LANDAU–GINZBURG MIRROR, QUANTUM DIFFERENTIAL EQUATIONS AND *qKZ* DIFFERENCE EQUATIONS FOR A PARTIAL FLAG VARIETY

VITALY TARASOV° AND ALEXANDER VARCHENKO*

*Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599-3250, USA

°Department of Mathematical Sciences, Indiana University – Purdue University Indianapolis
402 North Blackford St, Indianapolis, IN 46202-3216, USA

Key words: Dynamical differential equations, *qKZ* difference equations, quantum differential equations, q- hypergeometric solutions

2010 Mathematics Subject Classification: 14N35, 53D45, 14D05, 33C70

Abstract. We consider the system of quantum differential equations for a partial flag variety and construct a basis of solutions in the form of multidimensional hypergeometric functions, that is, we construct a Landau–Ginzburg mirror for that partial flag variety. In our construction, the solutions are labeled by elements of the K-theory algebra of the partial flag variety.

To establish these facts we consider the equivariant quantum differential equations for a partial flag variety and introduce a compatible system of difference equations, which we call the *qKZ* equations. We construct a basis of solutions of the joint system of the equivariant quantum differential equations and *qKZ* difference equations in the form of multidimensional hypergeometric functions. Then the facts about the non-equivariant quantum differential equations are obtained from the facts about the equivariant quantum differential equations by a suitable limit.

Analyzing these constructions we obtain a formula for the fundamental Levelt solution of the quantum differential equations for a partial flag variety.

Contents

1. Introduction 2
2. Dynamical and *qKZ* equations 6
2.1. Notations 6
2.2. Difference *qKZ* equations 6
2.3. Differential dynamical equations 7
3. Limit $h \to \infty$ 7
3.1. Limit of *qKZ* operators 7
3.2. Dynamical operators $X_1^\circ, \dots, X_N^\circ$ 9
3.3. Limit of dynamical Hamiltonians 10
3.4. Limiting equations 11
4. Hypergeometric solutions of the dynamical and *qKZ* equations 11

°*E-mail*: vtarasov@iupui.edu, supported in part by Simons Foundation grants 430235, 852996
**E-mail*: anv@email.unc.edu, supported in part by NSF grants DMS-1665239, DMS-1954266

This is the author's manuscript of the article published in final edited form as:

Tarasov, V., & Varchenko, A. (2023). Landau–Ginzburg mirror, quantum differential equations and qKZ difference equations for a partial flag variety. Journal of Geometry and Physics, 184, 104711. https://doi.org/10.1016/j.geomphys.2022.104711

4.1. Weight functions $W_I(\boldsymbol{t};\boldsymbol{z};h)$ 11
4.2. Master function 13
4.3. Jackson integrals 13
4.4. Solutions of the dynamical and qKZ equations 15
4.5. Proof of Theorem 4.9 17
4.6. Solutions of the dynamical and qKZ equations parametrized by Laurent polynomials 20
4.7. Levelt fundamental solution 22
4.8. Proofs of Theorem 4.21 and Proposition 4.22 23
4.9. The map $Б_{\boldsymbol{\lambda}}$ 26
4.10. Example $\boldsymbol{\lambda}=(1,n-1)$ 27
5. Limit $h\to\infty$ for solutions of the dynamical and qKZ equations 33
5.1. Limiting weight functions $W_I^\circ(\boldsymbol{t};\boldsymbol{z})$ 33
5.2. Limiting master function 34
5.3. Solutions of the limiting dynamical and qKZ equations 35
5.4. Solutions parametrized by Laurent polynomials 38
5.5. Levelt fundamental solution 40
5.6. The map $Б^\circ_{\boldsymbol{\lambda}}$ 42
5.7. Example $\boldsymbol{\lambda}=(1,n-1)$ 43
6. Equations for partial flag varieties 48
6.1. Equivariant cohomology of partial flag varieties 48
6.2. Stable envelope map 50
6.3. Quantum multiplication 52
6.4. Differential and difference equations 52
6.5. Solutions with values in cohomology 53
6.6. The map $Б^H_{\boldsymbol{\lambda}}$ 56
6.7. The nonequivariant case $\boldsymbol{z}=0$ 58
6.8. Topological-enumerative morphism and J-function 61
6.9. Example $\boldsymbol{\lambda}=(1,n-1)$, $\mathcal{F}_{\boldsymbol{\lambda}}=\mathbb{CP}^{n-1}$ 63
Appendix A. Polynomial identities 69
Appendix B. Schubert polynomials 71
Appendix C. Proofs of Lemmas 5.2 and 5.3 72
Appendix D. Stirling's formula 74
Appendix E. Polynomiality of solutions 74
References 76

1. Introduction

The genus zero Gromov–Witten invariants of a partial flag variety V answer enumerative questions about rational curves in V and are put together in various ways to define rich mathematical structures such as the quantum cohomology algebra and the system of quantum differential equations. Those structures are part of the so-called 'A-model' of V. Landau–Ginzburg mirror symmetry seeks to describe these structures in terms of a mirror dual 'B-model' associated with V. The way originated by A. Givental in [Gi1, Gi2] seeks to encode the data from the A-model by oscillating integrals of a superpotential function W, which is defined on a 'mirror dual' affine variety V^o. In particular, these oscillating integrals are supposed to give all solutions of the system of quantum differential equations.

This Landau–Ginzburg mirror symmetry has been established by A. Givental for full flag varieties and projective spaces in [Gi1, Gi2, GKi]. The construction for Grassmannians has been performed by R. J. Marsh and K. Rietsch in [MR]. A standard problem, related to generalizations of Givental's approach, is to check that the number of critical points of the superpotential equals the dimension of the cohomology algebra of V, and then check that the oscillating integrals, which are intrinsically labeled by the critical points, indeed generate a basis of solutions of the system of quantum differential equations. While the first part is of algebraic nature, the second part is of global topological nature (to determine integration cycles exiting critical points). For example, in [MR] a superpotential for a Grassmannian was constructed, and it was shown that the number of its critical points is the correct one, but the part that the oscillating integrals indeed give a basis of solutions was not addressed.

In this paper, we suggest a new 'hypergeometric Landau–Ginzburg mirror symmetry model' for a partial flag variety V and construct the full set of solutions of the system of quantum differential equation of V in the form of multidimensional hypergeometric functions. In these functions the role of Givental's exponential of the superpotential is played by what we call the master function, which is the product of Gamma functions multiplied by the exponential of a linear function, and the role of the mirror dual V^o is played by the complement in an affine space to the set of all poles of the product of Gamma functions composing the master function. In particular, our V^o *is not* an algebraic variety, but a complex analytic variety.

It is interesting to note that in our construction, the solutions of the system of quantum differential equations of V are naturally labeled by elements of the K-theory algebra of V. In particular, that observation suggests that the monodromy and Stokes phenomenon of the system of quantum differential equations of V may be described in terms of the K-theory algebra of V, in accordance with the philosophy of B. Dubrovin, see [D1, D2, CDG, TV6, TV7, TV2, CV].

Consider the example of the m-dimensional complex projective space $V = \mathbb{CP}^m$. In the body of the paper, this example corresponds to the case $n = m+1$, $N = 2$, $\boldsymbol{\lambda} = (1, m)$. The cohomology algebra is $H^*(V;\mathbb{C}) = \mathbb{C}[x]\big/\langle x^{m+1}\rangle$, where x is the first Chern class of the tautological line bundle. The quantum multiplication $*_p$ on $H^*(V;\mathbb{C})$ depends on a parameter p,

$$x^i *_p x^j = x^{i+j} \quad \text{for } i+j \leqslant m\,, \qquad x^i *_p x^j = p\, x^{i+j-m-1} \quad \text{for } i+j > m\,,$$

$i, j = 0, \dots, m$. The parameter p corresponds to p_2/p_1 in the body of the paper. The quantum differential equation is

$$-\,\kappa p \frac{\partial I}{\partial p} \,=\, x *_p I\,, \tag{1.1}$$

where κ is the parameter of the differential equation and I is the unknown $H^*(V;\mathbb{C})$-valued function. The K-theory algebra is $K(V;\mathbb{C}) = \mathbb{C}[X, X^{-1}]\big/\big\langle (X-1)^{m+1}\big\rangle$, where X is the class of the tautological line bundle.

The $H^*(V;\mathbb{C})$-valued weight function is $W(t,x) = \sum_{i=0}^m t^{m-i} x^i$. The master function is

$$\Phi(t,p,\kappa) \,=\, (\kappa^{m+1}/p)^{t/\kappa} \big(\Gamma(t/\kappa)\big)^{m+1}.$$

For a univariate Laurent polynomial $P(X)$, define

$$\Psi_P(p,\kappa) \,=\, \kappa^{-1}\sum_{r=0}^{\infty} \mathrm{Res}_{t=-r\kappa}\bigl(\Phi(t,p,\kappa)\,P(e^{2\pi\sqrt{-1}\,t/\kappa})\,W(t,x)\bigr)\,.$$

The function $\Psi_P(p,\kappa)$ depends only on the class of P in $K(V;\mathbb{C})$. Alternatively, $\Psi_P(p,\kappa)$ can be written as an integral over an appropriate contour C encircling the poles of $\Phi(t,p,\kappa)$ in the positive direction,

$$\Psi_P(p,\kappa) \,=\, \frac{1}{2\pi\sqrt{-1}}\int_C \Phi(t,p,\kappa)\,P(e^{2\pi\sqrt{-1}\,t/\kappa})\,W(t,x)\,dt\,.$$

For instance, one can take the parabola $C = \{\,\kappa\bigl(1-s^2+s\sqrt{-1}\,\bigr)\ \mid\ s\in\mathbb{R}\,\}$.

