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On the Construction of Kinematic Confidence Ellipsoids for Uncertain Spatial Displacements

Zihan Yu[1], **Qiaode Jeffrey Ge**[1], **Mark P. Langer**[2], **Mona Arbab**[3]

[1]Computational Design Kinematics Lab, Stony Brook University-SUNY, Stony Brook, NY 11794, USA

[2]Radiation Oncology Department, Indiana University, Indianapolis, IN 46202, USA

[3]Radiation Oncology Department, UT Southwestern Medical Center, Dallas, TX 75390, USA

Abstract

This paper deals with the problem of estimating confidence regions of a set of uncertain spatial displacements for a given level of confidence or probabilities. While a direct application of the commonly used statistic methods to the coordinates of the moving frame is straight-forward, it is also the least effective in that it grossly overestimate the confidence region. Based on the dual-quaternion representation, this paper introduces the notion of the kinematic confidence ellipsoids as an alternative to the existing method called rotation and translation confidence limit (RTCL). An example is provided to demonstrate how the kinematic confidence ellipsoids can be computed.

Keywords

spatial displacements; mean and variance; confidence level; confidence ellipsoids

1 Introduction

As an object in space has six degrees of freedom, a set of displacement data that captures variations and uncertainties in the spatial position and orientation of the object is a multivariate distribution from the viewpoint of statistics. Methods for estimating the mean, the variance, as well as its confidence region, i.e., the geometric range of uncertainties at a given confidence level, have found a number of applications that involve multivariate data fusion of position information of objects, such as SLAM in mobile robotics and autonomous vehicles [1, 2] and margin construction for planning target volume (PTV) design in image-guided radiotherapy [3–5].

This paper studies the computational-kinematic problem of determining the confidence regions from a set of spatial displacement that includes both translation and rotation uncertainties. The need and challenges to account for rotational margins for PTV construction in intensity modulated radiotherapy (IMRT) have been highlighted and extensively discussed in [4]. Recently, the planar kinematic problem of constructing

Qiaode.Ge@stonybrook.edu .

confidence regions from a set of planar displacements has been investigated in [6] using the planar quaternion formulation. The purpose of the current paper is to extend this work to spatial kinematics using dual quaternion formulation.

A spatial displacement can be decomposed into a translation of a point, $\mathbf{d} = (a, b, c)$, and a 3D rotation about that point, which can be further decomposed into three elemental rotations in terms of three Euler angles $\boldsymbol{\Omega} = (\alpha, \beta, \gamma)$. The set of six parameters $(\mathbf{d}, \boldsymbol{\Omega})$ has been used in [5] to develop the RTCL (rotation and translation confidence limit) method for the construction of a 6D ellipsoid as the confidence region for spatial displacements. The purpose of the current paper is to extend the method for constructing confidence regions from planar quaternion to dual quaternion formulation. This leads to kinematic covariance matrices and kinematic confidence ellipsoids that take into account the subgroup structure of $SE(3)$. Consequently, the resulting confidence region can be significantly smaller and thus more effective than the RTCL method.

The organization of the rest of the paper as follows. Section 2 summarizes the basics of dual quaternions as a representation of spatial displacements. Section 3 presents the dual quaternion approach to the computation of the mean of a set of uncertain spatial displacements. Section 4 introduces the notion of kinematic covariance matrices that characterizes the multivariate deviations relative to the mean displacement. Section 5 introduces and studies the notion of kinematic confidence ellipsoids. An example is provided in the end to show the kinematic confidence ellipsoids for a given set of clinical data in radiotherapy.

