c_{\ell+1}, b\} = \{1, 2, 3\} \right).$$

For type  $D$ ,

$$\begin{aligned} c_{\ell-1} = c_\ell = c_{\ell+1} : \quad A_\ell(z) &= q_{c_\ell} : \frac{S_\ell^+(s_b^{-1}z)}{S_\ell^+(s_b z)} : = q_{c_\ell} : \frac{S_\ell^-(s_c^{-1}z)}{S_\ell^-(s_c z)} : && \left( (c_\ell, b, c) = \text{cycl}(1, 2, 3) \right), \\ c_{\ell-1} \neq c_\ell = c_{\ell+1} : \quad A_\ell(z) &= s_b^{-1} : \frac{S_\ell^f(s_b^{-1}z)}{S_\ell^f(s_b z)} : && \left( \{c_{\ell-1}, c_\ell, b\} = \{1, 2, 3\} \right). \end{aligned}$$

The zeroth screening currents  $S_0^\pm(z)$  are defined in the same way (changing indices  $\ell, \ell - 1, \ell - 2$  to  $0, 1, 2$  respectively).

Then we have the following theorem.

**Theorem 4.7.** *Assume  $\hat{B}$  is stable. The screening operators  $S_i$  with  $i \neq 0$  commute with both  $E(z)$  and  $\mathbf{E}(z)$ :*

$$[S_i, E(z)] = [S_i, \mathbf{E}(z)] = 0, \quad i = 1, \dots, \ell.$$

For the zeroth screening we have

$$\begin{aligned} [S_0, \mathbf{E}(z)] &= [S_0, b_{c_1}(\Lambda_1(z) - \Lambda_1(\mu z))] && \text{for type } C, \\ [S_0, \mathbf{E}(z)] &= [S_0, b_{c_1}(\Lambda_1(z) - \Lambda_1(\mu z)) + b_{c_2}(\Lambda_2(z) - \Lambda_2(\mu z))] && \text{for type } D. \end{aligned}$$

*Proof.* For  $i = 1, \dots, \ell - 1$  this is a type A computation, so it reduces to Lemma 2.1.

For type C, one can check that the triples  $(A_\ell(z), \Lambda_\ell(z), \Lambda_{\bar{\ell}}(z))$  and  $(A_0(z), \Lambda_{\bar{1}}(z), \Lambda_1(\mu z))$  satisfy the conditions of Lemma 2.1.

Likewise, for type D, one checks that the triples  $(A_\ell(z), \Lambda_{\ell-1}(z), \Lambda_{\bar{\ell}}(z))$ ,  $(A_\ell(z), \Lambda_\ell(z), \Lambda_{\bar{\ell}-1}(z))$ ,  $(A_0(z), \Lambda_{\bar{2}}(z), \Lambda_1(\mu z))$ ,  $(A_0(z), \Lambda_{\bar{1}}(z), \Lambda_2(\mu z))$  satisfy the conditions of Lemma 2.1.

The remaining case is type B with  $i = \ell$ . The relevant contractions are

$$\begin{aligned} \mathcal{C}(A_\ell(z), \Lambda_\ell(w)) &= -\frac{t_1 t_2 t_3}{t_{c_\ell}} s_{c_\ell}^{-1} s_{c_{\ell+1}}^{-1/2}, & \mathcal{C}(\Lambda_\ell(z), A_\ell(w)) &= -\frac{t_1 t_2 t_3}{t_{c_\ell}} s_{c_\ell} s_{c_{\ell+1}}^{1/2}, \\ \mathcal{C}(A_\ell(z), \Lambda_0(w)) &= -\mathcal{C}(\Lambda_0(z), A_\ell(w)) = \frac{t_1 t_2 t_3}{s_{c_{\ell+1}}^{1/2} + s_{c_{\ell+1}}^{-1/2}}, \\ \mathcal{C}(A_\ell(z), \Lambda_{\bar{\ell}}(w)) &= \frac{t_1 t_2 t_3}{t_{c_\ell}} s_{c_\ell} s_{c_{\ell+1}}^{1/2}, & \mathcal{C}(\Lambda_{\bar{\ell}}(z), A_\ell(w)) &= \frac{t_1 t_2 t_3}{t_{c_\ell}} s_{c_\ell}^{-1} s_{c_{\ell+1}}^{-1/2}. \end{aligned}$$

Suppose  $c_\ell = c_{\ell+1}$ . We take  $c_\ell = c_{\ell+1} = 2$  for concreteness. From the above contractions we compute the singular parts of the operator products,

$$\begin{aligned} S_\ell^+(z) \Lambda_\ell(w) &= -\frac{t_1 s_2^{-3/2} w}{z - s_3 s_2^{-1/2} w} : S_\ell^+(s_3 s_2^{-1/2} w) \Lambda_\ell(w) : + O(1), \\ S_\ell^+(z) \Lambda_\ell(w) &= k \left( \frac{s_3^{-1} s_2^{-1/2} w}{z - s_3^{-1} s_2^{-1/2} w} : S_\ell^+(s_3^{-1} s_2^{-1/2} w) \Lambda_0(w) : - \frac{s_3 s_2^{1/2} w}{z - s_3 s_2^{1/2} w} : S_\ell^+(s_3 s_2^{1/2} w) \Lambda_0(w) : \right) + O(1), \\ S_\ell^+(z) \Lambda_{\bar{\ell}}(w) &= \frac{t_1 s_2^{3/2} w}{z - s_3^{-1} s_2^{1/2} w} : S_\ell^+(s_3^{-1} s_2^{1/2} w) \Lambda_{\bar{\ell}}(w) : + O(1), \end{aligned}$$

where  $k = t_1 (s_2^{1/2} - s_2^{-1/2}) / (s_1 s_2^{1/2} - s_1^{-1} s_2^{-1/2}) = t_1 b_2 / k_2^B$ . In view of the relations

$$\begin{aligned} : S_\ell^+(s_3 z) \Lambda_\ell(s_2^{1/2} z) : &= s_3^{-1} s_2 : S_\ell^+(s_3^{-1} z) \Lambda_0(s_2^{1/2} z) :, \\ : S_\ell^+(s_3^{-1} z) \Lambda_{\bar{\ell}}(s_2^{-1/2} z) : &= s_3 s_2^{-1} : S_\ell^+(s_3 z) \Lambda_0(s_2^{-1/2} z) :, \end{aligned}$$

we conclude that

$$[S_\ell^+, b_2 \Lambda_\ell(w) + k_2^B \Lambda_0(w) + b_2 \Lambda_{\bar{\ell}}(w)] = 0.$$

Note that we can define another screening current  $S_\ell^-(z)$  by interchanging  $q_1 \leftrightarrow q_3$  in  $S_\ell^+(z)$ . Note also that  $k_2^B = (1 + s_2)(s_3 - s_1) / \kappa_1$  changes sign under the swap  $q_1 \leftrightarrow q_3$ . So, the screening operator  $S_\ell^-$  commutes with a different current (which is also a representation of  $\mathcal{K}$ , see (4.6))

$$[S_\ell^-, b_2 \Lambda_\ell(w) - k_2^B \Lambda_0(w) + b_2 \Lambda_{\bar{\ell}}(w)] = 0.$$

That the present case admits only one screening has been observed in [FR1], see the end of Section 7 thereof.

The calculation for  $c_\ell \neq c_{\ell+1}$  is entirely similar, cf. [FR1], Theorem 3. Alternatively one can write the relevant three terms as a ‘‘fusion’’ and reduce the calculation to two-dimensional representation of  $\mathfrak{sl}_2$ , see the last line of the proof of Proposition B.1 below.  $\square$

For type **B**, the current  $\mathbf{E}(z)$  does not “close up” under  $S_0$ . However, definition (4.29) suggests that we introduce an operator

$$\mathbf{\Lambda}_{\bar{0}}(z) = : \mathbf{\Lambda}_{\bar{1}}(z) A_0^{-1}(\mu^{1/2} s_{c_0}^{-1/2} z) : = : \mathbf{\Lambda}_1(\mu z) A_0(\mu^{1/2} s_{c_0}^{1/2} z) :$$

and consider an extended current

$$(4.36) \quad \tilde{\mathbf{E}}(z) = \mathbf{E}(z) + k_{c_0}^{\mathbf{B}} \mathbf{\Lambda}_{\bar{0}}(z).$$

One can think that this current corresponds to a direct sum of the vector representation and the trivial representation, see (4.35).

**Theorem 4.8.** *Assume  $\hat{B}$  is stable. The screening operators  $S_i$  with  $i \neq 0$  commute with  $\tilde{\mathbf{E}}(z)$ :*

$$[S_i, \tilde{\mathbf{E}}(z)] = 0, \quad i = 1, \dots, \ell.$$

For the zeroth screening we have

$$[S_0, \tilde{\mathbf{E}}(z)] = b_{c_1} [S_0, \mathbf{\Lambda}_1(z) - \mathbf{\Lambda}_1(\mu z)].$$

*Proof.* The proof is parallel to that of Theorem 4.7. □

**4.8. Integrals of motion associated with  $\mathcal{K}$ .** In this subsection we show that algebra  $\mathcal{K}$  possesses a family of integrals of motion when  $C^2 = \mu q_{c_0}$ , namely if the zeroth node is of type **C** or **D**.

First, we prepare a lemma about matrix elements of products of  $\mathbf{E}(z)$ . Set

$$f(x) = \frac{(1 - C^2 x)(1 - C^{-2} x)}{(1 - x)^5} \times \prod_{s=1}^3 \frac{(q_s^{-1} x; \mu)_{\infty}}{(\mu q_s x; \mu)_{\infty}}.$$

Using the defining relations, we find

$$f(w/z) \mathbf{E}(z) \mathbf{E}(w) = f(z/w) \mathbf{E}(w) \mathbf{E}(z).$$

With the definition

$$(4.37) \quad \mathbf{E}(z_1, \dots, z_n) = \prod_{i < j} f(z_j/z_i) \times \mathbf{E}(z_1) \cdots \mathbf{E}(z_n),$$

the matrix elements of (4.37) have the form  $p(z_1, \dots, z_n) / \prod_{i < j} (z_i - z_j)^4$ , where  $p$  is a symmetric Laurent polynomial. Furthermore, the Serre relations entail the following zero conditions.