Theorem 1.1. *For any Laurent polynomial P, the $H^*(V)$-valued function $\Psi_P(p,\kappa)$ is an entire function of $\log p$ that solves the quantum differential equation (1.1). If the classes of Laurent polynomials $P_0,\dots,P_m$ give a basis of $K(V;\mathbb{C})$, then the functions $\Psi_{P_0}(p,\kappa),\dots,$ $\Psi_{P_m}(p,\kappa)$ give a basis of solutions of the quantum differential equation.*

Theorem 1.1 follows from Proposition 6.25, see also Proposition 5.17. The fact that solutions $\Psi_{P_0}(p,\kappa),\dots,\Psi_{P_m}(p,\kappa)$ of Theorem 1.1 form a basis follows from the following determinant formula.

Theorem 1.2. *We have*

$$\det\biggl(\sum_{r=0}^{\infty}\mathrm{Res}_{t=-r\kappa}\bigl(t^i e^{-2\pi\sqrt{-1}\,jt/\kappa}\,\Phi(t,p,\kappa)\bigr)\biggr)_{i,j=0}^{m} \,=\, \bigl(2\pi\sqrt{-1}\,\bigr)^{m(m+1)/2}\,\kappa^{(m+1)(m+2)/2}.$$

Theorem 1.2 follows from Theorem 5.18, see also formula (5.60).

Notice that in [Gu], D. Guzzetti considers a scalar linear differential equation of order $m+1$ equivalent to the quantum differential equation (1.1) and constructs a basis of solutions in the form of similar integrals.

Our construction of the basis of solutions of the system of quantum differential equations for a partial flag variety is done in several steps and is based on constructions from representation theory.

We consider the joint compatible system of rational qKZ difference equations and dynamical differential equations for sections of the trivial bundle $\pi\,:\,(\mathbb{C}^N)^{\otimes n}\times\mathbb{C}^n\times\mathbb{C}^N\,\to\,\mathbb{C}^n\times\mathbb{C}^N$. This system is defined in terms of the Yangian $Y_h(\mathfrak{gl}_N)$ action on $(\mathbb{C}^N)^{\otimes n}$ and depends on the Yangian deformation parameter h. In this paper, we describe the $h\to\infty$ limit of this system of difference and differential equations. This is our first main result, see Section 3. We call the obtained equations the limiting qKZ difference equations and limiting dynamical differential equations.

In [TV1, TV6] we constructed solutions of the joint system of initial qKZ difference equations and dynamical differential equations in the form of multidimensional hypergeometric functions, see Theorem 4.9. In this paper, we calculate the $h\to\infty$ limit of these solutions and obtain solutions of the joint system of limiting equations, see Theorem 5.13. This is our second main result.

According to the general theory in [MO] the initial system of qKZ difference equations and dynamical differential equations is identified with the system of qKZ difference equations and equivariant quantum differential equations in the equivariant cohomology of cotangent bundles of partial flag varieties. The precise formulae for that identification can be found in [GRTV, RTV1]. It was expected that a suitable $h \to \infty$ limit of these equations gives a system of difference and differential equations related to the equivariant quantum cohomology of partial flag varieties themselves instead of their cotangent bundles. In particular, it was shown in [BMO] that the $h \to \infty$ limit of the equivariant quantum differential equations for the cotangent bundle of the full flag variety gives the equivariant quantum differential equations of the corresponding full flag variety. Also, the appropriate $h \to \infty$ limit of the Yangian R-matrix associated with the quantum cohomology of the cotangent bundle of a Grassmannian, was used in [GK, GKS] to calculate the quantum multiplication in the cohomology of the Grassmannian itself.

In this paper, we identify our limiting dynamical differential equations for sections of the bundle π with the equivariant quantum differential equations for partial flag varieties. Under this identification, our limiting qKZ difference equations become a system of difference equations in the equivariant cohomology of partial flag varieties, hence, a new system of difference equations compatible with the equivariant quantum differential equations. At the same time, the multidimensional hypergeometric solutions of the limiting equations for sections of the bundle π become solutions of the equivariant quantum differential equation and qKZ difference equations for partial flag varieties. This is our third main result, see Theorem 6.17.

The particular case of a projective space was considered in [TV7, CV], where the joint system of the equivariant quantum differential equation and compatible qKZ difference equations, together with hypergeometric solutions, was used to analyze the Stokes phenomenon for the equivariant quantum differential equation of a projective space. We expect similar applications of our compatible qKZ equations for partial flag varieties.

The paper is organized as follows. In Section 2 we define the initial joint system of qKZ and dynamical equations. In Section 3 we describe the $h \to \infty$ limit of these equations. In Section 4 we describe multidimensional hypergeometric solutions of the initial qKZ and dynamical equations. In Section 5 we obtain solutions of the limiting equations by taking the $h \to \infty$ limit of the solutions of the initial equations. In Section 6 we identify the limiting equations with the equation in equivariant cohomology of partial flag varieties. Appendices A–D contain technical information.

In Appendix E we show that the monodromy of the system of dynamical differential equations (2.7) is abelian under a certain resonance condition for the equivariant parameters. This is analogous to the corresponding property of the quantum differential equation of the Hilbert scheme of points in the plane, see [OP].

Notice that Proposition 6.23 provides an equivariant Gamma theorem for a partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$, cf. Theorem B.2 and formula (11.19) in [TV6]. Notice also that in Theorems 4.10, 4.17, 5.14, 5.18 we prove different determinant formulas which imply that the functions entering the determinants form bases in the spaces of solutions of the corresponding differential and difference equations. The Levelt fundamental solutions are discussed in Sections 4.7 and 5.5.

The authors thank G. Cotti, A. Givental, V. Gorbounov, L. Mihalcea, K. Rietsch, R. Rimányi for useful discussions. The second author thanks Max Planck Institute for Mathematics in Bonn for hospitality in May–June of 2022.

2. Dynamical and qKZ equations

2.1. **Notations.** Fix $N, n \in \mathbb{Z}_{>0}$. Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = \lambda_1 + \ldots + \lambda_N = n$. Let $I = (I_1, \ldots, I_N)$ be a partition of $\{1, \ldots, n\}$ into disjoint subsets $I_1, \ldots, I_N$. Denote by $\mathcal{I}_{\boldsymbol{\lambda}}$ the set of all partitions I with $|I_j| = \lambda_j$, $j = 1, \ldots, N$.

Consider the space $\mathbb{C}^N$ with the standard basis $v_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)$, $i = 1, \ldots, N$, and the tensor product $(\mathbb{C}^N)^{\otimes n}$ with the basis $(v_I)_I$,

$$v_I \,=\, v_{i_1} \otimes \ldots \otimes v_{i_n}\,,$$

where $i_j = m$ if $j \in I_m$.

The space $(\mathbb{C}^N)^{\otimes n}$ is a module over the Lie algebra $\mathfrak{gl}_N$ with basis $e_{i,j}$, $i, j = 1, \ldots, N$. The $\mathfrak{gl}_N$-module $(\mathbb{C}^N)^{\otimes n}$ has weight decomposition $(\mathbb{C}^N)^{\otimes n} = \sum_{|\boldsymbol{\lambda}|=n} (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$, where $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ is the subspace with basis $(v_I)_{I \in \mathcal{I}_{\boldsymbol{\lambda}}}$.

2.2. **Difference *qKZ* equations.** Define the R-matrix acting on $(\mathbb{C}^N)^{\otimes 2}$,

$$R(u;h) \,=\, \frac{u - hP}{u - h}\,,$$

where P is the permutation of factors of $(\mathbb{C}^N)^{\otimes 2}$ and $u, h \in \mathbb{C}$. The R-matrix satisfies the Yang-Baxter and unitarity equations,

$$R^{(12)}(u-v;h)R^{(13)}(u;h)R^{(23)}(v;h) \,=\, R^{(23)}(v;h)R^{(13)}(u;h)R^{(12)}(u-v;h)\,, \tag{2.1}$$

$$R^{(12)}(u;h)R^{(21)}(-u;h) \,=\, 1\,. \tag{2.2}$$

The first equation is an equation in $\mathrm{End}((\mathbb{C}^N)^{\otimes 3})$. The superscript indicates the factors of $(\mathbb{C}^N)^{\otimes 3}$ on which the corresponding operators act.