2 Screw Displacement and Dual Quaternion

Just as a planar displacement is equivalent to a rotation about its pole, a spatial displacement is equivalent to a screw displacement about a fixed line, called the screw axis, with rotation angle θ about and translation distance h along the screw axis. The screw axis is represented by the so-called Plücker line coordinates, which consists of a unit vector $\mathbf{s} = (s_x, s_y, s_z)$ representing the direction of the screw axis and another vector $\mathbf{s}^0 = (s_x^0, s_y^0, s_z^0)$ perpendicular to $\mathbf{s}$, i.e., $\mathbf{s} \cdot \mathbf{s}^0 = 0$. The pair of Plücker vectors can be combined to form a dual vector $\hat{\mathbf{s}} = \mathbf{s} + \epsilon \mathbf{s}^0$, where ϵ is the dual unit with the property $\epsilon^2 = 0$. As $\hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = \mathbf{s} \cdot \mathbf{s} + 2\epsilon \mathbf{s} \cdot \mathbf{s}^0 = 1$, $\hat{\mathbf{s}}$ is called a *unit dual vector*.

Similarly, a dual angle can be defined by combining rotation angle and translation distance as $\hat{\theta} = \theta + \epsilon h$. In this way, a screw displacement can be represented by the dual quaternion (see [7,8]):

$$\widehat{\mathbf{Q}} = \hat{\mathbf{s}} \sin\frac{\hat{\theta}}{2} + \cos\frac{\hat{\theta}}{2}, \tag{1}$$

whose dual-number coordinates, $\hat{\mathbf{Q}} = \left(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3, \hat{Q}_4\right)$, are given by

$$\hat{Q}_1 = \hat{s}_x \sin\frac{\hat{\theta}}{2},\ \hat{Q}_2 = \hat{s}_y \sin\frac{\hat{\theta}}{2},\ \hat{Q}_3 = \hat{s}_z \sin\frac{\hat{\theta}}{2},\ \hat{Q}_4 = \cos\frac{\hat{\theta}}{2}, \tag{2}$$

where

$$\sin\frac{\hat{\theta}}{2} = \sin\frac{\theta}{2} + \epsilon\frac{h}{2}\cos\frac{\theta}{2},\ \cos\frac{\hat{\theta}}{2} = \cos\frac{\theta}{2} - \epsilon\frac{h}{2}\sin\frac{\theta}{2}. \tag{3}$$

Details on sinusoidal functions of dual angles can be found in [7]. The conjugate of $\hat{\mathbf{Q}}$ is simply $\hat{\mathbf{Q}}^* = -\hat{\mathbf{s}}\sin\frac{\hat{\theta}}{2} + \cos\frac{\hat{\theta}}{2}$ and it can be used to compute the norm squared of a dual quaternion as

$$\left|\hat{\mathbf{Q}}\right|^2 = \hat{\mathbf{Q}}^*\hat{\mathbf{Q}} = \hat{\mathbf{Q}}\hat{\mathbf{Q}}^* = \hat{Q}_1^2 + \hat{Q}_2^2 + \hat{Q}_3^2 + \hat{Q}_4^2, \tag{4}$$

When $\left|\hat{\mathbf{Q}}\right| = 1$, Eq. (1) is said to define a *unit dual quaternion*, which can be considered as defining a point on the unit dual hypersphere [8]. Let $\hat{Q}_i = Q_i + \epsilon Q_i^0$ and separate the real and dual parts in (4), it can be seen that a unit dual quaternon must satisfy:

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = 1, \tag{5}$$

and

$$Q_1Q_1^0 + Q_2Q_2^0 + Q_3Q_3^0 + Q_4Q_4^0 = 0. \tag{6}$$

Eq. (6) is also said to define the Study quadric [9]. Later in this paper, when introducing the covariance matrix, the following

$$\hat{\mathbf{R}} = \hat{\mathbf{s}}\tan\frac{\hat{\theta}}{2} \tag{7}$$

is used instead of $\hat{\mathbf{Q}}$. In kinematics literature, the components of $\hat{\mathbf{Q}}$ are also known as *dual Euler parameters* and those of $\hat{\mathbf{R}}$ are called *dual Rodriques parameters* [8, 10–12].