**Lemma 4.9.** *We have*

$$(4.38) \quad \mathbf{E}(z, q_{c_1} z, q_{c_1} q_{c_2} z) = 0 \quad (c_1 \neq c_2, \quad c_1, c_2 \in \{1, 2, 3\}),$$

$$(4.39) \quad \mathbf{E}(z, C^{\pm 2} z, C^{\pm 2} q_c^{\pm 1} z) = 0 \quad (c \in \{1, 2, 3\}),$$

$$(4.40) \quad \mathbf{E}(C^{-2} z, z, C^2 z) = 0.$$

*Proof.* We can write

$$\mathbf{E}(z_1, \dots, z_n) = E(z_1, \dots, z_n) \prod_{i=1}^n K_{\mu}^{+}(z_i)^{-1},$$

where

$$E(z_1, \dots, z_n) = \prod_{i < j} f_0(z_j/z_i) \times E(z_1) \cdots E(z_n),$$

$$f_0(x) = \frac{1 - C^2 x}{1 - x} \frac{1 - C^{-2} x}{1 - x} \prod_{s=1}^3 \frac{1 - q_s x}{1 - x}.$$

Since  $K_\mu^+(z)^{-1}$  on any vector is a Laurent polynomial in  $z^{-1}$ , it is enough to prove the statements for  $E(z_1, \dots, z_n)$ .

Consider the Serre relation (4.5).

The left hand side has the form

$$LHS = Sym_{z_1, z_2, z_3} \left[ \left( \frac{z_2}{z_3} - \frac{z_3}{z_2} - \frac{z_1}{z_2} + \frac{z_2}{z_1} \right) E(z_1) E(z_2) E(z_3) \right],$$

while the right hand side is

$$RHS = Sym_{z_1, z_2, z_3} \left[ X(z_1, z_2, z_3) \kappa_1^{-1} \delta(C^2 z_1/z_3) K^-(z_1) E(z_2) K^+(z_3) \right].$$

Let us consider the left hand side. Each term  $E(z_i) E(z_j) E(z_k)$  is defined in the region  $|z_i| \gg |z_j| \gg |z_k|$ . It can be rewritten as

$$E(z_i) E(z_j) E(z_k) = E(z_1, z_2, z_3) \times f_0(z_j/z_i)^{-1} f_0(z_k/z_i)^{-1} f_0(z_k/z_j)^{-1},$$

where all matrix elements of  $E(z_1, z_2, z_3)$  are symmetric Laurent polynomials multiplied by  $\prod_{i < j} (z_i - z_j)^{-4}$ . To find the zero conditions of  $E(z_1, z_2, z_3)$ , we bring all terms in both sides into expansions in the common domain  $|z_1| \gg |z_2| \gg |z_3|$  with additional delta functions. For example, the term  $E(z_2) E(z_1) E(z_3)$  gives rise to five delta functions  $\delta(q_c z_2/z_1)$  ( $c = 1, 2, 3$ ) and  $\delta(C^{\pm 2} z_2/z_1)$ .

We observe that the terms without delta functions cancel out, due to the identity of rational functions

$$\text{Skew}_{z_1, z_2, z_3} \left( \frac{z_2}{z_3} - \frac{z_3}{z_2} - \frac{z_1}{z_2} + \frac{z_2}{z_1} \right) \frac{1}{g_{12} g_{13} g_{23}} = 0,$$

where  $g_{ij} = g(z_i, z_j)$ .

The terms with one delta function also cancel out.

For example, the coefficient of  $\delta(q_1 z_1/z_2)$  comes from 3 terms and cancel out:

$$\left[ \left( \frac{z_1}{z_3} - \frac{z_3}{z_1} - \frac{z_2}{z_1} + \frac{z_1}{z_2} \right) \frac{1}{g_{13} g_{23}} + \left( \frac{z_3}{z_1} - \frac{z_1}{z_3} - \frac{z_2}{z_3} + \frac{z_3}{z_2} \right) \frac{1}{g_{23} g_{31}} + \left( \frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2} + \frac{z_2}{z_3} \right) \frac{1}{g_{31} g_{32}} \right] \Big|_{z_2=q_1 z_1} = 0.$$

Similarly, the coefficient of  $\delta(C^2 z_1/z_3)$  comes from 3 terms. In the left hand side, we collect all contributions to  $\kappa_1^{-1} \delta(C^2 z_1/z_3) K^-(z_1) E(z_2) K^+(z_3)$  and find

$$\begin{aligned} & - \left( \frac{z_3}{z_2} - \frac{z_2}{z_3} - \frac{z_1}{z_3} + \frac{z_3}{z_1} \right) \frac{g_{13} g_{12}}{g_{31} g_{21}} - \left( \frac{z_1}{z_2} - \frac{z_2}{z_1} - \frac{z_3}{z_1} + \frac{z_1}{z_3} \right) \frac{g_{13} g_{23}}{g_{31} g_{32}} + \left( \frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2} + \frac{z_2}{z_3} \right) \frac{g_{12} g_{13} g_{23}}{g_{21} g_{31} g_{32}} \\ & = \kappa_1 \left( \frac{z_2^2}{z_1 z_3} - 1 \right) \frac{z_2(z_2 + z_1)(z_2 + z_3)}{g_{21} g_{32}} g_{13}, \end{aligned}$$

which coincides with the corresponding term in the right hand side.

Finally, we consider terms with two delta functions.

Consider the term  $\delta(q_{c_1}z_1/z_2)\delta(q_{c_2}z_2/z_3)$  ( $c_1 \neq c_2$ ). It comes only from the last term on the left hand side with non-zero coefficient,

$$\left(\frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2} + \frac{z_2}{z_3}\right) \frac{1}{g_{31}} \Big|_{z_2=q_{c_1}z_1, z_3=q_{c_2}z_2} \neq 0,$$

which implies the zero condition (4.38).

Consider  $\delta(q_i z_1/z_2)\delta(C^2 z_2/z_3)$ . Again, only the last term from LHS contributes, with non-zero coefficient, which yields  $E(z, q_i z, C^2 q_i z) = 0$ .

Likewise, only the last term contributes to the coefficient of  $\delta(C^2 z_1/z_2)\delta(q_i z_2/z_3)$ , giving the relation  $E(z, q_i^{-1} z, C^{-2} q_i^{-1} z) = 0$ .

Consider  $\delta(C^2 z_1/z_2)\delta(C^2 z_2/z_3)$ . Again only the last term contributes and we find the equality  $E(C^{-2} z, z, C^2 z) = 0$ .  $\square$

**Remark 4.10.** *Since the zero conditions are important, let us check them directly on representations. For simplicity we consider the case  $c_1 = \dots = c_{\ell+1} = 2$ . The contractions between  $\Lambda_i(z)$  are given as follows, see (4.14):*

$$\begin{aligned} \overline{\Lambda_i(z)\Lambda_i(w)} &= \begin{cases} \frac{1-w/z}{1-q_1w/z} \frac{1-q_2^{-1}w/z}{1-q_3w/z} & (i=1, \dots, \ell, \bar{\ell}, \dots, \bar{1}), \\ \frac{1-w/z}{1-q_2w/z} \frac{1-q_3^{-1}w/z}{1-q_1w/z} \frac{1-q_3^{1/2}q_2w/z}{1-q_3^{1/2}w/z} \frac{1-q_3^{-1/2}q_2^{-1}w/z}{1-q_3^{-1/2}w/z} & (i=0), \end{cases} \\ \overline{\Lambda_i(z)\Lambda_j(w)} &= \begin{cases} \frac{\bar{g}(z, w)}{g(z, w)} & (i \prec j, i \neq \bar{j}), \\ 1 & (i \succ j, i \neq \bar{j}), \end{cases} \\ \overline{\Lambda_i(z)\Lambda_{\bar{i}}(w)} &= \frac{\bar{g}(z, w)}{g(z, w)} \frac{1-q_1^{-1}q_2^{-i}C^2w/z}{1-q_2^{-i}C^2w/z} \frac{1-q_3^{-1}q_2^{-i}C^2w/z}{1-q_2^{-i+1}C^2w/z} \quad (i=1, \dots, \ell), \\ \overline{\Lambda_{\bar{i}}(z)\Lambda_i(w)} &= \frac{1-q_1q_2^iC^{-2}w/z}{1-q_2^{i-1}C^{-2}w/z} \frac{1-q_3q_2^iC^{-2}w/z}{1-q_2^iC^{-2}w/z} \quad (i=1, \dots, \ell). \end{aligned}$$

As an example, consider (4.39). From the above we observe that

$$\begin{aligned} (1-C^{-2}z_2/z_1)\overline{\Lambda_i(z_1)\Lambda_j(z_2)} \Big|_{z_2=C^2z_1} &= 0 \quad ((i, j) \neq (\bar{1}, 1)), \\ \overline{\Lambda_1(z_2)\Lambda_k(z_3)} \Big|_{z_3=q_c z_2} &= 0 \quad (k \neq 1, c=1, 2, 3), \quad \overline{\Lambda_1(z_2)\Lambda_1(z_3)} \Big|_{z_3=q_2 z_2} = 0, \\ \overline{\Lambda_{\bar{1}}(z_1)\Lambda_1(z_3)} \Big|_{z_3=C^2q_c z_1} &= 0 \quad (c=1, 3). \end{aligned}$$

Therefore the operator  $\prod_{r < s} f_0(z_s/z_r) \cdot \Lambda_i(z_1)\Lambda_j(z_2)\Lambda_k(z_3)$  vanishes at  $(z_1, z_2, z_3) = (z, C^2z, C^2q_cz)$  for all  $i, j, k$ , and (4.39) follows. The other zero conditions can be verified similarly.