Let $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\boldsymbol{q} = (q_1, \ldots, q_N) \in \mathbb{C}^N$, and $\kappa \in \mathbb{C}$. Define the qKZ operators $K_1, \ldots, K_n$ acting on $(\mathbb{C}^N)^{\otimes n}$:

$$\begin{aligned} K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, & R^{(a,a-1)}(z_a - z_{a-1} + \kappa; h)\,\ldots\, R^{(a,1)}(z_a - z_1 + \kappa; h) \,\times \\ & \times\, q_1^{e^{(a)}_{1,1}} \ldots\, q_N^{e^{(a)}_{N,N}}\, R^{(a,n)}(z_a - z_n; h)\,\ldots\, R^{(a,a+1)}(z_a - z_{a+1}; h)\,. \end{aligned} \tag{2.3}$$

The qKZ operators preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and have the property

$$\begin{aligned} & K_a(z_1, \ldots, z_b + \kappa, \ldots, z_n; h; \boldsymbol{q}; \kappa)\, K_b(\boldsymbol{z}; h; \boldsymbol{q}; \kappa) \,= \\ & \qquad =\, K_b(z_1, \ldots, z_a + \kappa, \ldots, z_n; h; \boldsymbol{q}; \kappa)\, K_a(\boldsymbol{z}; h; \boldsymbol{q}; \kappa) \end{aligned} \tag{2.4}$$

for all $a, b = 1, \ldots, n$, see [FR]. We say that the collection of operators $K_1, \ldots, K_n$ with this property define a *discrete flat connection*, see [TV2].

The system of difference equations with step κ,

$$f(z_1,\dots,z_a+\kappa,\dots,z_n;h;\boldsymbol{q};\kappa) \,=\, K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa)\, f(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,, \qquad a=1,\dots,n, \tag{2.5}$$

for a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(\boldsymbol{z},h,\boldsymbol{q};\kappa)$ is called the *qKZ equations*.

Remark 2.1.

2.3. **Differential dynamical equations.** Define the linear operators $X_1,\dots,X_N$ acting on $(\mathbb{C}^N)^{\otimes n}$, called the dynamical Hamiltonians:

$$X_i(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \sum_{a=1}^n z_a e^{(a)}_{i,i} - h\Big(\frac{\tilde e_{i,i}(1-\tilde e_{i,i})}{2} + \sum_{1\leqslant a<b\leqslant n}\sum_{k=1}^N e^{(a)}_{i,k}\, e^{(b)}_{k,i} + {}$$
$$+ \sum_{\substack{j=1\\ j\neq i}}^N \frac{q_j}{q_i-q_j}\,(\tilde e_{i,j}\,\tilde e_{j,i}-\tilde e_{i,i})\Big),$$

where $\tilde e_{s,t}=\sum_{a=1}^n e^{(a)}_{s,t}$. The differential operators

$$\nabla_{\boldsymbol{q},\kappa,i} \,=\, \kappa\, q_i\frac{\partial}{\partial q_i} - X_i(\boldsymbol{z};h;\boldsymbol{q})\,, \qquad i=1,\dots,N, \tag{2.6}$$

preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and pairwise commute, see [TV3], also [GRTV, Section 3.4], [RTV1, Section 7.1], [MTV]. The operators $\nabla_{\boldsymbol{q},\kappa,i}$ define the $(\mathbb{C}^N)^{\otimes n}$-valued *dynamical connection*.

The system of differential equations

$$\kappa\, q_i\frac{\partial f}{\partial q_i} \,=\, X_i(\boldsymbol{z};h;\boldsymbol{q})\, f\,, \qquad i=1,\dots,N, \tag{2.7}$$

for a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is called the *dynamical equations*.

Theorem 2.2 ([TV3])**.** *The joint system of dynamical and qKZ equations with the same parameter κ is compatible.*

Lemma 2.3. *The dynamical Hamiltonians $X_1,\dots,X_N$ acting on $(\mathbb{C}^N)^{\otimes n}$ can be written in the form*

$$X_i(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \sum_{a=1}^n z_a e^{(a)}_{i,i} - h\sum_{1\leqslant a<b\leqslant n}\sum_{\substack{j=1\\ j\neq i}}^N\Big(\frac{e^{(a)}_{i,j}\,e^{(b)}_{j,i}}{1-q_j/q_i} - \frac{e^{(a)}_{j,i}\,e^{(b)}_{i,j}}{1-q_i/q_j}\Big). \tag{2.8}$$

Proof. The proof is by direct verification. □

3. Limit $h\to\infty$

3.1. **Limit of *qKZ* operators.** Introduce the R-matrix $R^\circ(u)$ acting on $(\mathbb{C}^N)^{\otimes 2}$:

$$R^\circ(u) \,=\, P + u\sum_{1\leqslant i<j\leqslant N} e_{i,i}\otimes e_{j,j}\,.$$

For example, for $N=2$ the matrix is

$$R^\circ(u)=\begin{pmatrix}1&0&0&0\\0&u&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Lemma 3.1. *We have*

$$R^\circ(u)\,=\,\lim_{h\to\infty}\bigl((-h)^{\sum_{i<j}e^{(1)}_{i,i}e^{(2)}_{j,j}}\,R(u;h)\,(-h)^{-\sum_{i<j}e^{(1)}_{j,j}e^{(2)}_{i,i}}\,\bigr)\,. \tag{3.1}$$

Moreover, $R^\circ(u)$ *satisfies the Yang-Baxter and unitarity equations,*

$$\bigl(R^\circ(u-v)\bigr)^{(12)}\bigl(R^\circ(u)\bigr)^{(13)}\bigl(R^\circ(v)\bigr)^{(23)}\,=\,\bigl(R^\circ(v)\bigr)^{(23)}\bigl(R^\circ(u)\bigr)^{(13)}\bigl(R^\circ(u-v)\bigr)^{(12)}, \tag{3.2}$$

$$\bigl(R^\circ(u)\bigr)^{(12)}\bigl(R^\circ(-u)\bigr)^{(21)}\,=\,1\,. \tag{3.3}$$

Proof. The proof of formula (3.1) is straightforward. Equations (3.2), (3.3) follow from (3.1) and (2.1), (2.2), respectively. □

Let $\boldsymbol{p}=(p_1,\dots,p_N)\in\mathbb{C}^N$. Define the *qKZ* operators $K^\circ_1,\dots,K^\circ_n$ acting on $(\mathbb{C}^N)^{\otimes n}$:

$$\begin{aligned}K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)\,=\,&\bigl(R^\circ(z_a-z_{a-1}+\kappa)\bigr)^{(a,a-1)}\dots\,\bigl(R^\circ(z_a-z_1+\kappa)\bigr)^{(a,1)}\times\\&\times\,p_1^{e^{(a)}_{1,1}}\dots\,p_N^{e^{(a)}_{N,N}}\,\bigl(R^\circ(z_i-z_n)\bigr)^{(a,n)}\dots\,\bigl(R^\circ(z_a-z_{a+1})\bigr)^{(a,a+1)}\,.\end{aligned} \tag{3.4}$$

Lemma 3.2. *For* $a=1,\dots,n$, *we have*

$$K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)\,=\lim_{h\to\infty}\bigl((-h)^{\sum_{b<c,\,i<j}e^{(b)}_{j,j}e^{(c)}_{i,i}}\,\widetilde{K}_i(\boldsymbol{z};h;\boldsymbol{p};\kappa)\,(-h)^{-\sum_{b<c,\,i<j}e^{(b)}_{j,j}e^{(c)}_{i,i}}\,\bigr)\,,$$

where the operators $\widetilde{K}_1,\dots,\widetilde{K}_n$ *are obtained from the qKZ operators* $K_1,\dots,K_n$, *see* (2.3), *by the substitution*

$$q_i\,\mapsto\,p_i\,(-h)^{\sum_{b=1}^n\bigl(\sum_{j>i}e^{(b)}_{j,j}-\sum_{j<i}e^{(b)}_{j,j}\bigr)}\,,\qquad i=1,\dots,N\,. \tag{3.5}$$

Proof. The lemma is proved by direct verification using Lemma 3.1. □

Corollary 3.3. *The qKZ operators* $K^\circ_1,\dots,K^\circ_n$ *preserve the weight decomposition of* $(\mathbb{C}^N)^{\otimes n}$ *and define a discrete flat connection,*

$$K^\circ_a(z_1,\dots,z_b+\kappa,\dots,z_n;\boldsymbol{p};\kappa)\,K^\circ_b(\boldsymbol{z};\boldsymbol{p};\kappa)\,=\,K^\circ_b(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{p};\kappa)\,K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)$$

for all $a,b=1,\dots,n$, *cf.* (2.4)*.*

Remark 3.4. The R-matrix $R^\circ(u)$ for $N=2$ is analogous to the R-matrix of the five-vertex model and the q-boson model, and to the R-matrices considered in [GK, GKS] in relation to the quantum cohomology theory for Grassmannians.

Remark 3.5. Since $\det R^\circ(u)=1$, the *qKZ* operators $K^\circ_1,\dots,K^\circ_n$ are invertible for all $\boldsymbol{z},\boldsymbol{p},\kappa$ provided $p_i\neq0$ for all $i=1,\dots,N$. That is, the discrete flat connection on $(\mathbb{C}^N)^{\otimes n}$ defined by $K^\circ_1,\dots,K^\circ_n$ is regular for all $\boldsymbol{z}$, unlike the discrete flat connection defined by the *qKZ* operators $K_1,\dots,K_n$, see (2.3).

