

Super Odd-Sum Labeling of a Cube

What an Odd Cube!

Jyotirmoy Sarkar

We propose a new puzzle: Label the eight vertices of a cube using *distinct* integers between 0 and 12 (both inclusive) such that the *induced* labeling of each edge, given by the sum of the labels of its end points, causes the 12 edges to be labeled with distinct odd numbers 1, 3, . . . , 23.

Any solution to the puzzle is called a super odd-sum labeling. We deftly discover all super odd-sum labeling of the cube.

Introduction

We propose a puzzle that any secondary-school student can try to solve. We find the solutions efficiently. The proof that we have found all solutions is accessible to high school graduates.

1. The Puzzle

Statement of the Puzzle: Label the eight vertices of a cube using eight **distinct** values chosen from the set $S = \{0, 1, 2, \dots, 12\}$ in such a way that when each edge receives an induced label given by the sum of the labels of its two end vertices, the 12 edge-labels constitute the first 12 (positive) odd numbers 1, 3, 5, . . . , 23.

Such a labeling, if one exists, is called a **super odd-sum labeling** of a cube, because the edge labels constitute **distinct, consecutive** odd numbers. Note that we only label the vertices; thereafter the edge labels are **induced**. Also note that for an edge to receive label 1, we must label its two end vertices 0 and 1. Likewise, for an edge to receive label 23, we must label its two end vertices 12 and 11. Hence, we allow the vertex labels to come from the set $S = \{0, 1, 2, \dots, 12\}$, even though we need exactly eight of them.



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You get to choose *any* 8 labels you need. But do not choose the same label more than once.

Keywords

Graph labeling, parity, path, rotation and reflection symmetries, navigation-wheel-with- $2n$ -spokes, hypercube.

Stop reading and start solving the puzzle.

To derive optimal benefit from this paper, the reader should stop reading and start solving the puzzle. As an aid to solving the puzzle a reader may photocopy Figure 1.

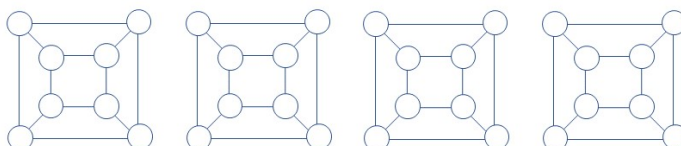


Figure 1. Label the vertices of a cube with eight *distinct* values chosen from $\{0, 1, \dots, 12\}$ such that when each edge receives the induced label obtained by adding the labels of its end vertices, the 12 edges receive distinct odd labels $1, 3, \dots, 23$.

Mathematicians demand an efficient solution, or a rigorous logical proof that no solution exists.

2. Solutions

Since there are $\binom{13}{8} = 1287$ ways to choose 8 elements out of S , and since for each such choice there are $7!/3! = 840$ ways to assign the chosen elements to the vertices (not distinguishing rotation and reflection symmetries), one may write a computer code to discover all super odd-sum labeling of a cube via a complete search. It is a good exercise for students studying computer programming. But students of mathematics ought to first seek and then demand all solutions (or at least their descriptions) using a more efficient manner, or to discover and write a rigorous logical proof that no such super odd-sum labeling exists. We satisfy such a legitimate demand from students of mathematics.

Dear reader, please do not peek at the solution.

In fact, we solve the puzzle by showing not just one super odd-sum labeling of a cube, but by listing four such labeling in Figure 3 towards the end of this article. Dear reader, please do not peek at the solution. If you have found one solution, well done! Now find the other three. If you have found none yet, don't give up. I have just assured you there are four solutions. However, if you do not trust me (it is okay to harbor a reasonable dose of skepticism), then go ahead and prove that there is no solution.

SPOILER ALERT

Drop this paper and get busy solving the puzzle.

Do not read any further lest you miss the joy of discovery. After copying Figure 1, drop this paper and get busy solving the puzzle. Return to the paper only after spending sufficient time solving the puzzle on your own or in collaboration with someone else.



Now we shall prove that the complete collection of all super odd-sum labeling of a cube consists of four solutions.