First, we consider the case  $C^2 = \mu q_2$ .

**Theorem 4.11.** *Assume that  $C^2 = \mu q_2$ . Then the following elements  $\{\mathbf{I}_n\}_{n=1}^\infty$  are mutually commutative,*

$$(4.41) \quad \mathbf{I}_n = \int \cdots \int \mathbf{E}(z_1) \cdots \mathbf{E}(z_n) \cdot \prod_{j < k} \omega_2(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{2\pi i z_j}.$$

Here the integral is taken on the unit circle  $|z_j| = 1$ ,  $j = 1, \dots, n$  in the region  $|q_1|, |q_3| > 1$  and extended by analytic continuation everywhere else.

Theorem 4.11 is proved in Appendix A.

In the case  $C^2 = \mu q_3$  or  $C^2 = \mu q_1$  we use a different kernel function. For example we have the following theorem.

**Theorem 4.12.** *Assume that  $C^2 = \mu q_3$ , and define*

$$(4.42) \quad \mathbf{I}'_n = \int \cdots \int \mathbf{E}(z_1) \cdots \mathbf{E}(z_n) \cdot \prod_{j < k} \omega_3(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{2\pi i z_j}.$$

We choose the contour for  $z_j$  for each  $j = 1, \dots, n$  in such a way that

$$\begin{aligned} \mu^s q_i^{-1} z_k \quad (s \geq 0, i = 1, 2, k < j) \text{ are inside,} \\ \mu^{-s} q_i z_k \quad (s \geq 0, i = 1, 2, k < j) \text{ are outside.} \end{aligned}$$

Then the elements  $\{\mathbf{I}'_n\}_{n=1}^\infty$  are mutually commutative.

*Proof.* The integrand of (4.42) are obtained from the one of (4.41) by interchanging the roles of  $q_2$  and  $q_3$ . The result of the previous subsection tells that, if the parameters satisfy  $|q_1|, |q_2| > 1$ , then the integrals (4.42) over the unit circle give a commutative family. Being an analytic continuation in the parameter  $q_2$  from  $|q_2| > 1$  to  $|q_2| < |q_1|^{-1}$ , the integrals (4.42) remain commutative.  $\square$

Thus in a generic admissible  $\mathcal{K}$  module, we have three commutative families of operators. We call them deformed integrals of motion due to the following theorem.

**Theorem 4.13.** *Assume  $\hat{B}$  is stable. For  $C^2 = \mu q_2$ , the elements  $\mathbf{I}_n$  commute with all screening operators,*

$$[S_i, \mathbf{I}_n] = 0, \quad i = 0, 1, \dots, \ell, \quad n \geq 1.$$

*Proof.* The argument for type C is similar to that in type A, see Theorem 3.7.

For type D the formulas become a little more cumbersome. We illustrate it in the case  $c_0 = c_1 = c_2 = 2$ . We start from

$$[S_0, \mathbf{E}(z)] = \text{const.} (\mathcal{A}^{(1)}(z) - \mathcal{A}^{(1)}(\mu z)),$$

where now  $\mathcal{A}^{(1)}(z)$  is a sum of two terms

$$\mathcal{A}^{(1)}(z) = z : S_0(s_3^{-1} \mu^{-1/2} z) (\mathbf{\Lambda}_1(z) + \mathbf{\Lambda}_2(z)) : .$$

Taking residues repeatedly we obtain currents for  $m = 1, 2, \dots$

$$\mathcal{A}^{(m)}(z) = z : S_0(s_3^{-1}q_1^{-m+1}\mu^{-1/2}z) \sum_{k=0}^m a_k^{(m)} \Lambda_2(q_3^{-m+1}z) \cdots \Lambda_2(q_3^{-m+k}z) \Lambda_1(q_3^{-m+k+1}z) \cdots \Lambda_1(z) :,$$

$$a_k^{(m)} = \prod_{j=1}^k \frac{1 - q_3^{m-j+1}}{1 - q_3^j} \frac{1 - q_1 q_3^j}{1 - q_1 q_3^{m-j+1}}.$$

The rest of the argument is the same.  $\square$

It would be interesting to study the spectrum of deformed integrals of motion.

Theorem 4.13 deals with affine nodes of types C and D. If the affine node is of type B, the integral of current  $\tilde{\mathbf{E}}(z)$ , see (3.7), commutes with all the screenings. In the case the  $\ell$ -th node is not of type B, this integral coincides with the integral of another current for which the affine node is of type C or D (the same as the  $\ell$ -th node for the original current, see (4.35)). Therefore, we can include the integral of  $\tilde{\mathbf{E}}(z)$  into a family of integral of motions corresponding to that current. We expect this family commutes with all screening operators.

## 5. ADDITIONAL REMARKS

**5.1. Integrals of motion of KZ type.** In this subsection we continue with the Cartan matrices in Table 1 except for types  $A_\ell^{(1)}$ ,  $D_{\ell+1}^{(2)}$ , and construct another set of commuting operators which commute with integrals  $\mathbf{I}_n$ . We hope that these integrals may be more convenient for Bethe ansatz study in the future.

The results in this section depend on the following technical statement which we do not discuss.

**Conjecture 5.1.** *The  $\mathcal{K}$  module  $\mathbb{F} = \mathcal{F}_2(u_1) \otimes \cdots \otimes \mathcal{F}_2(u_\ell) \otimes \mathcal{F}_c^X$  is irreducible for generic parameters  $u_1, \dots, u_\ell$ .  $\square$*

The vector space  $\mathbb{F}$  is an irreducible representation of the set of  $\ell+1$  bosons  $\{\mathbf{a}_{i,r} \mid i = 0, \dots, \ell, r \neq 0\}$ . We regard  $\{\mathbf{a}_{i,0} \mid i = 0, \dots, \ell\}$  as functions of the parameters  $u_1, \dots, u_\ell$  through (4.15), (4.16)–(4.18) and (4.29)–(4.31). Separating the zero modes we shall write  $A_i(z) = e^{\mathbf{a}_{i,0}} A_i^{osc}(z)$ ,  $Y_i(z) = e^{y_{i,0}} Y_i^{osc}(z)$ . To each  $i = 0, \dots, \ell$  we associate a reflection operator  $R_i \in \text{End } \mathbb{F}$  defined as follows.

**Proposition 5.2.** *There exist operators  $R_i \in \text{End } \mathbb{F}$  with the properties*

$$R_i Y_j(z) = Y_j(z) R_i \quad (j \neq i),$$

$$R_i \left( e^{y_{i,0}} Y_i^{osc}(z) + e^{y_{i,0} - \mathbf{a}_{i,0}} : \frac{Y_i^{osc}(z)}{A_i^{osc}(\hat{s}_i z)} : \right) = \left( e^{y_{i,0} - \mathbf{a}_{i,0}} Y_i^{osc}(z) + e^{y_{i,0}} : \frac{Y_i^{osc}(z)}{A_i^{osc}(\hat{s}_i z)} : \right) R_i,$$

where  $\hat{s}_i = s_2$  for  $i \neq 0, \ell$ ,  $\hat{s}_i = s_2 s_3^{1/2}$ ,  $s_2 s_3^{-1}$ ,  $s_2$  if  $i \in \{0, \ell\}$  is of type B, C, D, respectively.

In other words,  $R_i$  is an operator depending only on one boson  $\{\mathbf{a}_{i,r}\}_{r \neq 0}$  as well as the zero modes  $\{\mathbf{a}_{j,0}\}_{j=1, \dots, \ell}$ , and implements the Weyl reflection on the latter.

*Proof.* It is clear that for  $i = 1, \dots, \ell - 1$  the  $R$  matrix  $R_i = \tilde{R}_{i,i+1}(u_i/u_{i+1})$ , obtained from the universal  $R$  matrix of  $\mathcal{E}$ , has the required properties. For  $i = \ell$  and type B, C, we use Proposition B.1 to get  $R_\ell = \mathbf{K}_\ell(u_\ell)$ . In the case of type D, from the remark after Proposition B.1 we have  $\mathbf{K}_\ell A_{\ell-1}(z) \mathbf{K}_\ell = A_\ell(z)$ ,  $\mathbf{K}_\ell^2 = 1$ . Hence we can take  $R_\ell = \mathbf{K}_\ell \tilde{R}_{\ell-1, \ell}(u_{\ell-1} u_\ell) \mathbf{K}_\ell$ .

The case  $i = 0$  is similar.  $\square$

For  $i \neq 0$ , they are intertwiners of  $\mathcal{K}$  modules. Writing the  $\mathbf{E}(z)$  current as  $\mathbf{E}(z; u_1, \dots, u_\ell)$  we have

$$(5.1) \quad \begin{aligned} & \check{R}_{i,i+1}(u_i/u_{i+1})\mathbf{E}(z; u_1, \dots, u_i, u_{i+1}, \dots, u_\ell) \\ &= \mathbf{E}(z; u_1, \dots, u_{i+1}, u_i, \dots, u_\ell)\check{R}_{i,i+1}(u_i/u_{i+1}) \quad (i = 1, \dots, \ell - 1), \end{aligned}$$

$$(5.2) \quad \mathbf{K}_\ell(u_\ell)\mathbf{E}(z; u_1, \dots, u_{\ell-1}, u_\ell) = \mathbf{E}(z; u_1, \dots, u_{\ell-1}, u_\ell^{-1})\mathbf{K}_\ell(u_\ell) \quad (i = \ell).$$

For  $i = 0$  the corresponding relation takes the form

$$(5.3) \quad \mathbf{K}_1(u_1)\mathbf{E}^*(z; u_1, u_2, \dots, u_\ell) = \mathbf{E}^*(z; u_1^{-1}, u_2, \dots, u_\ell)\mathbf{K}_1(u_1),$$

where  $\mathbf{E}^*(z; u_1, \dots, u_\ell)$  means  $\mathbf{E}(z; u_1, \dots, u_\ell) - b_2(\Lambda_1(z) - \Lambda_1(\mu z))$  or  $\mathbf{E}(z; u_1, \dots, u_\ell) - b_2(\Lambda_1(z) - \Lambda_1(\mu z)) - b_2(\Lambda_2(z) - \Lambda_2(\mu z))$ , depending on whether the zeroth node is of type C or of type D.