3.2. **Dynamical operators** $X_1^\circ,\dots,X_N^\circ$**.** Recall $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$. Let $\lambda^{(i)}=\sum_{j=1}^{i}\lambda_i$, $i=1,\dots,N$, so that $\lambda^{(N)}=n$. For $1\leqslant i<j\leqslant n$, set $\lambda_{\langle i,j\rangle}=\sum_{k=i}^{j-1}(\lambda_k+\lambda_{k+1})=\lambda_i+\lambda_j+2\sum_{k=i+1}^{j-1}\lambda_k$.

Recall $I=(I_1,\dots,I_N)$ and the vectors $v_I\in(\mathbb{C}^N)^{\otimes n}$. For $\sigma\in S_n$ and $I=(I_1,\dots,I_N)$, set $\sigma(I)=\bigl(\sigma(I_1),\dots,\sigma(I_N)\bigr)$. For $a,b=1,\dots,n$, let $s_{a,b}$ be the transposition of a,b.

Given $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, let $a\in I_i$ and $b\in I_j$. A pair (a,b) is called *I-disordered* if either $a<b$, $i>j$, or $a>b$, $i<j$. A pair (a,b) is called *I-ordered* if either $a<b$, $i<j$, or $a>b$, $i>j$. A pair (a,b) is called *I-flat* if $i=j$.

Denote $M=\{\min(a,b)+1,\dots,\max(a,b)-1\}$, $k=\min(i,j)$, $l=\max(i,j)$. If $i\neq j$, so that $k<l$, set $m_{a,b,I}=\lambda_{\langle k,l\rangle}$ and

$$r_{a,b,I}\,=\,|M\cap I_i|+|M\cap I_j|+2\Bigl|M\cap\bigcup_{r=k+1}^{l-1}I_r\Bigr|.$$

Clearly, $m_{a,b.I}\geqslant 2$ and $0\leqslant r_{a,b,I}\leqslant m_{a,b,I}-2$ if $i\neq j$. Set $I_{[a,b]}=\bigcup_{r=k}^{l}I_r$.

Set

$$I_{\boldsymbol{\lambda}}^{\min}\,=\,\bigl((\lambda_1,\dots,\lambda^{(1)}),(\lambda^{(1)}+1,\dots,\lambda^{(2)}),\,\dots\,(\lambda^{(N-1)}+1,\dots,n)\bigr).\tag{3.6}$$

For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, let $\sigma_I\in S_n$ be the element of minimal length such that $\sigma_I(I_{\boldsymbol{\lambda}}^{\min})=I$. Notice that

$$|\sigma_I|\,=\,\bigl|\{(a,b)\mid a\in I_i\,,\ b\in I_j\,,\ a<b\,,\ i>j\}\bigr|.\tag{3.7}$$

Lemma 3.6. *We have*

a) *If the pair* (a,b) *is I-disordered, then* $|\sigma_{s_{a,b}(I)}|=|\sigma_I|-r_{a,b,I}-1$.

b) *If the pair* (a,b) *is I-ordered, then* $|\sigma_{s_{a,b}(I)}|=|\sigma_I|+r_{a,b,I}+1$.

c) *If the pair* (a,b) *is I-flat, then* $s_{a,b}(I)=I$.

Corollary 3.7. *If* $|\sigma_{s_{a,b}(I)}|<|\sigma_I|$, *then the pair* (a,b) *is I-disordered. If* $|\sigma_{s_{a,b}(I)}|>|\sigma_I|$, *then the pair* (a,b) *is I-ordered.*

A pair (a,b) is called *I-admissible of the first kind* if $|\sigma_{s_{a,b}(I)}|=|\sigma_I|-1$, and *I-admissible of the second kind* if $|\sigma_{s_{a,b}(I)}|=|\sigma_I|+m_{a,b,I}-1$.

Lemma 3.8. *A pair* (a,b) *is I-admissible of the first kind if and only if it is I-disordered and the intersection* $\{\min(a,b)+1,\dots,\max(a,b)-1\}\cap I_{[a,b]}$ *is empty.*

Lemma 3.9. *A pair* (a,b) *is I-admissible of the second kind if and only if it is I-ordered and the intersection* $\{1,\dots,\min(a,b)-1,\,\max(a,b)+1,\dots,n\}\cap I_{[a,b]}$ *is empty.*

Example. Let $N=4$, $n=5$, $\boldsymbol{\lambda}=(2,1,1,1)$, $I=\bigl(\{1,3\},\{4\},\{2\},\{5\}\bigr)$. Then the I-admissible pairs of the first kind are $(3,2)$, $(2,3)$, $(4,2)$, $(2,4)$, and the I-admissible pairs of the second kind are $(1,4)$, $(4,1)$, $(1,5)$, $(5,1)$, $(2,5)$, $(5,2)$.

For all i,j,a,b, define linear operators $Q_{i,j}^{a,b}$ acting on $(\mathbb{C}^N)^{\otimes n}$ by the rule

$$\begin{aligned}Q_{i,j}^{a,b}v_I\,&=\,v_{s_{a,b}(I)}\,, &&\text{if } a\in I_i\,,\ b\in I_j\,,\text{ and the pair } (a,b) \text{ is } I\text{-admissible},\\ Q_{i,j}^{a,b}v_I\,&=\,0\,, &&\text{otherwise.}\end{aligned}$$

Recall the linear operators $e^{(a)}_{i,i}$ acting on $(\mathbb{C}^N)^{\otimes n}$.

The dynamical operators $X^\circ_1,\dots,X^\circ_N$, acting on $(\mathbb{C}^N)^{\otimes n}$ are given by the formula

$$X^\circ_i(\boldsymbol{z};\boldsymbol{p}) \,=\sum_{a=1}^n z_a\, e^{(a)}_{i,i} \,+ \sum_{1\leqslant b<a\leqslant n}\biggl(\,\sum_{j=i+1}^N Q^{a,b}_{i,j} - \sum_{j=1}^{i-1}\frac{p_i}{p_j}\,Q^{a,b}_{i,j}\biggr) \,+ \\ +\sum_{1\leqslant a<b\leqslant n}\biggl(\,\sum_{j=i+1}^N \frac{p_j}{p_i}\,Q^{a,b}_{i,j} - \sum_{j=1}^{i-1} Q^{a,b}_{i,j}\biggr). \tag{3.8}$$

Notice that the operators $X^\circ_1,\dots,X^\circ_N$ preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$.

Example. Let $N=2$, $n=3$. The operators X°_1, X°_2 preserve the subspace spanned by the vectors

$$v_{(\{1\},\{2,3\})} \,=\, v_1\otimes v_2\otimes v_2\,,\qquad v_{(\{2\},\{1,3\})} \,=\, v_2\otimes v_1\otimes v_2\,,\qquad v_{(\{3\},\{1,2\})} \,=\, v_2\otimes v_2\otimes v_1\,.$$

of weight $\boldsymbol{\lambda}=(1,2)$. The matrices of X°_1, X°_2 in this basis are

$$X^\circ_1 = \begin{pmatrix} z_1 & 1 & 0\\ 0 & z_2 & 1\\ p_2/p_1 & 0 & z_3\end{pmatrix},\qquad X^\circ_2 = \begin{pmatrix} z_2+z_3 & -1 & 0\\ 0 & z_1+z_3 & -1\\ -p_2/p_1 & 0 & z_1+z_2\end{pmatrix}.$$

Example. Let $N=5$, $n=6$, $I=\bigl(\{2,5\},\{6\},\{3\},\{1\},\{4\}\bigr)$. Then

$$X^\circ_1(\boldsymbol{z};\boldsymbol{p})\,v_I \,=\, (z_2+z_5)\,v_I + v_{(\{1,5\},\{6\},\{3\},\{2\},\{4\})} + v_{(\{2,4\},\{6\},\{3\},\{1\},\{5\})} \,+ \\ +\, v_{(\{2,3\},\{6\},\{5\},\{1\},\{4\})} + \frac{p_2}{p_1}\,v_{(\{5,6\},\{2\},\{3\},\{1\},\{4\})}\,,$$

$$X^\circ_2(\boldsymbol{z};\boldsymbol{p})\,v_I \,=\, z_5\,v_I + v_{(\{2,5\},\{4\},\{3\},\{1\},\{6\})} + v_{(\{2,5\},\{3\},\{6\},\{1\},\{4\})} - \frac{p_2}{p_1}\,v_{(\{5,6\},\{2\},\{3\},\{1\},\{4\})}\,.$$

3.3. **Limit of dynamical Hamiltonians.** Let the operators $\widetilde X_1,\dots,\widetilde X_N$ be obtained from the dynamical Hamiltonians $X_1,\dots,X_N$, see (2.8), by substitution (3.5). In more detail, the operators $\widetilde X_1,\dots,\widetilde X_N$ preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and the action of $\widetilde X_i(\boldsymbol{z};h;\boldsymbol{p})$ on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ coincides with the action of $X_i(\boldsymbol{z};h;\boldsymbol{q})$ for $\boldsymbol{q}=(q_1,\dots,q_N)$,

$$q_j \,=\, p_j\,(-h)^{\sum_{k>j}\lambda_k-\sum_{k<j}\lambda_k}\,,\qquad j=1,\dots,N\,. \tag{3.9}$$