3 Mean and Variance of a Set of Uncertain Spatial Displacements

Given a set of uncertain spatial displacements, computing its mean and variance is the first step in statistical analysis of the displacement data. In [5], this computation is carried out by treating each of the six displacement parameters in $(\mathbf{d}, \boldsymbol{\Omega})$ separately. Consequently, the resulting confidence region is always six dimensional regardless whether underlying the displacement data belong to any subgroups of $SE(3)$, whose dimensions could range from one to six. Furthermore, the underlying assumption in this approach is that the relative displacement between two given displacements, say $(\mathbf{d}_i, \boldsymbol{\Omega}_i)(i = 1,2)$, is simply the difference between them, i.e., $(\mathbf{d}_2 - \mathbf{d}_1, \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1)$, which is in general not true.

Ge et al. in [13] used the dual-quaternion representation to formulate the problem of computing a mean spatial displacement as one that minimizes a multivariate measure of deviations based on relative displacements. Two dual-quaternion based methods were presented in [13]. This paper adopts the following method for computing the mean displacement.

Given a set of unit dual quaternions $\hat{\mathbf{Q}}_i\left(i = 1,2,\cdots,n\right)$ representing a set of uncertain displacements, the dual quaternion $\hat{\mathbf{V}}$ that represents the mean displacement is obtained using the following formula:

$$\widehat{\mathbf{V}} = \frac{1}{\widehat{S}}\sum_{i=1}^{n}\widehat{\mathbf{Q}}_i, \text{ where } \widehat{S} = \left|\sum_{i=1}^{n}\widehat{\mathbf{Q}}_i\right|. \tag{8}$$

$\hat{S}$ is the dual-number norm of the sum of dual quaternions $\hat{\mathbf{Q}}_i\left(i = 1,2,\cdots,n\right)$ so that the resulting mean $\hat{\mathbf{V}}$ is a unit dual quaternion.

To illustrate the difference between the dual-quaternion based method for computing the mean displacement and that based on $(\mathbf{d}, \boldsymbol{\Omega})$ representation, we now consider the special case of a set of screw displacements about the same screw axis $\hat{\mathbf{s}}$ with various dual angles $\hat{\theta}_i\left(i = 1,2,\cdots,n\right)$. It follows from (1) and (8) that

$$\widehat{\mathbf{V}} = \hat{\mathbf{s}}\sin\frac{\hat{\theta}_V}{2} + \cos\frac{\hat{\theta}_V}{2}, \tag{9}$$

where

$$\sin\frac{\hat{\theta}_V}{2} = \frac{1}{\widehat{S}}\sum_{i=1}^{n}\sin\frac{\hat{\theta}_i}{2}, \ \cos\frac{\hat{\theta}_V}{2} = \frac{1}{\widehat{S}}\sum_{i=1}^{n}\cos\frac{\hat{\theta}_i}{2}. \tag{10}$$

Therefore, the resulting mean displacement is a screw displacement about the same screw axis $\hat{S}$. This is in general not the case, however, if we were to compute the mean of $(\mathbf{d}_i, \boldsymbol{\Omega}_i)$ associated with the same set of screw displacements.

4 Kinematic Covariance Matrices

For a set of one-dimensional data, the variance is the average of the squared difference from its mean. The covariance matrix generalizes the notion of variance to multiple dimensions. As spatial displacements are represented by dual quaternions, in this section, we seek to extend the notion of covariance matrix to the dual quaternion formulation.

For a set of unit dual quaternions $\hat{\mathbf{Q}}_i(i = 1, 2, \cdots, n)$ and its mean $\hat{\mathbf{V}}$, the first step is to determine the deviation between $\hat{\mathbf{Q}}_i$ and the mean $\hat{\mathbf{V}}$. Kinematically, this means to compute the relative displacement between them. Let $\hat{\mathbf{W}}_i$ be the dual quaternion that represents the relative displacement. Then $\hat{\mathbf{W}}_i$ is given by the dual quaternion product

$$\widehat{\mathbf{W}}_i = \widehat{\mathbf{V}}^* \widehat{\mathbf{Q}}_i, \tag{11}$$

where $\hat{\mathbf{V}}^*$ is the conjugate of $\hat{\mathbf{V}}$. In matrix form, the dual quaternion product can be expressed as [8]:

$$\widehat{\mathbf{W}}_i = \left[\widehat{V}^*\right] \widehat{\mathbf{Q}}_i \tag{12}$$

where

$$\left[\widehat{V}^*\right] = \begin{bmatrix} \widehat{V}_4 & \widehat{V}_3 & -\widehat{V}_2 & -\widehat{V}_1 \\ -\widehat{V}_3 & \widehat{V}_4 & \widehat{V}_1 & -\widehat{V}_2 \\ \widehat{V}_2 & -\widehat{V}_1 & \widehat{V}_4 & -\widehat{V}_3 \\ \widehat{V}_1 & \widehat{V}_2 & \widehat{V}_3 & \widehat{V}_4 \end{bmatrix}. \tag{13}$$

In the matrix form above, the symbols $\hat{\mathbf{W}}_i$ and $\hat{\mathbf{Q}}_i$ are considered as four-dimensional dual-number vectors as opposed to dual quaternions. For the rest of the paper, we use the same symbol for both interchangeably wherever appropriate.

As the set of all unit dual quaternions defines a unit dual hypersphere, one may project points represented by four-dimensional dual-number vectors $\hat{\mathbf{W}}_i = \left(\hat{W}_{i1}, \hat{W}_{i2}, \hat{W}_{i3}, \hat{W}_{i4}\right)$ onto its dual tangent hyperplane $\hat{R}_{i4} = 1$ using the dual Rodriques parameters $\hat{\mathbf{R}}_i = \left[\hat{R}_{i1} \ \hat{R}_{i2} \ \hat{R}_{i3}\right]^T$:

$$\hat{R}_{i1} = \frac{\widehat{W}_{i1}}{\widehat{W}_{i4}}, \ \hat{R}_{i2} = \frac{\widehat{W}_{i2}}{\widehat{W}_{i4}}, \ \hat{R}_{i3} = \frac{\widehat{W}_{i3}}{\widehat{W}_{i4}}.$$

(14)

After separating the real and dual parts of the resulting dual vector, i.e., $\hat{\mathbf{R}}_i = \mathbf{R}_i + \epsilon\mathbf{R}_i^0\left(i = 1, \ldots, n\right)$, one can define two 3×3 covariance matrices:

$$[\Sigma_R] = \frac{1}{n}[\mathbf{R}_1\ \ \mathbf{R}_2\cdots\ \ \mathbf{R}_n]\begin{bmatrix}\mathbf{R}_1^T\\ \mathbf{R}_2^T\\ \vdots\\ \mathbf{R}_n^T\end{bmatrix},\ \left[\Sigma_R^0\right] = \frac{1}{n}[\mathbf{R}_1^0\mathbf{R}_2^0\cdots\mathbf{R}_n^0]\begin{bmatrix}\mathbf{R}_1^{0T}\\ \mathbf{R}_2^{0T}\\ \vdots\\ \mathbf{R}_n^{0T}\end{bmatrix}.$$

(15)

The covariance matrix $[\Sigma_R]$ corresponds to variances in the directions of rotation and the angles of rotation in the relative displacement data with respect to the mean displacement $\hat{\mathbf{V}}$, while the matrix $\left[\Sigma_R^0\right]$ captures the variance in the locations of the screw axes as well as the translations along the screw axes.

5 Kinematic Confidence Ellipsoids

Let $\lambda_i, \lambda_i^0\left(i = 1,2,3\right)$ denote the eigenvalues of $[\Sigma_R]$ and $\left[\Sigma_R^0\right]$, respectively. They correspond variances in the principal directions, which are defined by their respective eigenvectors. It follows that they define two ellipsoids:

$$\frac{r_1^2}{\lambda_1} + \frac{r_2^2}{\lambda_2} + \frac{r_3^2}{\lambda_3} \leq \alpha^2,\ \frac{\left(r_1^0\right)^2}{\lambda_1^0} + \frac{\left(r_2^0\right)^2}{\lambda_2^0} + \frac{\left(r_3^0\right)^2}{\lambda_3^0} \leq \alpha^2,$$