We now introduce operators  $T_i$ ,  $i = 1 \dots, \ell$ , by

$$\begin{aligned} T_i &= T_i^+ T_i^-, \\ T_i^+ &= \check{R}_{i-1,i}(u_i/u_{i-1}) \cdots \check{R}_{1,2}(u_i/u_1)\mathbf{K}_1(u_i)\check{R}_{1,2}(u_1 u_i) \cdots \check{R}_{i-1,i}(u_{i-1} u_i), \\ T_i^- &= \check{R}_{i,i+1}(u_i u_{i+1}) \cdots \check{R}_{\ell-1,\ell}(u_i u_\ell)\mathbf{K}_\ell(u_i)\check{R}_{\ell-1,\ell}(u_i/u_\ell) \cdots \check{R}_{i,i+1}(u_i/u_{i+1}). \end{aligned}$$

We call  $T_i$  integrals of motion of KZ type for the following reason.

**Theorem 5.3.** *The operators  $T_i$  and  $\mathbf{I}_n$  in Theorems 4.11, 4.12 are mutually commutative:*

$$\begin{aligned} [T_i, T_j] &= 0 \quad (i, j = 1, \dots, \ell), \\ [T_i, \mathbf{I}_n] &= 0 \quad (i = 1, \dots, \ell, n \geq 1). \end{aligned}$$

*Proof.* The commutativity of  $T_i$ 's is a simple consequence of the (ordinary and boundary) Yang-Baxter equations.

To see the second statement, we use (5.1), (5.2) and (5.3). The only issue is to check that one can safely shift  $\Lambda_1(z_j)$  to  $\Lambda_1(\mu z_j)$  without encountering poles in between. This is straightforward.  $\square$

**5.2. Exceptional types.** The  $\mathcal{W}$  algebras and integrals of motion of type A are obtained from the quantum toroidal algebra  $\mathcal{E}$ . In this paper we have introduced an algebra  $\mathcal{K}$  which allows us to treat deformed  $\mathcal{W}$  algebras of non-exceptional types uniformly. A natural question is what happens in exceptional types.

For an exceptional type, one can consider a similar algebra by taking the current  $T(z)$  in the sense of [FR1], together with a vertex operator  $Z(z)$  in one extra boson such that  $Z(z)T(w) = T(w)Z(z)$  and such that all terms in  $E(z) = T(z)Z(z)$  have rational contractions. This gives the quantum algebra in this type similar to  $\mathcal{E}$  and  $\mathcal{K}$ .

For example, in the case of the seven dimensional representation of  $\mathbf{G}_2$ , one obtains the relation with four  $\delta$ -functions

$$\begin{aligned} & g(z, w)E(z)E(w) + g(w, z)E(w)E(z) \\ &= c_1 \left( \delta\left(q_1^3 q_2^6 \frac{w}{z}\right) w^3 K(w) + \delta\left(q_1^3 q_2^6 \frac{z}{w}\right) z^3 K(z) \right) \\ &+ c_2 \left( \delta\left(q_1^2 q_2^4 \frac{w}{z}\right) w^3 : E(q_1 q_2^2 w) \tilde{K}(w) : + \delta\left(q_1^2 q_2^4 \frac{z}{w}\right) z^3 : E(q_1 q_2^2 z) \tilde{K}(z) : \right), \end{aligned}$$

where  $K(z) = : Z(z)Z(q_1^3 q_2^6 z) :$ ,  $\tilde{K}(z) = : Z(z)Z(q_1^2 q_2^4 z)Z^{-1}(q_1 q_2^2 z) :$ , and  $c_1, c_2$  are constants and

$$\begin{aligned}\mathcal{C}(Z(z), Z(w)) &= (1 - q_1)(1 - q_3) \frac{1 + q_1 q_2^2}{1 + q_1^3 q_2^6} (q_1 q_2^2 - 1 + q_2), \\ \mathcal{C}(K(z), E(w)) &= (1 - q_1)(1 - q_3)(1 + q_1 q_2^2)(q_1 q_2^2 - 1 + q_2), \\ \mathcal{C}(\tilde{K}(z), E(w)) &= (1 - q_1)(1 - q_3)(q_1 q_2^2 - 1 + q_2).\end{aligned}$$

The role of this algebra is not clear since  $\mathbf{G}_2$  is not a part of any family. In particular, we do not expect any comodule or coalgebra structure.

#### APPENDIX A. PROOF OF THEOREM 4.11

In this Section we prove Theorem A. We have  $C^2 = \mu q_2$ .

**A.1. Commutativity**  $[\mathbf{I}_1, \mathbf{I}_2] = 0$ . As an illustration, let us verify the commutativity of  $\mathbf{I}_1$  and  $\mathbf{I}_2$ .

We note that, by making use of the decomposition

$$\omega_2(x) = q_2 C^2 f(x) \sigma_2(x) \sigma_2(x^{-1}), \quad \sigma_2(x) = (1 - x)^3 \frac{(\mu x, \mu^2 q_2 x)_\infty}{(q_1^{-1} x, q_3^{-1} x)_\infty},$$

the integral (4.41) can be rewritten in terms of the currents (4.37) as

$$\text{const. } \mathbf{I}_n = \int \cdots \int \mathbf{E}(z_1, \dots, z_n) \cdot \prod_{j \neq k} \sigma_2(z_k/z_j) \prod_{j=1}^n \frac{dz_j}{z_j}.$$

Consider the products

$$\begin{aligned}\mathbf{I}_1 \mathbf{I}_2 &= \iiint \mathbf{E}(z_1, z_2, z_3) \times f(z_2/z_1)^{-1} f(z_3/z_1)^{-1} \sigma_2(z_2/z_3) \sigma_2(z_3/z_2) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \\ \mathbf{I}_2 \mathbf{I}_1 &= \iiint \mathbf{E}(z_1, z_2, z_3) \times f(z_1/z_2)^{-1} f(z_1/z_3)^{-1} \sigma_2(z_2/z_3) \sigma_2(z_3/z_2) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}.\end{aligned}$$

The integral in  $\mathbf{I}_1 \mathbf{I}_2$  is initially defined for  $|z_1| \gg |z_2| = |z_3| = 1$ , while in  $\mathbf{I}_2 \mathbf{I}_1$  it is defined for  $|z_1| \ll |z_2| = |z_3| = 1$ . In both cases we move the contour for  $z_1$  to the unit circle. Along the way we pick up residues at the poles  $z_1 = q_2^{-1} z_i, \mu^{-1} q_2^{-1} z_i$  or  $z_1 = q_2 z_i, \mu q_2 z_i$ ,  $i = 2, 3$ , respectively.

When all variables are on the unit circle, the two integrals coincide thanks to the identity (3.24).

Let us compare the residues at  $z_1 = q_2^{\pm 1} z_3$ . We obtain respectively

$$\begin{aligned}J_1 &= \iint_{|z_2|=|z_3|=1} \mathbf{E}(z_2, q_2^{-1} z_3, z_3) f(q_2 z_2/z_3)^{-1} \sigma_2(z_2/z_3) \sigma_2(z_3/z_2) \prod_{i=2}^3 \frac{dz_i}{2\pi i z_i}, \\ J_2 &= \iint_{|z_2|=|z_3|=1} \mathbf{E}(z_2, z_3, q_2 z_3) f(q_2 z_3/z_2)^{-1} \sigma_2(z_2/z_3) \sigma_2(z_3/z_2) \prod_{i=2}^3 \frac{dz_i}{2\pi i z_i}.\end{aligned}$$

If we rename  $z_3$  in  $J_1$  to  $q_2 z_3$  (so that  $q_2 z_3$  is on the unit circle), then the two integrands become the same thanks to the identity

$$(A.1) \quad f(x)^{-1} \sigma_2(q_2^{-1} x) = \sigma_2(x) \frac{(1-x)^2 (1-q_2^{-1} x)^2}{(1-\mu x)(1-\mu^{-1} q_2^{-1} x)(1-q_1 x)(1-q_3 x)}.$$

The integrand of  $J_2$  has poles at (see Figure 1 below)

$$\begin{aligned} z_3 &= \mu^m q_s^{-1} z_2 \quad (s = 1, 3, m \geq 0); \\ z_3 &= \mu^{-m} q_s q_2^{-1} z_2 \quad (s = 1, 3, m \geq 1); \\ z_3 &= q_s z_2 \quad (s = 1, 3). \end{aligned}$$

Among them the points  $z_3 = q_s z_2$  are inside the contour for  $J_1$  (after renaming) and outside that for  $J_2$ . However these poles are actually absent due to the zero condition (4.38). Hence we have  $J_1 = J_2$ .