Lemma 3.10. *For $i=1,\dots,N$, we have*

$$X^\circ_i(\boldsymbol{z};\boldsymbol{p}) \,=\, \lim_{h\to\infty}\Bigl((-h)^{\sum_{b<c,\,j<k}e^{(b)}_{k,k}e^{(c)}_{j,j}}\,\widetilde X_i(\boldsymbol{z};h;\boldsymbol{p})\,(-h)^{-\sum_{b<c,\,j<k}e^{(b)}_{k,k}e^{(c)}_{j,j}}\Bigr)\,. \tag{3.10}$$

Proof. The lemma is proved by direct verification using Lemma 2.3, the equality

$$\sum_{b<c,\,j<k} e^{(b)}_{k,k}\,e^{(c)}_{j,j}\,v_I \,=\, |\sigma_I|\,v_I\,,$$

following from (3.7), and Lemma 3.6. ∎

Corollary 3.11. *The differential operators*

$$\nabla^\circ_{\boldsymbol{p},\kappa,i} \,=\, \kappa p_i \frac{\partial}{\partial p_i} - X^\circ_i(\boldsymbol{z};\boldsymbol{p})\,, \qquad i=1,\dots,N\,, \tag{3.11}$$

preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ *and pairwise commute.*

3.4. **Limiting equations.** The system of difference equations with step κ,

$$f(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{p}) \,=\, K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)\, f(\boldsymbol{z};\boldsymbol{p})\,, \qquad a=1,\dots,n\,, \tag{3.12}$$

for a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(\boldsymbol{z},\boldsymbol{p})$ will be called the *limiting qKZ equations*.

The system of differential equations with parameter κ,

$$\kappa p_i \frac{\partial f}{\partial p_i} \,=\, X^\circ_i(\boldsymbol{z};\boldsymbol{p})\, f\,, \qquad i=1,\dots,N\,, \tag{3.13}$$

for a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(\boldsymbol{z};\boldsymbol{p})$ will be called the *limiting dynamical equations*.

Theorem 3.12. *The joint systems of limiting dynamical and limiting qKZ equations with the same parameter* κ *is compatible.*

Proof. The theorem follows from Theorem 2.2 and Lemmas 3.2, 3.10. □

4. Hypergeometric solutions of the dynamical and *qKZ* equations

4.1. **Weight functions $W_I(\boldsymbol{t};\boldsymbol{z};h)$.** For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, we define the weight functions $W_I(\boldsymbol{t};\boldsymbol{z};h)$, cf. [TV1, TV4, RTV1, TV6]. The functions $W_I(\boldsymbol{t};\boldsymbol{z};h)$ here coincide with those defined in [TV6, Section 3.4].

Recall $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$ and $I=(I_1,\dots,I_N)$. Set $\bigcup_{k=1}^j I_k \,=\, \{i^{(j)}_1<\,\dots\,<i^{(j)}_{\lambda^{(j)}}\}$. Consider the variables $t^{(j)}_a$, $j=1,\dots,N-1$, $a=1,\dots,\lambda^{(j)}$. Set $t^{(N)}_a=z_a$, $a=1,\dots,n$. Denote $\boldsymbol{t}^{(j)}=(t^{(j)}_1,\dots,t^{(j)}_{\lambda^{(j)}})$ and $\boldsymbol{t}=(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)})$.

For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, define

$$\Sigma_I \,=\, (z_{i^{(1)}_1},\dots,z_{i^{(1)}_{\lambda^{(1)}}},z_{i^{(2)}_1},\dots,z_{i^{(2)}_{\lambda^{(2)}}},\,\dots\,,\,z_{i^{(N-1)}_1},\dots,z_{i^{(N-1)}_{\lambda^{(N-1)}}})\,, \tag{4.1}$$

so that, $\boldsymbol{t}=\Sigma_I$ reads in detail as

$$t^{(k)}_a \,=\, z_{i^{(k)}_a}\,, \qquad k=1,\dots,N-1\,, \quad i=a,\dots,\lambda^{(k)}.$$

For a function $f(x_1,\dots,x_k)$ of some variables, denote

$$\mathrm{Sym}_{x_1,\dots,x_k} f(x_1,\dots,x_k) \,=\, \sum_{\sigma\in S_k} f(x_{\sigma(1)},\dots,x_{\sigma(k)})\,.$$

The weight functions are

$$W_I(\boldsymbol{t};\boldsymbol{z};h) \,=\, \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}}\,\dots\,\mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}} U_I(\boldsymbol{t};\boldsymbol{z};h)\,, \tag{4.2}$$

$$U_I(\boldsymbol{t};\boldsymbol{z};h) \,=\,$$
$$=\,\prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\biggl(\prod_{\substack{c=1\\ i_c^{(j+1)}<i_a^{(j)}}}^{\lambda^{(j+1)}}(t_a^{(j)}-t_c^{(j+1)})\prod_{\substack{d=1\\ i_d^{(j+1)}>i_a^{(j)}}}^{\lambda^{(j+1)}}(t_a^{(j)}-t_d^{(j+1)}-h)\prod_{b=a+1}^{\lambda^{(j)}}\frac{t_b^{(j)}-t_a^{(j)}-h}{t_b^{(j)}-t_a^{(j)}}\biggr).$$

Example. Let $N=2$, $n=2$, $\boldsymbol{\lambda}=(1,1)$, $I=(\{1\},\{2\})$, $J=(\{2\},\{1\})$. Then

$$W_I(\boldsymbol{t};\boldsymbol{z};h)\,=\,t_1^{(1)}-z_2-h\,,\qquad W_J(\boldsymbol{t};\boldsymbol{z};h)\,=\,t_1^{(1)}-z_1\,.$$

Lemma 4.1 ([RTV1, TV6]). *For any $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, $i=1,\dots,N-1$, and $a=1,\dots,n-1$, we have*

$$W_{s_{a,a+1}(I)}(\boldsymbol{t};\boldsymbol{z};h)\,=\, \tag{4.3}$$
$$=\,\frac{z_a-z_{a+1}-h}{z_a-z_{a+1}}\,W_I(\boldsymbol{t};z_1,\dots,z_{a+1},z_a,\dots,z_n;h)\,+\,\frac{h}{z_a-z_{a+1}}\,W_I(\boldsymbol{t};\boldsymbol{z};h)\,.$$

Remark 4.2. Define the operators $\hat{s}_1,\dots,\hat{s}_{n-1}$ acting on functions of $z_1,\dots,z_n$:

$$\hat{s}_a f(\boldsymbol{z})\,=\,\frac{z_a-z_{a+1}-h}{z_a-z_{a+1}}\,f(z_1,\dots,z_{a+1},z_a,\dots,z_n)\,+\,\frac{h}{z_a-z_{a+1}}\,f(\boldsymbol{z})\,. \tag{4.4}$$

The assignment $s_{a,a+1}\mapsto\hat{s}_a$, $a=1,\dots,n-1$, yields a representation of the symmetric group S_n.

Set

$$c_{\boldsymbol{\lambda}}(\boldsymbol{t};h)\,=\,\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}(t_a^{(i)}-t_b^{(i)}-h)\,. \tag{4.5}$$

Recall the permutations σ_I, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, defined in Section 3.2.

Lemma 4.3 ([RTV1]). *We have $W_J(\Sigma_I;\boldsymbol{z};h)=0$ unless $I=J$ or $|\sigma_I|>|\sigma_J|$, and*

$$W_I(\Sigma_I;\boldsymbol{z};h)\,=\,c_{\boldsymbol{\lambda}}(\Sigma_I;h)\,\prod_{j=1}^{N-1}\prod_{k=j+1}^{N}\prod_{a\in I_j}\biggl(\prod_{\substack{b\in I_k\\ b<a}}(z_a-z_b)\prod_{\substack{b\in I_k\\ b>a}}(z_a-z_b-h)\biggr). \tag{4.6}$$

Lemma 4.4. *For any $J\in\mathcal{I}_{\boldsymbol{\lambda}}$, the polynomial $W_J(\Sigma_I;\boldsymbol{z};h)$ is divisible by*

$$c_{\boldsymbol{\lambda}}(\Sigma_I;h)\,\prod_{j=1}^{N-1}\prod_{k=j+1}^{N}\prod_{a\in I_j}\prod_{\substack{b\in I_k\\ b>a}}(z_a-z_b-h)\,.$$

Lemma 4.4 is a version of [RTV2, Lemma 3.1, item (I)]. A stronger more technical analogue of Lemma 4.4 is given by Lemma 4.14 in Section 4.5.

Let σ_0 be the longest permutation, $\sigma_0(i)=n+1-i$, $i-1,\dots,n$. For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, define

$$\check{W}_I(\boldsymbol{t};\boldsymbol{z};h)\,=\,W_{\sigma_0(I)}(\boldsymbol{t};z_n,\dots,z_1;h)\,. \tag{4.7}$$

Set

$$R_{\boldsymbol\lambda}(\boldsymbol z) = \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}(z_a-z_b)\,,\qquad Q_{\boldsymbol\lambda}(\boldsymbol z;h) = \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}(z_a-z_b-h)\,, \tag{4.8}$$

For $\sigma\in S_n$, denote $\boldsymbol z_\sigma=(z_{\sigma(1)},\dots,z_{\sigma(n)})$.