(16)

where $\mathbf{r} = \left(r_1, r_2, r_3\right)$ and $\mathbf{r}^0 = \left(r_1^0, r_2^0, r_3^0\right)$ denote points on each of the two ellipsoids, α is the scaling coefficient associated with a confidence level and degree of freedom (DOF) of the multivariate distribution [5] [14]. For a confidence level of 95% and three DOF data, $\alpha^2 = 7.815$. For each pair of $\mathbf{r}$ and $\mathbf{r}^0$ obtained from the two ellipsoids, the corresponding dual quaternions of relative displacements can be found by reversing the equation (14).

6 Example

In this section, we use a set of clinical data (Table 1) for spatial displacements to illustrate how the kinematic confidence ellipsoids can be computed.

1. Convert the displacement data $\left(\mathbf{d}_i, \boldsymbol{\Omega}_i\right)\left(i = 1,2, \cdots, 7\right)$ to dual quaternions $\hat{\mathbf{Q}}_i$. The results are shown in Table 2. The mean displacement, $\hat{\mathbf{V}}$, is then computed and appended to the end of Table 2.
2. Use Eq. (11) to compute dual quaternions $\hat{\mathbf{W}}_i$ that describe the relative displacements from $\hat{\mathbf{Q}}_i$ to $\hat{\mathbf{V}}$. After that, the dual Rodriques parameters are obtained using Eq. (14). The results are shown in Tables 3 and 4, respectively.

3. Separate the real and dual parts of the dual Rodriques parameters and then compute the two kinematic variance matrices, $[\Sigma_R]$ and $[\Sigma_R^0]$, according to (15). After that, compute the eigenvalues λ_i, λ_i^0 of $[\Sigma_R]$ and $[\Sigma_R^0]$, respectively. The results are shown in Table 5.

4. Construct two kinematic confidence ellipsoids using Eq. (16) and $\alpha^2 = 7.815$. The two ellipsoids are shown in Figures 1 and 2, respectively. As can be seen, the variance in r_1 and r_1^0 directions are significantly less than the other directions.

Conclusions

In this paper we presented a dual-quaternion formulation for constructing the confidence region for a set of uncertain spatial displacements. This leads to the notion of kinematic covariance matrices and kinematic confidence ellipsoids. Future work is to expand this work to study the swept volumes resulting from the kinematic confidence ellipsoids based the clinical data of tumor shapes and their displacements.