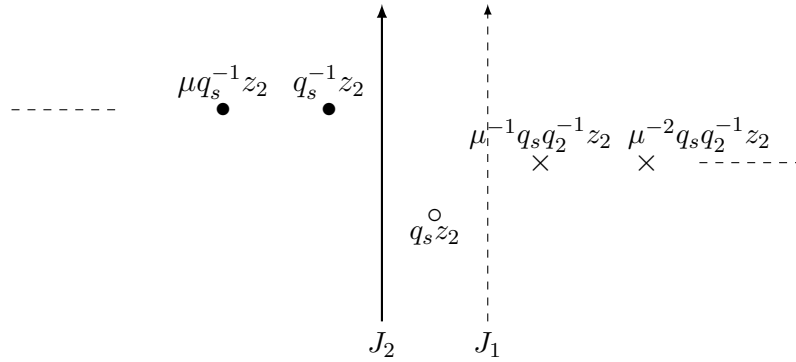


FIGURE 1. Integration contours on the  $z_3$ -plane ( $s = 1, 3$ ) for  $J_1, J_2$ .

Next let us consider the residues at  $z_1 = (\mu q_2)^{\pm 1} z_3$ . Similarly as above we obtain

$$\begin{aligned} J'_1 &= \iint_{|z_2|=|z_3|=1} \mathbf{E}(z_2, \mu^{-1} q_2^{-1} z_3, z_3) f(\mu q_2 z_2 / z_3)^{-1} \sigma_2(z_2 / z_3) \sigma_2(z_3 / z_2) \prod_{i=2}^3 \frac{dz_i}{2\pi i z_i}, \\ J'_2 &= \iint_{|z_2|=|z_3|=1} \mathbf{E}(z_2, z_3, \mu q_2 z_3) f(\mu q_2 z_3 / z_2)^{-1} \sigma_2(z_2 / z_3) \sigma_2(z_3 / z_2) \prod_{i=2}^3 \frac{dz_i}{2\pi i z_i}. \end{aligned}$$

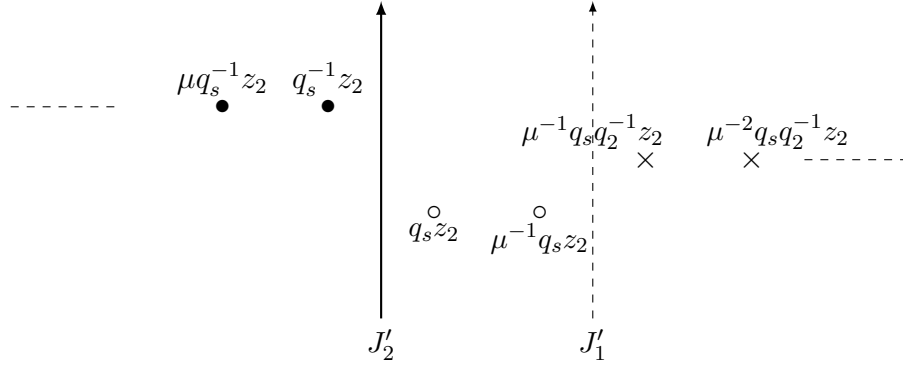
We have another identity

$$(A.2) \quad f(x)^{-1} \sigma_2(\mu^{-1} q_2^{-1} x) = \sigma_2(x) \frac{(1-x)^2 (1-\mu^{-1} q_2^{-1} x)^2}{(1-\mu^{-1} q_1 x)(1-\mu^{-1} q_3 x)(1-q_1 x)(1-q_3 x)}.$$

After renaming  $z_3 \rightarrow \mu q_2 z_3$  in  $J'_1$ , the two integrands become the same. The integrand of  $J'_2$  has poles at (see Figure 2)

$$\begin{aligned} z_3 &= \mu^m q_s^{-1} z_2 \quad (m \geq 0, s = 1, 3); \\ z_3 &= \mu^{-m} q_2^{-1} q_s z_2 \quad (m \geq 1, s = 1, 3); \\ z_3 &= q_s z_2, \mu^{-1} q_s z_2 \quad (s = 1, 3). \end{aligned}$$

The last ones  $z_3 = q_s z_2, \mu^{-1} q_s z_2$  ( $s = 1, 3$ ) are inside the contour for  $J'_1$ , while they are outside for  $J'_2$ . Using the zero conditions (4.38), (4.39), we conclude that  $J'_1 = J'_2$ .  $\square$


 FIGURE 2. Integration contours on the  $z_3$ -plane ( $s = 1, 3$ ) for  $J'_1, J'_2$ .

**A.2. The general case.** We consider the general case  $[\mathbf{I}_m, \mathbf{I}_n] = 0$ . Call the integration variables  $z_1, \dots, z_m$  for  $\mathbf{I}_m$  and  $w_1, \dots, w_n$  for  $\mathbf{I}_n$ . We proceed in the same way, rewriting  $\mathbf{I}_m \mathbf{I}_n$  and  $\mathbf{I}_n \mathbf{I}_m$  as integrals over the unit circle and picking residues with respect to some groups of variables.

First consider  $\mathbf{I}_m \mathbf{I}_n$ . In view of symmetry and zeros on the diagonal, it is sufficient to consider residues from

$$\begin{aligned} z_i &= q_2^{-1}w_i \quad (1 \leq i \leq k), \\ z_i &= \mu^{-1}q_2^{-1}w_i \quad (k+1 \leq i \leq k+l). \end{aligned}$$

The result is (we write only the integrand)

$$J_1 = \mathbf{E}(\{q_2^{-1}w_i, w_i\}_{i=1}^k, \{\mu^{-1}q_2^{-1}w_i, w_i\}_{i=k+1}^{k+l}, \{z_j\}_{j=k+l+1}^m, \{w_j\}_{j=k+l+1}^n) \times F_1 G_1 H_1,$$

with

$$\begin{aligned} F_1 &= \prod_{1 \leq i \neq j \leq k} f(q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i)^2 \prod_{k+1 \leq i \neq j \leq k+l} f(\mu q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i)^2 \\ &\times \prod_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq k+l}} f(q_2 w_j / w_i)^{-1} \sigma_2(\mu^{-1} w_j / w_i) \sigma_2(w_i / w_j) \prod_{\substack{k+1 \leq i \leq k+l \\ 1 \leq j \leq k}} f(\mu q_2 w_j / w_i)^{-1} \sigma_2(\mu w_j / w_i) \sigma_2(w_i / w_j), \\ G_1 &= \prod_{j=k+l+1}^m \left( \prod_{1 \leq i \leq k} f(w_i / z_j)^{-1} \sigma_2(q_2^{-1} w_i / z_j) \sigma_2(q_2 z_j / w_i) \prod_{k+1 \leq i \leq k+l} f(w_i / z_j)^{-1} \sigma_2(\mu^{-1} q_2^{-1} w_i / z_j) \sigma_2(\mu q_2 z_j / w_i) \right) \\ &\times \prod_{j=k+l+1}^n \left( \prod_{1 \leq i \leq k} f(q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i) \sigma_2(w_i / w_j) \prod_{k+1 \leq i \leq k+l} f(\mu q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i) \sigma_2(w_i / w_j) \right) \\ H_1 &= \prod_{\substack{k+l+1 \leq i \leq m \\ k+l+1 \leq j \leq n}} f(w_j / z_i)^{-1} \prod_{k+l+1 \leq i \neq j \leq m} \sigma_2(z_j / z_i) \prod_{k+l+1 \leq i \neq j \leq n} \sigma_2(w_j / w_i). \end{aligned}$$

For  $\mathbf{I}_n \mathbf{I}_m$  we work similarly, picking poles at

$$\begin{aligned} z_i &= q_2 w_i \quad (1 \leq i \leq k), \\ z_i &= \mu q_2 w_i \quad (k+1 \leq i \leq k+l), \end{aligned}$$

ending up with

$$J_2 = \mathbf{E}(\{q_2 w_i, w_i\}_{i=1}^k, \{\mu q_2 w_i, w_i\}_{i=k+1}^{k+l}, \{z_j\}_{j=k+l+1}^m, \{w_j\}_{j=k+l+1}^n) \times F_2 G_2 H_2,$$

where

$$\begin{aligned} F_2 &= \prod_{1 \leq i \neq j \leq k} f(q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i)^2 \prod_{k+1 \leq i \neq j \leq k+l} f(\mu q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i)^2 \\ &\times \prod_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq k+l}} f(q_2 w_i / w_j)^{-1} \sigma_2(w_i / w_j) \sigma_2(\mu w_j / w_i) \prod_{\substack{k+1 \leq i \leq k+l \\ 1 \leq j \leq k}} f(\mu q_2 w_j / w_i)^{-1} \sigma_2(w_j / w_i) \sigma_2(\mu^{-1} w_i / w_j), \\ G_2 &= \prod_{i=k+l+1}^m \left( \prod_{1 \leq j \leq k} f(z_i / w_j)^{-1} \sigma_2(q_2 w_j / z_i) \sigma_2(q_2^{-1} z_i / w_j) \prod_{k+1 \leq i \leq k+l} f(z_i / w_j)^{-1} \sigma_2(\mu q_2 w_j / z_i) \sigma_2(\mu^{-1} q_2^{-1} z_i / w_j) \right) \\ &\times \prod_{j=k+l+1}^n \left( \prod_{1 \leq i \leq k} f(q_2 w_i / w_j)^{-1} \sigma_2(w_i / w_j) \sigma_2(w_j / w_i) \prod_{k+1 \leq i \leq k+l} f(\mu q_2 w_i / w_j)^{-1} \sigma_2(w_i / w_j) \sigma_2(w_j / w_i) \right), \\ H_2 &= \prod_{\substack{k+l+1 \leq i \leq m \\ k+l+1 \leq j \leq n}} f(z_i / w_j)^{-1} \prod_{k+l+1 \leq i \neq j \leq m} \sigma_2(z_j / z_i) \prod_{k+l+1 \leq i \neq j \leq n} \sigma_2(w_j / w_i). \end{aligned}$$

Using the identities (A.1),(A.2) one can check that

$$J_1 \Big|_{w_i \rightarrow q_2 w_i (1 \leq i \leq k), w_j \rightarrow \mu q_2 w_j (k+1 \leq j \leq k+l)} = J_2.$$