Proposition 4.5 ([RTV1, Lemma 3.4])**.** *The functions $W_I(\boldsymbol t;\boldsymbol z;h)$ and $\check W_J(\boldsymbol t;\boldsymbol z;h)$ are biorthogonal,*

$$\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{W_J(\Sigma_I;\boldsymbol z;h)\,\check W_K(\Sigma_I;\boldsymbol z;h)}{c^2_{\boldsymbol\lambda}(\Sigma_I;h)\,R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_I})\,Q_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_I};h)} = \delta_{J,K}\,. \tag{4.9}$$

Corollary 4.6. *We have*

$$\sum_{I\in\mathcal I_{\boldsymbol\lambda}}W_I(\Sigma_J;\boldsymbol z;h)\,\check W_I(\Sigma_K;\boldsymbol z;h) = \delta_{J,K}\,c^2_{\boldsymbol\lambda}(\Sigma_J;h)\,R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J})\,Q_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};h)\,. \tag{4.10}$$

4.2. **Master function.** Let $\phi(x,h,\kappa)=\Gamma(x/\kappa)\,\Gamma\bigl((h-x)/\kappa\bigr)$. Define the *master function*:

$$\begin{aligned}\Phi_{\boldsymbol\lambda}(\boldsymbol t;\boldsymbol z;h;\boldsymbol q;\kappa) &= \bigl(e^{\pi\sqrt{-1}\,(n-\lambda_N)}q_N\bigr)^{\sum_{a=1}^n z_a/\kappa}\prod_{i=1}^{N-1}\Bigl(e^{\pi\sqrt{-1}\,(\lambda_{i+1}-\lambda_i)}\,\frac{q_i}{q_{i+1}}\Bigr)^{\sum_{a=1}^{\lambda^{(i)}}t_a^{(i)}/\kappa}\times\\ &\times\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Bigl(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\frac{1}{(t^{(i)}_a-t^{(i)}_b-h)\,\phi(t^{(i)}_a-t^{(i)}_b,h,\kappa)}\prod_{c=1}^{\lambda^{(i+1)}}\phi(t^{(i)}_a-t^{(i+1)}_c,h,\kappa)\Bigr)\,,\end{aligned} \tag{4.11}$$

where $\lambda^{(N)}=n$ and $t^{(N)}_a=z_a$, $a=1,\dots,n$. The function $\Phi_{\boldsymbol\lambda}(\boldsymbol t;\boldsymbol z;h;\boldsymbol q;\kappa)$ is symmetric in the variables $t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}}$ for each $i=1,\dots,N-1$.

4.3. **Jackson integrals.** Let $\lambda^{\{1\}}=\sum_{i=1}^{N-1}\lambda^{(i)}$. Consider the space $\mathbb C^{\lambda^{\{1\}}}\times\mathbb C^n\times\mathbb C\times\mathbb C^N$ with coordinates $\boldsymbol t,\boldsymbol z,h,\boldsymbol q$. The lattice $\kappa\mathbb Z^{\lambda^{\{1\}}}$ naturally acts on this space by shifting the $\boldsymbol t$-coordinates.

For a function of $f(\boldsymbol t)$ and a point $\boldsymbol s\in\mathbb C^{\lambda^{\{1\}}}$, define $\operatorname{Res}_{\boldsymbol t=\boldsymbol s}f(\boldsymbol t)$ to be the iterated residue,

$$\operatorname{Res}_{\boldsymbol t=\boldsymbol s}f(\boldsymbol t) = \operatorname{Res}_{t^{(1)}_1=s^{(1)}_1}\dots\operatorname{Res}_{t^{(1)}_{\lambda^{(1)}}=s^{(1)}_{\lambda^{(1)}}}\dots\operatorname{Res}_{t^{(N-1)}_1=s^{(N-1)}_1}\dots\operatorname{Res}_{t^{(N-1)}_{\lambda^{(N-1)}}=s^{(N-1)}_{\lambda^{(N-1)}}}f(\boldsymbol t)\,.$$

Let L' be the complement in $\mathbb C^n\times\mathbb C$ of the union of the hyperplanes

$$h=m\kappa\,,\qquad z_a-z_b=m\kappa\,,\qquad z_a-z_b+h=m\kappa\,, \tag{4.12}$$

for all $a,b=1,\dots,n$, $a\neq b$, and all $m\in\mathbb Z_{\leqslant 0}$. Let $L''\subset\mathbb C^N$ be the domain

$$|q_{i+1}/q_i|<1\,,\qquad i=1,\dots,N-1\,, \tag{4.13}$$

with additional cuts fixing a branch of $\log q_i$ for all $i=1,\dots,N$. Set $L=L'\times L''\subset\mathbb C^n\times\mathbb C\times\mathbb C^N$.

Let $\boldsymbol{l} = \bigl(l_1^{(1)}, \dots, l_{\lambda^{(1)}}^{(1)}, \dots, l_1^{(N-1)}, \dots, l_{\lambda^{(N-1)}}^{(N-1)}\bigr) \in \mathbb{Z}^{\lambda^{\{1\}}}$. By convention, set $l_a^{(i)} = 0$ for $a > \lambda^{(i)}$ or $i = N$.

Let $f(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q})$ be a polynomial in $\boldsymbol{t}$ and a holomorphic function of $\boldsymbol{z}, h, \boldsymbol{q}$ in L. For $(\boldsymbol{z};h;\boldsymbol{q}) \in L$, define

$$\mathcal{M}_J(\Phi_{\boldsymbol{\lambda}} f)(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \sum_{\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}} \mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa} \bigl(\Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)\, f(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q})\bigr)\,. \tag{4.14}$$

This sum is called the *Jackson integral over the discrete cycle* $\widehat{\Sigma}_J \subset \mathbb{C}^{\lambda^{\{1\}}} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^N$,

$$\widehat{\Sigma}_J \,=\, \{(\Sigma_J - \boldsymbol{l}\kappa;\boldsymbol{z};h;\boldsymbol{q}) \ \mid\ \boldsymbol{l} \in \mathbb{Z}^{\lambda^{\{1\}}},\ \ (\boldsymbol{z};h;\boldsymbol{q}) \in L\}\,.$$

Notice that the master function $\Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)$ has only simple poles, and

$$\begin{aligned}\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa} \bigl(\Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)\, f(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q})\bigr) \,=\\ =\, f(\Sigma_J - \boldsymbol{l}\kappa;\boldsymbol{z};h;\boldsymbol{q})\ \mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\, \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)\,.\end{aligned}$$

A closed expression for the residue $\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\, \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is given by Lemma 4.7 below.

Recall $c_{\boldsymbol{\lambda}}(\boldsymbol{t};h)$, see (4.5). Set

$$M_{\boldsymbol{\lambda}}(\boldsymbol{z};\kappa) \,=\, \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \frac{\sin\bigl(\pi\,(z_a - z_b)/\kappa\bigr)}{\pi\, e^{\pi\sqrt{-1}\,(z_a+z_b)/\kappa}}\,, \tag{4.15}$$

and

$$A_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\kappa) \,=\, \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \Biggl(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}} \frac{\Gamma\bigl(1 + (t_b^{(i)} - t_a^{(i)})/\kappa\bigr)}{\Gamma\bigl((t_b^{(i)} - t_a^{(i)} + h)/\kappa\bigr)} \prod_{c=1}^{\lambda^{(i+1)}} \frac{\Gamma\bigl((t_c^{(i+1)} - t_a^{(i)} + h)/\kappa\bigr)}{\Gamma\bigl(1 + (t_c^{(i+1)} - t_a^{(i)})/\kappa\bigr)} \Biggr). \tag{4.16}$$

Lemma 4.7. *If* $\boldsymbol{l} \notin \mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}$, *then* $\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\, \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa) = 0$. *For* $\boldsymbol{l} \in \mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}$,

$$\begin{aligned}&\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\, \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\\ &=\, \frac{\kappa^{\lambda^{\{1\}}} A_{\boldsymbol{\lambda}}(\Sigma_J - \boldsymbol{l}\kappa;\boldsymbol{z};h;\kappa)}{M_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J};\kappa)\, c_{\boldsymbol{\lambda}}(\Sigma_J - \boldsymbol{l}\kappa;h)} \prod_{i=1}^{N} q_i^{\sum_{a\in J_i} z_a/\kappa} \prod_{i=1}^{N-1} (q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}.\end{aligned} \tag{4.17}$$

In particular,

$$\begin{aligned}\mathrm{Res}_{\boldsymbol{t}=\Sigma_J} \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa) \,&=\, \frac{\bigl(\kappa\,\Gamma(h/\kappa)\bigr)^{\lambda^{\{1\}}}}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)} \prod_{i=1}^{N} \bigl(e^{\pi\sqrt{-1}\,(n-\lambda_i)}\, q_i\bigr)^{\sum_{a\in J_i} z_a/\kappa} \,\times\\ &\times\, \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \prod_{a\in J_i} \prod_{b\in J_j} \Gamma\bigl((z_a - z_b)/\kappa\bigr)\, \Gamma\bigl((z_b - z_a + h)/\kappa\bigr)\,.\end{aligned} \tag{4.18}$$