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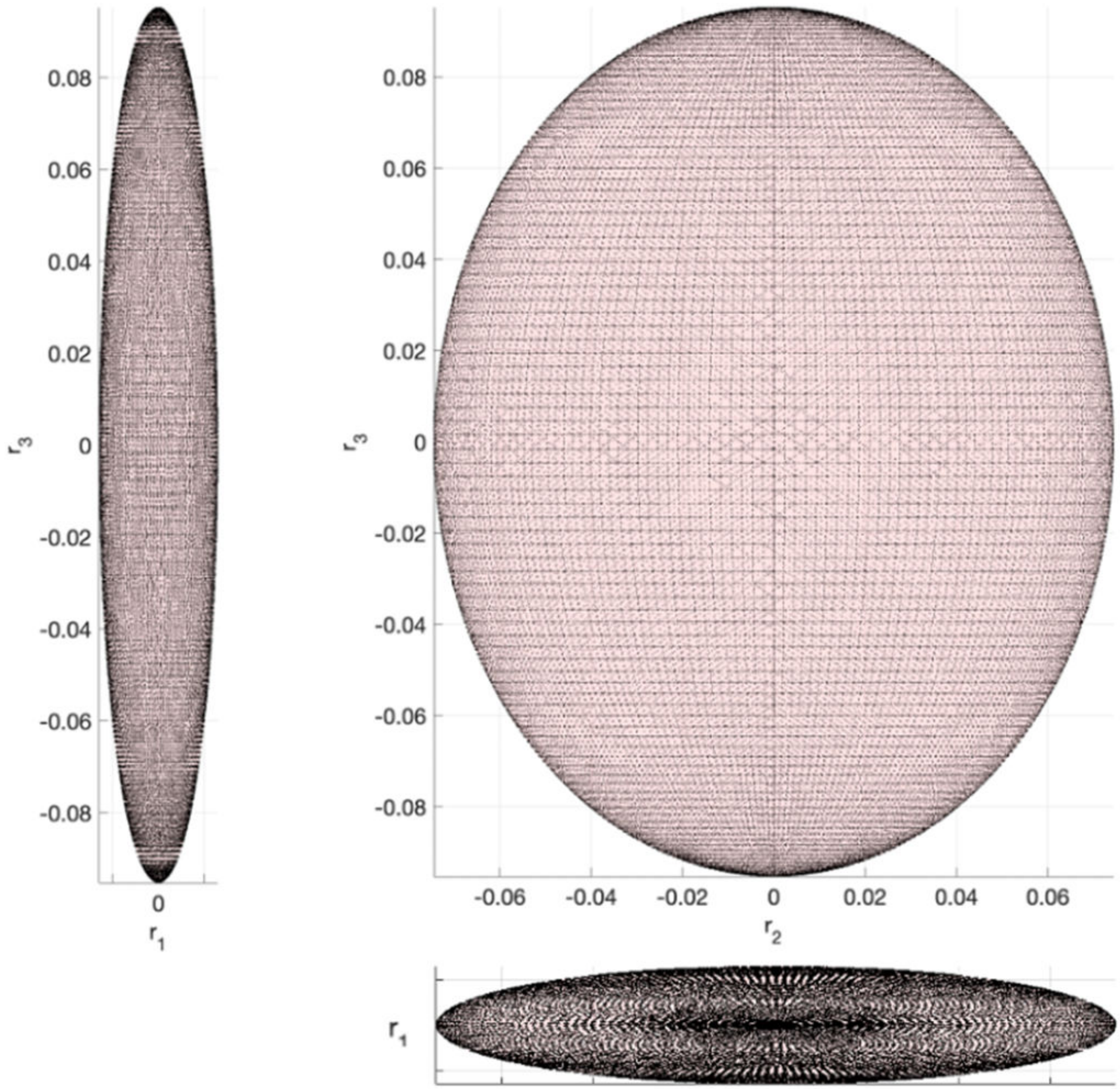

Fig. 1.
Kinematic confidence ellipsoid $[\Sigma_R]$ in three orthographic views, i.e., on the r_1r_3, r_2r_3 and r_1r_2 planes.