Theorem 1. *There are four super odd-sum labeling of a cube.*

Proof. For any super odd-sum labeling of a cube, the following three properties hold, thereby reducing the search space:

1. Since all edges receive an odd label, adjacent vertices must be given labels of opposite parity—one odd, one even. In particular, the edge labeled 1 must have adjacent vertices 0 and 1, and the edge labeled 23 must have adjacent vertices 12 and 11, as we have already mentioned earlier. Collectively, there must be four odd and four even labels for the eight vertices of the cube, with the parity alternating as one traces along the edges.
2. Each vertex of the cube contributes its label to the induced labeling of exactly three edges which are incident at that vertex. Therefore, three times the sum of labels of all vertices must equal the sum of all edge labels, or $1 + 3 + \dots + 23 = 144$. Hence, the eight vertex labels must add to 48.
3. To accommodate an edge labeled 3, we must have three vertices along a path P labeled either $(0, 1, 2)$ or $(1, 0, 3)$ in order. Without loss of generality, we assign labels $(0, 1, 2)$ to the bottom face of the cube at corners north-west, south-west, south-east in order; and we assign labels $(1, 0, 3)$ to the bottom face of the cube at corners south-west, north-west, north-east in order. Thus, 0 is always in bottom-north-west corner and 1 is in bottom-south-west corner. Likewise, to accommodate an edge labeled 21, we must have three vertices along a path Q labeled either $(12, 11, 10)$ or $(11, 12, 9)$ in order. How the two paths P and Q relate to each other is yet to be determined.

These properties help reduce the search space.

Property 1 says, "Adjacent vertices must be given labels of opposite parity."

Property 2 says, "The eight vertex labels must add to 48."

Property 3 says, "There is a path P labeled either $(0, 1, 2)$ or $(1, 0, 3)$, and a path Q labeled either $(12, 11, 10)$ or $(11, 12, 9)$."

Property 3 gives rise to four possible cases, depending on which labeling of P and Q are chosen. We study these cases separately. For each case, we ask, "What are the remaining two labels that will satisfy Properties 1 and 2?" These remaining two labels must have appropriate parity and a constant sum. The complete list of labels satisfying Properties 1–3 are given in Table 1, where

The pair (P, Q) can be chosen in four ways.

What are the remaining two labels?



parentheses indicate that the labels (being of opposite parity) must be placed uniquely and braces indicate that the labels (being of same parity) can be interchanged.

Table 1. For cases A–D, the remaining two labels that satisfy Properties 1 and 2

Labels of Path P	Labels of Path Q	
	(12, 11, 10)	(11, 12, 9)
(0, 1, 2)	{3, 9}	(3, 10)
	Case A {5, 7}	Case B (5, 8) (7, 6)
(1, 0, 3)	(2, 9)	{2, 10}
	Case C (4, 7) (6, 5)	Case D {4, 8}

It suffices to consider only 10 subsets of labels!

From Table 1 we learn that, instead of considering all $\binom{13}{8} = 1287$ possible subsets of labels, it suffices to consider only 10. (The preceding sentence deserves to end in a mark of exclamation; however, we forfeit it lest it be confused with the factorial sign.)

We ask some strategic questions—in succession.

Next, for each of the 10 (partially ordered) subsets of labels given in Table 1, we ask some strategic questions—in succession—to determine at which vertex we should place a particular label or set of labels. The answers to these questions are documented in Figure 2, where NA[u] denotes the placement is not allowed because edge label u is repeated. We will explain how to read Figure 2, after stating a convention and issuing an alert.