By Lemma 4.7, the actual summation in formula (4.14) is only over the positive cone of the lattice,

$$\mathcal{M}_J(\Phi_{\boldsymbol{\lambda}} f)(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \sum_{\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}} f(\Sigma_J - \boldsymbol{l}\kappa;\boldsymbol{z};h;\boldsymbol{q})\ \mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\, \Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa)\,. \tag{4.19}$$

4.4. **Solutions of the dynamical and qKZ equations.** For $J\in\mathcal{I}_{\boldsymbol{\lambda}}$, define

$$\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \bigl(\kappa\,\Gamma(h/\kappa)\bigr)^{-\lambda^{\{1\}}}\,\Omega_{\boldsymbol{\lambda}}(h;\boldsymbol{q};\kappa)\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathcal{M}_J(\Phi_{\boldsymbol{\lambda}}W_I)(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v_I\,, \tag{4.20}$$

where

$$\Omega_{\boldsymbol{\lambda}}(h;\boldsymbol{q};\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-q_j/q_i)^{h\lambda_i/\kappa}. \tag{4.21}$$

Definition 4.8. *Say that a function $f(\boldsymbol{q})$ is holomorphic in the unit polydisk around $\boldsymbol{q}=\mathbb{0}$ if $f(\boldsymbol{q})=g(q_2/q_1,\dots,q_N/q_{N-1})$ for a function $g(s_1,\dots,s_{N-1})$ holomorphic in $s_1,\dots,s_{N-1}$, provided $|s_i|<1$ for all $i=1,\dots,N-1$. Denote $f(\mathbb{0})=g(0,\dots,0)$.*

In the given definition, we described homogeneous functions of $\boldsymbol{q}$ that have behave regularly as the ratios of the subsequent coordinates approaches zero. The symbol $\mathbb{0}$ used in the definition is formal and does not represent any point of $\mathbb{C}^n$.

Theorem 4.9. *The $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued function $\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is a solution of the joint system of dynamical differential equations (2.7) and qKZ difference equations (2.5). It has the form*

$$\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,\prod_{i=1}^{N}\bigl(e^{\pi\sqrt{-1}\,(n-\lambda_i)}\,q_i\bigr)^{\sum_{a\in J_i}z_a/\kappa}\,\times \tag{4.22}$$

$$\times\,\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\biggl(\prod_{b\in J_j}\frac{\Gamma\bigl(1+(z_b-z_a+h)/\kappa\bigr)}{\sin\bigl(\pi\,(z_a-z_b)/\kappa\bigr)}\prod_{\substack{c\in J_j\\ c<a}}\frac{1}{z_a-z_c-h}\biggr),$$

where the function $\Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is entire in $\boldsymbol{z},h$ and holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\boldsymbol{q}=\mathbb{0}$. In more detail,

$$\Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{b\in J_j}\frac{-\pi\,\kappa}{\Gamma\bigl(1+(z_b-z_a)/\kappa\bigr)}\,\times \tag{4.23}$$

$$\times\,\biggl(\Psi^{\diamond}_{J,0}(\boldsymbol{z};h)\,+\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^{N-1}_{\geqslant 0}\\ \boldsymbol{m}\neq 0}}\Psi^{\diamond}_{J,\boldsymbol{m}}(\boldsymbol{z};h;\kappa)\,\prod_{i=1}^{N-1}(q_{i+1}/q_i)^{m_i}\biggr),$$

where

$$\Psi^{\diamond}_{J,0}(\boldsymbol{z};h) \,=\, \frac{1}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{\substack{b\in J_j\\ b>a}}\frac{1}{z_a-z_b-h}\,\times \tag{4.24}$$

$$\times\,\biggl(W_J(\Sigma_J;\boldsymbol{z};h)\,v_J\,+\sum_{\substack{I\in\mathcal{I}_{\boldsymbol{\lambda}}\\ |\sigma_I|<|\sigma_J|}}W_I(\Sigma_J;\boldsymbol{z};h)\,v_I\biggr)$$

is a polynomial in $\boldsymbol{z},h$, and $\Psi^{\diamond}_{J,\boldsymbol{m}}(\boldsymbol{z};h;\kappa)$ for $\boldsymbol{m}\neq 0$ are rational functions of $\boldsymbol{z},h,\kappa$ with at most simple poles on the hyperplanes $z_a-z_b\in\kappa\mathbb{Z}_{>0}$ for $a\in J_i$, $b\in J_j$, $1\leqslant i<j\leqslant N$.

Furthermore, for any transposition $s_{a,b}\in S_n$,

$$\Psi^{\diamond}_J(\boldsymbol z;h;\boldsymbol q;\kappa)\big|_{z_a=z_b} = \Psi^{\diamond}_{s_{a,b}(J)}(\boldsymbol z;h;\boldsymbol q;\kappa)\big|_{z_a=z_b}. \tag{4.25}$$

Proof. The fact that $\Psi_J(\boldsymbol z;h;\boldsymbol q;\kappa)$ solves the dynamical equations (2.7) is proved in [TV6, Theorem 8.4], and the fact that $\Psi_J(\boldsymbol z;h;\boldsymbol q;\kappa)$ solves the qKZ equations (2.5) is proved in [TV1, Theorem 1.5.2], cf. [TV4].

Analytic properties of $\Psi^{\diamond}_J(\boldsymbol z;h;\boldsymbol q;\kappa)$ are proved in Section 4.5. The fact that the right-hand side of formula (4.24) is a polynomial in $\boldsymbol z,h$ follows from Lemmas 4.3, 4.4. □

The functions $\Psi_J(\boldsymbol z;h;\boldsymbol q;\kappa)$ are called the *multidimensional hypergeometric solutions* of the dynamical equations. In [TV5], we constructed another type of solutions of the dynamical equations using multidimensional hypergeometric integrals.

The next theorem computes the determinant of coordinates of solutions $\Psi_J(\boldsymbol z;h;\boldsymbol q;\kappa)$ and is analogous to [TV6, Theorem 11.3].

Theorem 4.10. *Let* $n\geqslant 2$. *Then*

$$\begin{aligned}\det\big(\Omega_{\boldsymbol\lambda}(\boldsymbol q,\kappa)\,\mathcal M_J(\Phi_{\boldsymbol\lambda}W_I)(\boldsymbol z;h;\boldsymbol q;\kappa)\big)_{I,J\in\mathcal I_{\boldsymbol\lambda}} &= \\ = \big(\kappa\,\Gamma(h/\kappa)\big)^{\lambda^{\{1\}}d_{\boldsymbol\lambda}}\Big(e^{2\pi\sqrt{-1}\,(n-1)\,d^{(2)}_{\boldsymbol\lambda}}\prod_{i=1}^N q_i^{d^{(1)}_{\boldsymbol\lambda,i}}\Big)^{\sum_{a=1}^n z_a/\kappa}&\times \\ \times\prod_{a=1}^{n-1}\prod_{b=a+1}^n\bigg(\frac{\pi\kappa^2\,\Gamma\big((z_a-z_b+h)/\kappa\big)\,\Gamma\big(1+(z_b-z_a+h)/\kappa\big)}{\sin\big(\pi\,(z_a-z_b)/\kappa\big)}\bigg)^{d^{(2)}_{\boldsymbol\lambda}}&,\end{aligned}\tag{4.26}$$

where $\lambda^{\{1\}}=\sum_{i=1}^{N-1}(N-i)\,\lambda_i$,

$$d_{\boldsymbol\lambda} = \frac{n!}{\lambda_1!\dots\lambda_N!}\,,\qquad d^{(1)}_{\boldsymbol\lambda,i} = \frac{\lambda_i\,(n-1)!}{\lambda_1!\dots\lambda_N!}\,,\qquad d^{(2)}_{\boldsymbol\lambda} = \frac{(n-2)!}{\lambda_1!\dots\lambda_N!}\sum_{i=1}^{N-1}\sum_{j=i+1}^N\lambda_i\,\lambda_j\,. \tag{4.27}$$

Proof. Denote by $F(\boldsymbol z;\boldsymbol q)$ the determinant in the left-hand side of formula (4.26). By Theorem 4.9, it solves the differential equations

$$\Big(\kappa\,q_i\frac{\partial}{\partial q_i} - \operatorname{tr}X_i(\boldsymbol z;h;\boldsymbol q)|_{(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}}\Big)F(\boldsymbol z;\boldsymbol q) = 0\,,\qquad i=1,\dots,N\,,$$

where $X_i(\boldsymbol z;h;\boldsymbol q)|_{(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}}$ are the restrictions of dynamical Hamiltonians (2.8) to the invariant subspace $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$. Since $\operatorname{tr}X_i(\boldsymbol z;h;\boldsymbol q)|_{(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}} = d^{(1)}_{\boldsymbol\lambda,i}\sum_{a=1}^n z_a$, the function $F(\boldsymbol z;\boldsymbol q)$ equals the product of powers of $q_1,\dots,q_n$ in the right-hand side of formula (4.26) multiplied by a factor that does not depend on $\boldsymbol q$. This factor can be found by taking the limit $q_{i+1}/q_i\to 0$ for all $i=1,\dots,N-1$, using Theorem 4.9. □

Remark 4.11. By Theorem 4.9, the determinant $F(\boldsymbol{z};\boldsymbol{q})$ in Theorem 4.10 solves the difference equations

$$F(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{q}) \,=\, \det K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa)|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}} F(\boldsymbol{z};\boldsymbol{q})\,, \qquad a=1,\dots,n\,, \tag{4.28}$$

where $K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa)|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}$ are the restrictions of the qKZ operators (2.3) to the invariant subspace $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. Equations (4.28) determine the product of Gamma functions in the right-hand side of formula (4.26) up to a κ-periodic function of $z_1,\dots,z_n$.