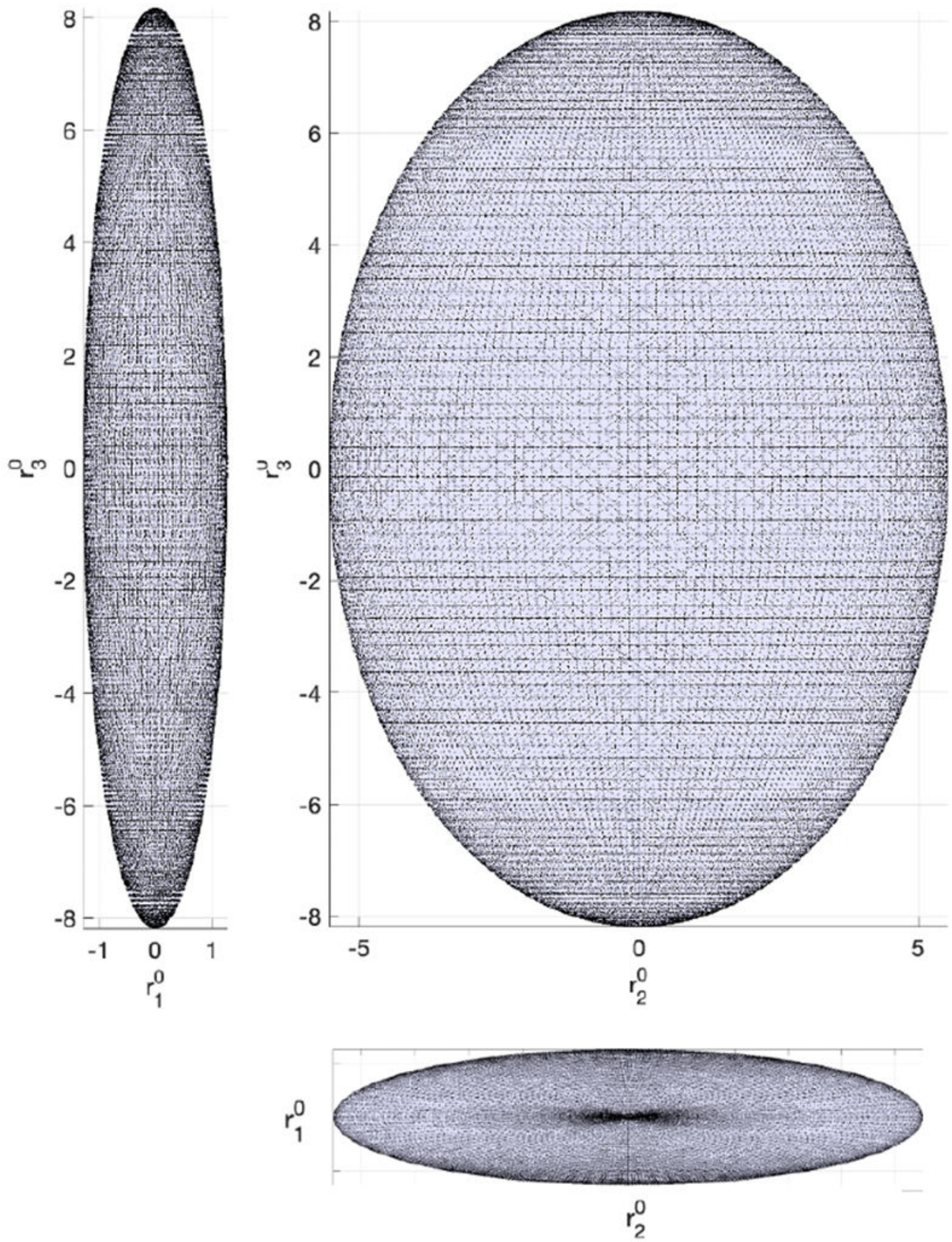

Fig. 2.

Kinematic confidence ellipsoid $[\Sigma_R^0]$ in three orthographic views, i.e., on the $r_1^0 r_3^0$, $r_2^0 r_3^0$ and $r_1^0 r_2^0$ planes.

Table 1.

Seven spatial displacements M_i, where translations $\mathbf{d}_i = (a_i, b_i, c_i)$ are in millimeter and rotations $\boldsymbol{\Omega}_i = (\alpha_i, \beta_i, \gamma_i)$ are in degree.

M_i	a_i	b_i	c_i	α_i	β_i	γ_i
1	2.60	−5.20	3.60	0.80	−0.60	−1.20
2	0.50	−0.30	10.10	3.60	−0.08	−0.20
3	−0.90	0.40	3.00	0.90	−0.30	0.40
4	−3.40	3.60	1.90	0.50	−0.30	1.60
5	−5.30	5.70	0.40	0.50	0.40	2.30
6	2.90	−6.90	3.60	0.70	−0.30	−1.80
7	2.60	−7.50	1.00	−0.40	−0.40	−1.50

Table 2.

The dual quaternions form of input displacements and the resulted average. The real part $\mathbf{Q}_i$ and $\mathbf{V}$ are to be scaled by a factor of 10^{-1}.