The contours can be chosen to be the unit circle by the same mechanism observed above. It remains to show that under symmetrization with respect to  $\{z_i\}_{i=k+l+1}^m$  and  $\{w_j\}_{j=k+l+1}^n$  we have

$$\text{Sym } H_1 = \text{Sym } H_2.$$

This reduces to identity (3.24). The proof of Theorem 4.11 is now complete.  $\square$

## APPENDIX B. $K$ MATRICES

It is well known that the Hopf algebra  $\mathcal{E}$  is equipped with the universal  $R$  matrix, which gives rise to an intertwiner of  $\mathcal{E}$  modules

$$(B.1) \quad \check{R}(u_1/u_2) : \mathcal{F}_2(u_1) \otimes \mathcal{F}_2(u_2) \rightarrow \mathcal{F}_2(u_2) \otimes \mathcal{F}_2(u_1).$$

Since  $\check{R}(u_1/u_2)$  commutes with the diagonal Heisenberg  $\Delta h_r$ , it is written in terms of a single boson  $\{\mathbf{a}_r^\Lambda\}$  entering the root current  $A(z) = : \Lambda_1(s_2 z) \Lambda_2(s_2 z)^{-1} :$ . Exhibiting the dependence on parameters we shall write

$$\check{R}(u_1/u_2) = \check{R}(u_1/u_2; q_1, q_2, q_3, \{\mathbf{a}_r^\Lambda\}).$$

**Proposition B.1.** *Consider a  $\mathcal{K}$  module  $\mathcal{F}_2(u) \otimes \mathcal{F}_c^X$  ( $X = B, CD$ ), and let  $A(z) = : e^{\sum_{r \in \mathbb{Z}} \mathbf{a}_r z^{-r}} :$  be the root current associated with  $E(z)$ . Then there exists an intertwiner of  $\mathcal{K}$  modules*

$$\mathcal{K}(u) : \mathcal{F}_2(u) \otimes \mathcal{F}_c^X \rightarrow \mathcal{F}_2(u^{-1}) \otimes \mathcal{F}_c^X$$

in the following cases.

$$(B.2) \quad \text{type B } (X = B, c = 3) : \quad K(u) = \check{R}(u; q_1, q_2 q_3^{1/2}, q_3^{1/2}, \{\mathbf{a}_r\}),$$

$$(B.3) \quad \text{type C } (X = CD, c = 3) : \quad K(u) = \check{R}(u^2; q_1, q_2 q_3^{-1}, q_3^2, \{\mathbf{a}_r\}),$$

$$(B.4) \quad \text{type D } (X = CD, c = 2) : \quad K(u) = (-1)^N.$$

In the last line,  $N$  denotes the number operator  $\sum_{r>0} \nu_r^{-1} \mathbf{a}_{-r} \mathbf{a}_r$ ,  $\nu_r = [\mathbf{a}_r, \mathbf{a}_{-r}]$ . Note that  $K(u)$  is independent of  $u$  in this case.

*Proof.* First note that, by extracting the diagonal Heisenberg, the intertwining relation (B.1) for type A reduces to

$$\check{R}(u_1/u_2)(u_1 \Lambda_+^A(z) + u_2 \Lambda_-^A(z)) = (u_2 \Lambda_+^A(z) + u_1 \Lambda_-^A(z)) \check{R}(u_1/u_2),$$

where  $\Lambda_{\pm}^A(z)$  are vertex operators in  $\{\mathbf{a}_r^A\}_{r \neq 0}$  such that

$$(B.5) \quad \Lambda_-^A(z) = : \Lambda_+^A(q_2 z)^{-1} :,$$

$$(B.6) \quad \mathfrak{C}(\Lambda_+^A(z), \Lambda_+^A(w)) = -\frac{(1-q_3)(1-q_1)}{1+q_2} q_2.$$

For type C and type D we proceed the same way as in type A to obtain the reduced intertwining relation

$$\mathbf{K}(u)(u \Lambda_+(z) + u^{-1} \Lambda_-(z)) = (u^{-1} \Lambda_+(z) + u \Lambda_-(z)) \mathbf{K}(u),$$

where  $\Lambda_{\pm}(z)$  are vertex operators in  $\{\mathbf{a}_r\}_{r \neq 0}$  such that

$$(B.7) \quad \Lambda_-(z) = : \Lambda_+(q_2 q_c^{-1} z)^{-1} :.$$

For type C ( $c = 3$ ), we have further

$$(B.8) \quad \mathfrak{C}(\Lambda_+(z), \Lambda_+(w)) = -\frac{(1-q_3^2)(1-q_1)}{1+q_2 q_3^{-1}} q_2 q_3^{-1}.$$

Comparing (B.7), (B.8) with (B.5), (B.6), we obtain (B.3).

For type D ( $c = 2$ ), (B.7) becomes  $\Lambda_-(z) = : \Lambda_+(z)^{-1} :$ , so that the intertwining relation reduces further to  $\mathbf{K}(u) \mathbf{a}_r = -\mathbf{a}_r \mathbf{K}(u)$  for all  $r \neq 0$ . This leads to the solution (B.4).

For type B, the reduced intertwining relation involves three terms

$$\mathbf{K}(u)(u \Lambda_{++}(z) + k \Lambda_0(z) + u^{-1} \Lambda_{--}(z)) = (u^{-1} \Lambda_{++}(z) + k \Lambda_0(z) + u \Lambda_{--}(z)) \mathbf{K}(u),$$

which corresponds to the  $qq$  character of the three dimensional representation of  $U_q \widehat{\mathfrak{sl}}_2$ . One can reduce it further to intertwining relation for the two dimensional one

$$\mathbf{K}(u)(u^{1/2} \Lambda_+(z) + u^{-1/2} \Lambda_-(z)) = (u^{-1/2} \Lambda_+(z) + u^{1/2} \Lambda_-(z)) \mathbf{K}(u)$$

by introducing  $\Lambda_{\pm}(z)$  such that  $\Lambda_{\pm\pm}(z) = : \Lambda_{\pm}(s_3^{1/2} z) \Lambda_{\pm}(s_3^{-1/2} z) :$ ,  $\Lambda_0(z) = : \Lambda_+(s_3^{1/2} z) \Lambda_-(s_3^{-1/2} z) :$ . The rest is similar to type C.  $\square$

On tensor products (4.9), (4.10), we write the intertwiners as  $\check{R}_{i,i+1}(u_i/u_{i+1})$  ( $i = 1, \dots, \ell - 1$ ) indicating the tensor components where they act non-trivially. For the intertwiners involving boundary Fock modules we write as  $K_\ell(u_\ell)$ . The standard argument (with  $\ell = 2$ ) leads to the boundary Yang-Baxter equation

$$(B.9) \quad \check{R}_{1,2}(u_1/u_2)K_2(u_1)\check{R}_{1,2}(u_1u_2)K_2(u_2) = K_2(u_2)\check{R}_{1,2}(u_1u_2)K_2(u_1)\check{R}_{1,2}(u_1/u_2).$$

As noted above, the  $K$  matrix  $K_\ell$  of type D is independent of  $u_\ell$  and satisfies  $K_\ell^2 = 1$ . Comparing the intertwining relation with the definition of the currents  $A_{\ell-1}(z)$  (4.15) and  $A_\ell(z)$  (4.18), we see that

$$K_\ell A_{\ell-1}(z) = A_\ell(z)K_\ell.$$

Though the zeroth node of the Dynkin diagram is not associated with boundary Fock modules, one can consider  $K$  matrices depending only on  $A_0(z)$  and satisfying the intertwining relations, for example for type C

$$K_1(u_1)(\Lambda_{\bar{1}}(z) + \Lambda_1(\mu z)) = (u_1^2 \Lambda_{\bar{1}}(z) + u_1^{-2} \Lambda_1(\mu z))K_1(u_1).$$

### APPENDIX C. THE LIBRARY OF CARTAN MATRICES

**C.1. Conventions.** The matrix of contractions  $\hat{B}$  is the deformed version of the symmetrized Cartan matrix. We will give a list of explicit deformed Cartan matrices  $\hat{C}$  of low rank which can be used to write all others as explained in Appendix C.4.

The deformed symmetrized Cartan matrix  $\hat{B}$  and the deformed Cartan matrix  $\hat{C}$  are related by a diagonal matrix  $\hat{D}$ , namely  $\hat{B} = \hat{D}\hat{C}$ , where the diagonal entries  $d_i$  of  $\hat{B}$  and the diagonal entries of  $\hat{C}$  are given as follows.