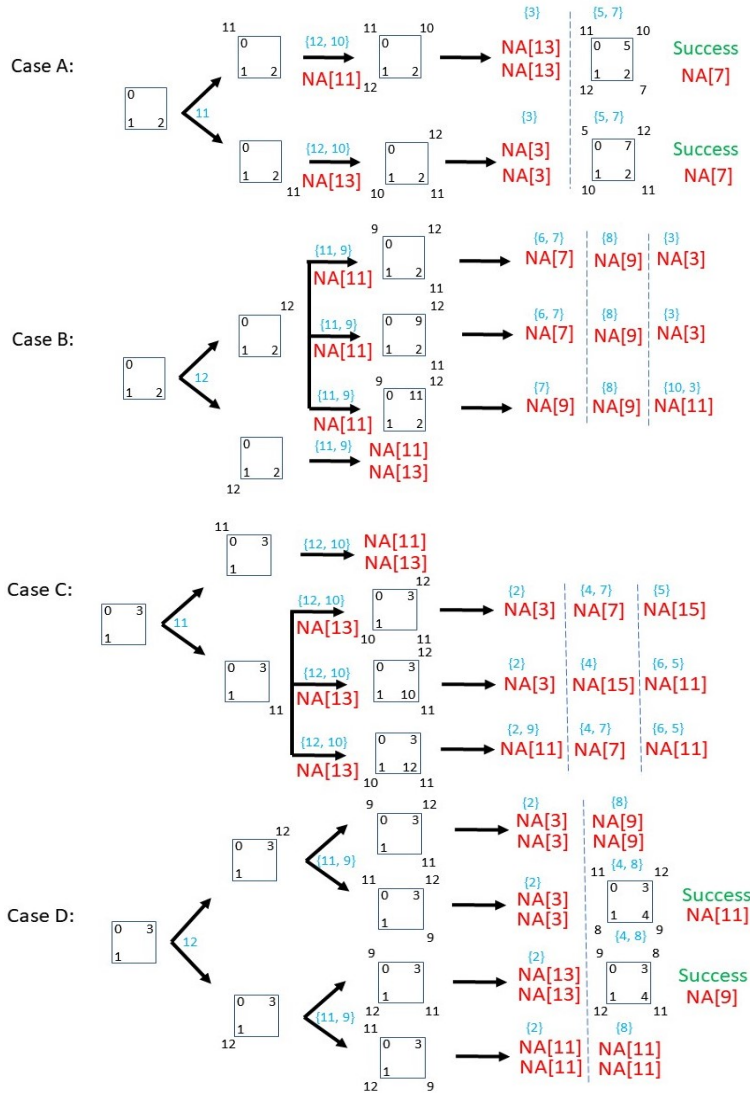
We have simplified the picture of a cube.

Convention: To save space, we have simplified the picture of a cube: The labels inside the depicted square go with the bottom square of the cube and the labels outside the depicted square go with the top square of the cube.

Where to place the central label of Q ? And where to place the peripheral labels?

Alert: While the labels form path P by design, to ensure that the labels also form path Q , we first determine where to place the central label of Q (keeping in mind its centrality on Q and its parity, there are two positions), and then we determine where to place the peripheral vertices (also called leaves) of Q . The permissible positions are shown in Figure 2.





Let us explain some branches of the trees in Figure 2, leaving the remaining branches to the reader.

1. Case A, Upper Branch: The central label 11 of Q is at the north-west corner of the upper square of the cube. Since the north-west corner of the lower square is already occupied by 0, we can place $\{10, 12\}$ only at north-east and south-west corners of the upper square. Only one such placement is shown; switching 10 and 12 is not allowed since then two edges would receive induced label

Figure 2. In each case A–D, we label the vertices: First we form path Q ; and then we use the remaining two labels shown in Table 1. We label as far as possible—until we reach a not allowed (NA[u]) state, or until we achieve a success.

On this branch, we check 5 (partial) arrangements.		11 (that is, it will cause NA[11]). The remaining two labels, according to Table 1, are either {3, 9} or {5, 7}. But label 3 is not allowed at either of the empty vertices because it will cause NA[3]. The only permissible placement of {5, 7} is shown as a success. In this successful configuration, switching labels 5 and 7 will cause NA[7].
On this branch, we check 12 (partial) arrangements.		2. Case B, Upper Branch: The central label 12 of Q is at the north-east corner of the upper square of the cube. We can place {9, 11} in three possible ways as shown; in each case, switching 9 and 11 will cause NA[11]. For each of these three subbranches, placing any of the three (ordered) pairs of remaining labels displayed in Table 1 causes NA[u], where u is documented in the figure.
Here, we check only 2 partial arrangements.		3. Case C, Upper Branch: The central label 11 of Q is at the north-west corner of the upper square of the cube. Since the north-west corner of the lower square is already occupied by 0, we must place {10, 12} only at north-east and south-west corners of the upper square. However, neither placement (by switching 10 and 12) is allowed since one causes NA[11] and the other NA[13].
On this branch, we check 8 arrangements.		4. Case D, Upper Branch: The central label 12 of Q is at the north-east corner of the upper square of the cube. Since the north-east corner of the lower square is occupied by 5, we can place {9, 11} in two possible ways as shown occupying north-west and south-east corners of the upper square. On the upper subbranch, when the remaining two labels are 2 and 10, it is impossible to place label 2; when the remaining two labels are 4 and 8, it is impossible to place label 8. On the lower subbranch, it is impossible to place label 2, but placement of labels 4 and 8 leads to a success as shown. In this successful configuration, switching labels 4 and 8 will cause NA[11].