$\hat{\mathbf{Q}}_i$	Q_{i1}	Q_{i2}	Q_{i3}	Q_{i4}	Q_{i1}^0	Q_{i2}^0	Q_{i3}^0	Q_{i4}^0
1	0.0693	−0.0531	−0.1051	9.9991	1.3368	−2.5736	1.8109	−0.0039
2	0.3141	−0.0075	−0.0177	9.9950	0.2539	0.0091	5.0520	0.0010
3	0.0786	−0.0259	0.0347	9.9996	−0.4454	0.2133	1.4995	−0.0011
4	0.0440	−0.0256	0.1395	9.9989	−1.6723	1.8277	0.9463	−0.0012
5	0.0429	0.0358	0.2008	9.9978	−2.5929	2.9035	0.1782	−0.0028
6	0.0607	−0.0271	−0.1572	9.9985	1.5089	−3.4158	1.8167	0.0101
7	−0.0354	−0.0344	−0.1308	9.9990	1.3506	−3.7344	0.4822	−0.0018
$\hat{\mathbf{V}}_i$	V_1	V_2	V_3	V_4	V_1^0	V_2^0	V_3^0	V_4^0
V	0.0820	−0.0197	−0.0051	9.9996	−0.0053	−0.0974	0.2406	0.0000

Table 3.

The relative dual quaternions. The real part $\mathbf{W}_i$ is to be scaled by a factor of 10^{-1}.

$\hat{\mathbf{W}}_i$	$\boldsymbol{W}_{i1}$	$\boldsymbol{W}_{i2}$	$\boldsymbol{W}_{i3}$	$\boldsymbol{W}_{i4}$	$\boldsymbol{W}^0_{i1}$	$\boldsymbol{W}^0_{i2}$	$\boldsymbol{W}^0_{i3}$	$\boldsymbol{W}^0_{i4}$
1	−0.0126	−0.0326	−0.1003	9.9994	1.3395	−2.4901	1.5526	0.0091
2	0.2321	0.0122	−0.0120	9.9973	0.2496	0.0724	4.8150	−0.0001
3	−0.0035	−0.0065	0.0397	9.9999	−0.4426	0.3005	1.2606	−0.0050
4	−0.0383	−0.0070	0.1445	9.9989	−1.6686	1.9192	0.7179	−0.0154
5	−0.0395	0.0538	0.2063	9.9976	−2.5892	3.0017	−0.0432	−0.0255
6	−0.0211	−0.0062	−0.1522	9.9988	1.5110	−3.3326	1.5517	0.0247
7	−0.1172	−0.0137	−0.1260	9.9985	1.3552	−3.6425	0.2134	0.0136

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Table 4.

The dual Rodriques parameters $\hat{\mathbf{R}}_i$. The real part $\mathbf{R}_i$ is to be scaled by a factor of 10^{-1}.

$\hat{\mathbf{R}}_i$	R_{i1}	R_{i2}	R_{i3}	R_{i1}^0	R_{i2}^0	R_{i3}^0
1	−0.0126	−0.0326	−0.1003	1.3396	−2.4902	1.5527
2	0.2322	0.0122	−0.0120	0.2497	0.0724	4.8163
3	−0.0035	−0.0065	0.0397	−0.4426	0.3005	1.2606
4	−0.0383	−0.0070	0.1445	−1.6688	1.9194	0.7182
5	−0.0395	0.0538	0.2064	−2.5899	3.0025	−0.0426
6	−0.0211	−0.0062	−0.1522	1.5112	−3.3330	1.5523
7	−0.1172	−0.0137	−0.1260	1.3555	−3.6430	0.2136

Table 5.

Kinematic covariance matrices and their corresponding eigenvalues. The numbers in the table below are to be scaled by a factor of 10^{-6}.

$[\boldsymbol{\Sigma}_R]$			$[\boldsymbol{\Sigma}_R^0]$		
101.83	4.47	3.75	2.24	−3.49	0.61
4.47	6.32	22.30	−3.49	6.20	−1.12
3.75	22.30	163.30	0.61	−1.12	4.31
λ_1	λ_2	λ_3	λ_1^0	λ_2^0	λ_3^0
3.05	101.69	166.70	0.21	3.93	8.60