The nodes of type A (corresponding to  $\mathcal{F}_{c_i} \otimes \mathcal{F}_{c_{i+1}}$ ).

$$\begin{aligned} \text{A, } c_i = c_{i+1} : \quad d_i &= -\frac{t_1 t_2 t_3}{t_{c_i}}, & C_{i,i} &= s_{c_i} + s_{c_i}^{-1}. & \circ & \\ & & & & c_i & \\ \text{A, } c_i \neq c_{i+1} : \quad d_i &= t_b, & C_{i,i} &= t_b \quad (b \neq c_i, c_{i+1}). & \otimes & \\ & & & & b & \end{aligned}$$

The nodes of type B (corresponding to  $\mathcal{F}_{c_\ell} \otimes \mathcal{F}_{c_{\ell+1}}^B$ ).

$$\begin{aligned} \text{B, } c_\ell = c_{\ell+1} : \quad d_\ell &= -\frac{t_1 t_2 t_3}{t_{c_{\ell+1}}(s_{c_{\ell+1}}^{1/2} + s_{c_{\ell+1}}^{-1/2})}, & C_{\ell,\ell} &= s_{c_{\ell+1}}^{3/2} + s_{c_{\ell+1}}^{-3/2}. & \bullet & \\ & & & & c_\ell & \\ \text{B, } c_\ell \neq c_{\ell+1} : \quad d_\ell &= -t_b(s_{c_{\ell+1}}^{1/2} - s_{c_{\ell+1}}^{-1/2}), & C_{\ell,\ell} &= s_{c_\ell}^{1/2} s_b^{-1/2} + s_{c_\ell}^{-1/2} s_b^{1/2} \quad (b \neq c_\ell, c_{\ell+1}). & \circ & \\ & & & & b & \end{aligned}$$

The nodes of type C (corresponding to  $\mathcal{F}_{c_\ell} \otimes \mathcal{F}_{c_{\ell+1}}^{\text{CD}}$ ,  $c_{\ell+1} \neq c_\ell$ ).

$$\text{C, } c_\ell \neq c_{\ell+1} : \quad d_\ell = -t_b(s_{c_{\ell+1}}^2 - s_{c_{\ell+1}}^{-2}), \quad C_{\ell,\ell} = s_{c_{\ell+1}} s_{c_\ell}^{-1} + s_{c_{\ell+1}^{-1}} s_{c_\ell} \quad (b \neq c_\ell, c_{\ell+1}). \quad \circ$$

The nodes of type D (corresponding to  $\mathcal{F}_{c_{\ell-1}} \otimes \mathcal{F}_{c_\ell} \otimes \mathcal{F}_{c_{\ell+1}}^{\text{CD}}$ ,  $c_{\ell+1} = c_\ell$ ).

$$\begin{array}{ll} \text{D, } c_{\ell+1} = c_{\ell-1}: & d_\ell = -\frac{t_1 t_2 t_3}{t_{c_{\ell+1}}}, \quad C_{\ell,\ell} = s_{c_{\ell+1}} + s_{c_{\ell+1}}^{-1} & \circ \\ & & c_\ell \\ \text{D, } c_{\ell+1} \neq c_{\ell-1}: & d_\ell = t_b, \quad C_{\ell,\ell} = t_b \quad (b \neq c_{\ell+1}, c_{\ell-1}) & \otimes \\ & & b \end{array}$$

Affine nodes (corresponding to dressing in non A types) are the same as B, C, D nodes after the change

$$c_i \rightarrow c_{\ell+1-i}, \quad d_i \rightarrow d_{\ell-i}, \quad C_{i,i} \rightarrow C_{\ell-i,\ell-i}.$$

The deformed affine Cartan matrices we obtain have “local” form:  $C_{i,j} = 0$  if  $|i - j|$  is sufficiently large. Moreover, the non-zero terms stabilize with  $\ell$  increased. We give here a number of explicit deformed Cartan matrices of low rank which can be used to write all others as explained in Appendix C.4.

We use the following notation. We write  $X(c_0; c_1, \dots, c_{\ell-1}; c_{\ell+1})Y$  for the choice of the module and the dressing and give the  $(\ell + 1) \times (\ell + 1)$  matrix  $\hat{C}$ . Here  $Y$  can be B, C, or D depending on which boundary module we consider  $\mathcal{F}_{c_{\ell+1}}^{\text{B}}$  or  $\mathcal{F}_{c_{\ell+1}}^{\text{CD}}$ . As always, in the latter case we choose D if  $c_{\ell+1} = c_\ell$  and C otherwise. Similarly  $X$  can be B, C, or D depending on which dressing we choose. Namely  $C^2/\mu = q_{c_0}^{-1/2}$  corresponds to  $X = \text{B}$  while  $C^2/\mu = q_{c_0}$  corresponds to  $X = \text{C}$ , if  $c_0 \neq c_1$  and to  $X = \text{D}$  if  $c_0 = c_1$ .

We skip writing  $X$  and  $c_0$  in finite types. In addition we also skip  $c_{\ell+1}$  and  $Y$  in A type.

## C.2. Finite types.

### Type A.

$$\begin{array}{ll} (2, 2, 2) & \begin{pmatrix} s_2 + s_2^{-1} & -1 \\ -1 & s_2 + s_2^{-1} \end{pmatrix} \\ (1, 2, 3) & \begin{pmatrix} t_3 & t_1 \\ t_3 & t_1 \end{pmatrix} \\ (1, 2, 1) & \begin{pmatrix} t_3 & t_1 \\ t_1 & t_3 \end{pmatrix} \\ (2, 2, 1) & \begin{pmatrix} s_2 + s_2^{-1} & -1 \\ t_1 & t_3 \end{pmatrix} \\ (1, 2, 2) & \begin{pmatrix} t_3 & t_1 \\ -1 & s_2 + s_2^{-1} \end{pmatrix} \end{array}$$

**Type B.**

$$\begin{aligned}
(2, 2; 2)\text{B} & \begin{pmatrix} s_2 + s_2^{-1} & -1 \\ -s_2^{1/2} - s_2^{-1/2} & s_2^{3/2} + s_2^{-3/2} \end{pmatrix} \\
(2, 2; 3)\text{B} & \begin{pmatrix} s_2 + s_2^{-1} & -1 \\ -s_3^{1/2} - s_3^{-1/2} & s_1^{1/2} s_2^{-1/2} + s_1^{-1/2} s_2^{1/2} \end{pmatrix} \\
(1, 2; 2)\text{B} & \begin{pmatrix} t_3 & t_1 \\ -s_2^{1/2} - s_2^{-1/2} & s_2^{3/2} + s_2^{-3/2} \end{pmatrix} \\
(1, 2; 1)\text{B} & \begin{pmatrix} t_3 & t_1 \\ -s_1^{1/2} - s_1^{-1/2} & s_3^{1/2} s_2^{-1/2} + s_3^{-1/2} s_2^{1/2} \end{pmatrix} \\
(1, 2; 3)\text{B} & \begin{pmatrix} t_3 & t_1 \\ -s_3^{1/2} - s_3^{-1/2} & s_1^{1/2} s_2^{-1/2} + s_1^{-1/2} s_2^{1/2} \end{pmatrix}
\end{aligned}$$

**Type C.**

$$\begin{aligned}
(2, 2; 3)\text{C} & \begin{pmatrix} s_2 + s_2^{-1} & -s_3 - s_3^{-1} \\ -1 & s_2 s_3^{-1} + s_2^{-1} s_3 \end{pmatrix} \\
(1, 2; 1)\text{C} & \begin{pmatrix} t_3 & s_1^2 - s_1^{-2} \\ -1 & s_2 s_1^{-1} + s_2^{-1} s_1 \end{pmatrix} \\
(1, 2; 3)\text{C} & \begin{pmatrix} t_3 & -t_1(s_3 + s_3^{-1}) \\ -1 & s_2 s_3^{-1} + s_2^{-1} s_3 \end{pmatrix}
\end{aligned}$$

**Type D.**

$$\begin{aligned}
(2, 2, 2; 2)\text{D} & \begin{pmatrix} s_2 + s_2^{-1} & -1 & -1 \\ -1 & s_2 + s_2^{-1} & 0 \\ -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
(1, 2, 2; 2)\text{D} & \begin{pmatrix} t_3 & t_1 & t_1 \\ -1 & s_2 + s_2^{-1} & 0 \\ -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
(2, 1, 2; 2)\text{D} & \begin{pmatrix} t_3 & t_2 & t_2 \\ t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix} \\
(1, 1, 2; 2)\text{D} & \begin{pmatrix} s_1 + s_1^{-1} & -1 & -1 \\ t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix} \\
(3, 1, 2; 2)\text{D} & \begin{pmatrix} t_2 & t_3 & t_3 \\ t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix}
\end{aligned}$$

**C.3. Affine types.**

2 × 2 cases.

**B – B types.**

$$\begin{aligned}
 \text{B}(2; 2; 2)\text{B} & \begin{pmatrix} s_2^{3/2} + s_2^{-3/2} & -s_2^{1/2} - s_2^{-1/2} \\ -s_2^{1/2} - s_2^{-1/2} & s_2^{3/2} + s_2^{-3/2} \end{pmatrix} \\
 \text{B}(2; 2; 1)\text{B} & \begin{pmatrix} s_2^{3/2} + s_2^{-3/2} & -s_2^{1/2} - s_2^{-1/2} \\ -s_1^{1/2} - s_1^{-1/2} & s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} \end{pmatrix} \\
 \text{B}(1; 2; 1)\text{B} & \begin{pmatrix} s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} & -s_1^{1/2} - s_1^{-1/2} \\ -s_1^{1/2} - s_1^{-1/2} & s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} \end{pmatrix} \\
 \text{B}(1; 2; 3)\text{B} & \begin{pmatrix} s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} & -s_1^{1/2} - s_1^{-1/2} \\ -s_3^{1/2} - s_3^{-1/2} & s_2^{1/2} s_1^{-1/2} + s_2^{-1/2} s_1^{1/2} \end{pmatrix}
 \end{aligned}$$

**B – C types.**

$$\begin{aligned}
 \text{B}(2; 2; 1)\text{C} & \begin{pmatrix} s_2^{3/2} + s_2^{-3/2} & -(s_1 + s_1^{-1})(s_2^{1/2} + s_2^{-1/2}) \\ -1 & s_1 s_2^{-1} + s_1^{-1} s_2 \end{pmatrix} \\
 \text{B}(1; 2; 1)\text{C} & \begin{pmatrix} s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} & -(s_1 + s_1^{-1})(s_1^{1/2} + s_1^{-1/2}) \\ -1 & s_1 s_2^{-1} + s_1^{-1} s_2 \end{pmatrix} \\
 \text{B}(3; 2; 1)\text{C} & \begin{pmatrix} s_1^{1/2} s_2^{-1/2} + s_1^{-1/2} s_2^{1/2} & -(s_1 + s_1^{-1})(s_3^{1/2} + s_3^{-1/2}) \\ -1 & s_1 s_2^{-1} + s_1^{-1} s_2 \end{pmatrix}
 \end{aligned}$$

**C – C types.**

$$\begin{aligned}
 \text{C}(1; 2; 1)\text{C} & \begin{pmatrix} s_1 s_2^{-1} + s_1^{-1} s_2 & -s_1 - s_1^{-1} \\ -s_1 - s_1^{-1} & s_1 s_2^{-1} + s_1^{-1} s_2 \end{pmatrix} \\
 \text{C}(1; 2; 3)\text{C} & \begin{pmatrix} s_1 s_2^{-1} + s_1^{-1} s_2 & -s_3 - s_3^{-1} \\ -s_1 - s_1^{-1} & s_3 s_2^{-1} + s_3^{-1} s_2 \end{pmatrix}
 \end{aligned}$$

3 × 3 cases.