We have logically exhausted all possible labeling.

Figure 2 shows two successful labeling for case A (where interchanging the last two labels not on P or Q causes NA[7]), two successful labeling for case D (where interchanging the last two labels not on P or Q causes NA[11] or N[9]), and no successful labeling for cases B and C. Since we have logically exhausted all possible labeling satisfying Properties 1–3 and found exactly four super odd-sum labeling, the proof is now complete. \square



The four successful solutions are extracted from Figure 2 and shown in Figure 3 for easy access. We urge the reader to check that the induced labels of the 12 edges do span $1, 3, 5, \dots, 23$.

Check the induced labels of the 12 edges.

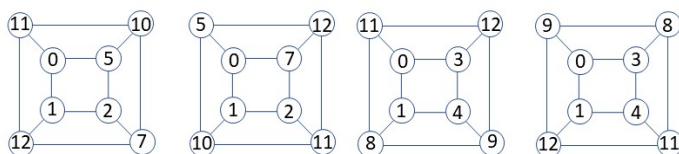


Figure 3. Four super odd-sum labeling of a cube

3. Afterthoughts

A naïve computer program may scan all $13!/5! = 51,891,840$ permutations of 13 elements taken 8 at a time to check if each cube labeling is super odd-sum. A smarter computer program, that can avoid duplicating permutations identical with respect to rotation and reflection transformations on the cube, will check $\binom{13}{8} \times 7!/3! = 1,081,080$ possible arrangements. In contrast, by using mathematical reasoning, we had to test only 54 (partial) labeling (in the proof we mentioned 27 labeling, leaving the other 27 to the reader to test.) to search for all super edge-odd labeling of a cube!

We mentioned 27 labeling, leaving the other 27 to the reader.

In all four super odd-sum labeling, the vertex labels actually chosen from S form a symmetric subset R of S ; that is, $x \in R$ implies $(12-x) \in R$. We say x and $(12-x)$ are mirror images of each other. Note that label 6 (which is its own mirror image) is never used in any solution. Therefore, eliminating label 6 and then subtracting 6 from all remaining labels, the puzzle takes on an alternative formulation:

Label 6 is its own mirror-image, and is never used.

Alternative Statement of the Puzzle: Label the eight vertices of a cube using eight distinct values chosen from the set $T = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$ in such a way that the 12 induced edge labels, given by the sum of the labels of the two end vertices, constitute the central 12 odd numbers $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11$.

You may choose any 8 distinct labels from T as you need.



Labeling the vertices and/or the edges of a graph is a popular topic in graph theory and combinatorics. Interested readers may read the book [1], the papers [2]–[5], the annually updated survey [6], and references therein.

Extend super odd-sum labeling to other graphs.

We invite the astute reader to extend the concept of super odd-sum labeling of a cube to other graphs such as (i) the navigation-wheel-and- $2n$ -spokes ($n \geq 3$), and (ii) the p -dimensional hypercube ($p \geq 4$). Then either find all (or at least one) such super odd-sum labeling, or prove that no such labeling exists.

Acknowledgement

I thank the 2020 IUPUI High School Math Contest Committee for soliciting contest problems and then demanding solution(s) with proof. I dedicate this paper to Professor George Gopen who opened my eyes to “see” the challenges a reader faces when reading, and then equipped me to adequately address these challenges even as I write.

Suggested Reading

- [1] F. Harary, *Graph Theory*, Narosa Publication House, New Delhi, 1988.
- [2] D. Ramya, R. Ponraj and P. Jeyanthi, Super mean labeling of graphs, *Ars Combinatoria*, Vol.112, pp.65–72, 2013.
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- [5] J. Rajeshni Golda, S. S. Sandhya and S. Somasundaram, Super contra harmonic mean labeling of graphs, *International Journal of Engineering Science and Mathematics*, Vol.7, No.6, pp.168–176, 2018.
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