**B – D types.**

$$\begin{aligned}
\text{B}(2; 2, 2; 2)\text{D} & \begin{pmatrix} s_2^{3/2} + s_2^{-3/2} & -s_2^{1/2} - s_2^{-1/2} & -s_2^{1/2} - s_2^{-1/2} \\ -1 & s_2 + s_2^{-1} & 0 \\ -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
\text{B}(1; 2, 2; 2)\text{D} & \begin{pmatrix} s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} & -s_1^{1/2} - s_1^{-1/2} & -s_1^{1/2} - s_1^{-1/2} \\ -1 & s_2 + s_2^{-1} & 0 \\ -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
\text{B}(2; 1, 2; 2)\text{D} & \begin{pmatrix} s_1^{1/2} s_3^{-1/2} + s_1^{-1/2} s_3^{1/2} & -s_2^{1/2} - s_2^{-1/2} & -s_2^{1/2} - s_2^{-1/2} \\ t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix} \\
\text{B}(1; 1, 2; 2)\text{D} & \begin{pmatrix} s_1^{3/2} + s_1^{-3/2} & -s_1^{1/2} - s_1^{-1/2} & -s_1^{1/2} - s_1^{-1/2} \\ t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix} \\
\text{B}(1; 3, 2; 2)\text{D} & \begin{pmatrix} s_2^{1/2} s_3^{-1/2} + s_2^{-1/2} s_3^{1/2} & -s_1^{1/2} - s_1^{-1/2} & -s_1^{1/2} - s_1^{-1/2} \\ t_2 & t_1 & s_3 s_2^{-1} - s_3^{-1} s_2 \\ t_2 & s_3 s_2^{-1} - s_3^{-1} s_2 & t_1 \end{pmatrix}
\end{aligned}$$

**C – D types.**

$$\begin{aligned}
\text{C}(1; 2, 2; 2)\text{D} & \begin{pmatrix} s_1 s_2^{-1} + s_1^{-1} s_2 & -1 & -1 \\ -s_1 - s_1^{-1} & s_2 + s_2^{-1} & 0 \\ -s_1 - s_1^{-1} & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
\text{C}(1; 3, 2; 2)\text{D} & \begin{pmatrix} s_1 s_3^{-1} + s_1^{-1} s_3 & -1 & -1 \\ (s_1 + s_1^{-1}) t_2 & t_1 & s_2^{-1} s_3 - s_2 s_3^{-1} \\ (s_1 + s_1^{-1}) t_2 & s_2^{-1} s_3 - s_2 s_3^{-1} & t_1 \end{pmatrix} \\
\text{C}(2; 1, 2; 2)\text{D} & \begin{pmatrix} s_1 s_2^{-1} + s_1^{-1} s_2 & -1 & -1 \\ s_2^2 - s_2^{-2} & t_3 & s_2^{-1} s_1 - s_2 s_1^{-1} \\ s_2^2 - s_2^{-2} & s_2^{-1} s_1 - s_2 s_1^{-1} & t_3 \end{pmatrix}
\end{aligned}$$

**D – D types.**

$$\begin{aligned}
\text{D}(2; 2, 2; 2)\text{D} & \begin{pmatrix} s_2 + s_2^{-1} & 0 & -s_2 - s_2^{-1} \\ 0 & s_2 + s_2^{-1} & 0 \\ -s_2 - s_2^{-1} & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
\text{D}(1; 1, 2; 2)\text{D} & \begin{pmatrix} t_3 & s_1^{-1} s_2 - s_1 s_2^{-1} & -t_3 \\ s_1^{-1} s_2 - s_1 s_2^{-1} & t_3 & -s_1^{-1} s_2 + s_1 s_2^{-1} \\ -t_3 & -s_1^{-1} s_2 + s_1 s_2^{-1} & t_3 \end{pmatrix}
\end{aligned}$$

**D – D types.**

$$\begin{aligned}
 \text{D}(2; 2, 2, 2; 2)\text{D} & \begin{pmatrix} s_2 + s_2^{-1} & 0 & -1 & -1 \\ 0 & s_2 + s_2^{-1} & -1 & -1 \\ -1 & -1 & s_2 + s_2^{-1} & 0 \\ -1 & -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
 \text{D}(2; 2, 1, 2; 2)\text{D} & \begin{pmatrix} t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_2 & t_2 \\ s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 & t_2 & t_2 \\ t_2 & t_2 & t_3 & s_1 s_2^{-1} - s_1^{-1} s_2 \\ t_2 & t_2 & s_1 s_2^{-1} - s_1^{-1} s_2 & t_3 \end{pmatrix} \\
 \text{D}(1; 1, 2, 2; 2)\text{D} & \begin{pmatrix} t_3 & s_1^{-1} s_2 - s_1 s_2^{-1} & t_1 & t_1 \\ s_1^{-1} s_2 - s_1 s_2^{-1} & t_3 & t_1 & t_1 \\ -1 & -1 & s_2 + s_2^{-1} & 0 \\ -1 & -1 & 0 & s_2 + s_2^{-1} \end{pmatrix} \\
 \text{D}(1; 1, 3, 2; 2)\text{D} & \begin{pmatrix} t_2 & s_1^{-1} s_3 - s_1 s_3^{-1} & t_1 & t_1 \\ s_1^{-1} s_3 - s_1 s_3^{-1} & t_2 & t_1 & t_1 \\ t_2 & t_2 & t_1 & s_2^{-1} s_3 - s_2 s_3^{-1} \\ t_2 & t_2 & s_2^{-1} s_3 - s_2 s_3^{-1} & t_1 \end{pmatrix}
 \end{aligned}$$

**C.4. General case.** In Appendix C.2 we gave several deformed Cartan matrices of finite type, and in Appendix C.3 several deformed Cartan matrices of affine type. In fact Appendix C.2 contains all maximal submatrices of stable deformed affine Cartan matrices whose upper right and bottom left corner elements are both non zero. Appendix C.3 contains all non-stable deformed affine Cartan matrices appearing in our construction.

It is important to keep in mind that we immediately obtain many more examples by permuting  $s_1, s_2, s_3$ . Another set of examples is obtained by reading the data from right to left. More precisely, the matrices  $\hat{C}$  corresponding to  $X(c_0; c_1, \dots, c_\ell; c_{\ell+1})Y$  and to  $Y(c_{\ell+1}; c_\ell, \dots, c_1; c_0)X$  are related by conjugation by the matrix  $(\delta_{i+j, \ell})_{i, j=0}^\ell$ .

The matrices listed in Appendices C.2 and C.3 allow us to write a deformed affine Cartan matrix corresponding to arbitrary data  $X(c_0; c_1, \dots, c_\ell; c_{\ell+1})Y$ , since for larger  $\ell$  the deformed Cartan matrices stabilize. One has to follow the following procedure.

First, search the affine examples, keeping in mind the symmetries. If the matrix is found, stop. If the matrix is not in the list, conclude that  $\hat{C}$  is stable, that is  $C_{0, \ell} = C_{\ell, 0} = 0$ .

Second, find the listed matrix of the finite type and of the largest size corresponding to right most colors:  $(c_{\ell-i}, \dots, c_\ell; c_{\ell+1})Y$  (it is  $2 \times 2$  for all cases except  $Y = \text{D}$  when it is  $3 \times 3$ ). That gives the right bottom submatrix of  $\hat{C}$ .

Third, repeat for left most colors, that is, find the listed matrix of the finite type and of the largest size corresponding to  $X(c_0; c_1, \dots, c_i)$ . This gives the left upper submatrix.

The rest nonzero entries of the matrix are recovered from matrices of type A.

The final result is the superposition of matrices which are linked via diagonal entries.

For example, the matrix  $\hat{C}$  corresponding to  $\text{B}(2; 3, 1; 3)\text{C}$  is a superposition of the last matrix of type B in our list and of the second matrix of type C with necessary symmetries (the  $C_{1,1}$  entry  $t_2$  is

common):

$$\begin{pmatrix} s_2^{1/2} s_3 + s_2^{-1/2} s_3^{-1} & -s_2^{1/2} - s_2^{-1/2} & 0 \\ t_1 & t_2 & s_3^2 - s_3^{-2} \\ 0 & -1 & s_2 s_3^{-1} + s_2^{-1} s_3 \end{pmatrix}.$$

Similarly, matrix  $\hat{C}$  corresponding to  $\mathbf{B}(2; 3, 2, 1; 3)\mathbf{C}$  is a superposition of three: the fourth on type  $\mathbf{B}$  list, the second on type  $\mathbf{A}$  list and the last on type  $\mathbf{C}$  list. The first two share a common  $C_{1,1}$  entry  $t_1$  and the last two share a common  $C_{2,2}$  entry  $t_3$ .

$$\begin{pmatrix} s_2^{1/2} s_3 + s_2^{-1/2} s_3^{-1} & -s_3^{1/2} - s_3^{-1/2} & 0 & 0 \\ t_3 & t_1 & t_3 & 0 \\ 0 & t_1 & t_3 & -t_2(s_3 + s_3^{-1}) \\ 0 & 0 & -1 & s_1 s_3^{-1} + s_1^{-1} s_3 \end{pmatrix}.$$

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