4.5. **Proof of Theorem 4.9.** Recall $\lambda^{\{1\}}=\sum_{i=1}^{N-1}\lambda^{(i)}$. Set $\lambda^{\{2\}}=\sum_{i=1}^{N-1}\big(\lambda^{(i)}\big)^2$ and $\lambda_{\{2\}}=\sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j$. Notice the homogeneity properties

$$\begin{aligned}\Phi_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{q};\kappa) \,&=\, \kappa^{\lambda^{\{1\}}-\lambda^{\{2\}}}\,\Phi_{\boldsymbol{\lambda}}(\boldsymbol{t}/\kappa;\boldsymbol{z}/\kappa;h/\kappa;\boldsymbol{q};1)\,,\\ \Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,&=\, \kappa^{\lambda^{\{1\}}+\lambda_{\{2\}}}\,\Psi_J(\boldsymbol{z}/\kappa;h/\kappa;\boldsymbol{q};1)\,.\end{aligned} \tag{4.29}$$

To simplify writing, we assume in this section that $\kappa=1$ and omit the corresponding argument in all functions. The general case can be recovered by the homogeneity.

Recall $I^{\min}_{\boldsymbol{\lambda}}$, $A_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h)$, see (3.6), (4.16). For $\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}$, define

$$B_{\boldsymbol{l}}(\boldsymbol{z};h) \,=\, \frac{A_{\boldsymbol{\lambda}}(\Sigma_{I^{\min}_{\boldsymbol{\lambda}}}-\boldsymbol{l};\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_{I^{\min}_{\boldsymbol{\lambda}}}-\boldsymbol{l};h)}\,. \tag{4.30}$$

Set

$$\mathcal{Z}_{\boldsymbol{\lambda}} \,=\, \{\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}\ |\ l^{(i)}_a\geqslant l^{(i+1)}_a,\ \ i=1,\dots,N-1\,,\ \ a=1,\dots,\lambda^{(i)}\}\,. \tag{4.31}$$

Lemma 4.12. *If $\boldsymbol{l}\notin\mathcal{Z}_{\boldsymbol{\lambda}}$, then $B_{\boldsymbol{l}}(\boldsymbol{z};h)=0$. For $\boldsymbol{l}\in\mathcal{Z}_{\boldsymbol{\lambda}}$,*

$$\begin{aligned}B_{\boldsymbol{l}}(\boldsymbol{z};h) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Bigg(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}} &\frac{\Gamma(z_b-z_a+l^{(i)}_a-l^{(i)}_b+1)\,\Gamma(z_b-z_a+l^{(i)}_a-l^{(i+1)}_b+h)}{\Gamma(z_b-z_a+l^{(i)}_a-l^{(i)}_b+h+1)\,\Gamma(z_b-z_a+l^{(i)}_a-l^{(i+1)}_b+1)}\ \times\\ &\times\ \frac{\Gamma(l^{(i)}_a-l^{(i+1)}_a+h)}{(l^{(i)}_a-l^{(i+1)}_a)!}\prod_{c=\lambda^{(i)}+1}^{\lambda^{(i+1)}}\frac{\Gamma(z_c-z_a+l^{(i)}_a-l^{(i+1)}_c+h)}{\Gamma(z_c-z_a+l^{(i)}_a-l^{(i+1)}_c+1)}\Bigg).\end{aligned} \tag{4.32}$$

Recall $\boldsymbol{z}_\sigma=(z_{\sigma(1)},\dots,z_{\sigma(n)})$ and the permutations σ_I, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, defined in Section 3.2. Let $\boldsymbol{r}=(r_1,\dots,r_{N-1})$. For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, define

$$\mathcal{B}_{I,J}(\boldsymbol{z};h;\boldsymbol{r}) \,=\, \sum_{\boldsymbol{l}\in\mathcal{Z}_{\boldsymbol{\lambda}}} B_{\boldsymbol{l}}(\boldsymbol{z};h)\,W_I(\Sigma_{I^{\min}_{\boldsymbol{\lambda}}}-\boldsymbol{l};\boldsymbol{z}_{\sigma_J^{-1}};h)\prod_{i=1}^{N-1} r_i^{\sum_{a=1}^{\lambda^{(i)}}l^{(i)}_a}. \tag{4.33}$$

Set

$$\widetilde{M}_{\boldsymbol{\lambda}}(\boldsymbol{z};h) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Bigg(\sin(\pi h)\prod_{b=a+1}^{\lambda^{(i)}}\frac{\sin\big(\pi\,(z_a-z_b)\big)}{z_a-z_b}\prod_{c=\lambda^{(i)}+1}^{\lambda^{(i+1)}}\sin\big(\pi\,(z_a-z_c-h)\big)\Bigg), \tag{4.34}$$

Proposition 4.13. *For any $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, the function $\widetilde{M}_{\boldsymbol{\lambda}}(\boldsymbol{z};h)\mathcal{B}_{I,J}(\boldsymbol{z};h;\boldsymbol{r})$ is entire in $\boldsymbol{z},h$ and holomorphic in $\boldsymbol{r}$ provided $|r_i|<1$ for all $i=1,\dots,N-1$.*

Proof. By Stirling's formula, ratios of Gamma functions appearing in formula (4.32) have the following asymptotics as $k \to +\infty$ over integers,

$$\frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} \,=\, k^{\alpha-\beta}\bigl(1+o(1)\bigr)\,, \qquad \frac{\Gamma(\alpha-k)}{\Gamma(\beta-k)} \,=\, k^{\alpha-\beta}\,\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}\bigl(1+o(1)\bigr)\,, \tag{4.35}$$

provided α and β are not integers in the second case, and these asymptotics can be differentiated with respect to α and β.

The right-hand side of formula (4.33) allows one to present $\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,\mathcal{B}_{I,J}(\boldsymbol z;h;\boldsymbol r)$ as a power series in $\boldsymbol r$:

$$\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,\mathcal{B}_I(\boldsymbol z_{\sigma_J};\boldsymbol z;h;\boldsymbol r) \,=\, \sum_{\boldsymbol k\in\mathbb{Z}_{\geqslant 0}^{N-1}} \widetilde{B}_{I,J,\boldsymbol k}(\boldsymbol z;h)\,\prod_{i=1}^{N-1} r_i^{k_i}\,.$$

Formulae (4.32), (4.35) show that for given $\boldsymbol z, h$, this series and its formal derivatives with respect to $\boldsymbol z, h$ converge provided $|r_i|<1$ for all $i=1,\dots,N-1$. Therefore, the function $\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,\mathcal{B}_{I,J}(\boldsymbol z;h;\boldsymbol r)$ is holomorphic in $\boldsymbol r$ provided $|r_i|<1$ for all $i=1,\dots,N-1$, and is holomorphic in $\boldsymbol z, h$ whenever all the coefficients $\widetilde{B}_{I,J,\boldsymbol k}(\boldsymbol z;h)$ are holomorphic in $\boldsymbol z, h$.

It remains to show that all the coefficients $\widetilde{B}_{I,J,\boldsymbol k}(\boldsymbol z;h)$ are entire in $\boldsymbol z, h$. To this end, it suffices to show that for any $\boldsymbol l\in\mathcal{Z}_{\boldsymbol\lambda}$, the product $\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,B_{\boldsymbol l}(\boldsymbol z;h)\,W_I(\Sigma_{I^{\min}_{\boldsymbol\lambda}}-\boldsymbol l;\boldsymbol z_{\sigma_J^{-1}};h)$ is an entire function of $\boldsymbol z, h$.

Take $\boldsymbol l\in\mathcal{Z}_{\boldsymbol\lambda}$ and $a\neq b$. Since $l^{(i)}_a\geqslant l^{(i+1)}_a$ and $l^{(i)}_b\geqslant l^{(i+1)}_b$, the ratio

$$\frac{\sin\bigl(\pi\,(z_a-z_b)\bigr)\,\Gamma(z_a-z_b+l^{(i)}_b-l^{(i)}_a+1)\,\Gamma(z_b-z_a+l^{(i)}_a-l^{(i)}_b+1)}{(z_a-z_b)\,\Gamma(z_a-z_b+l^{(i)}_b-l^{(i+1)}_a+1)\,\Gamma(z_b-z_a+l^{(i)}_a-l^{(i+1)}_b+1)}$$

is an entire function of z_a, z_b, and if $l^{(i)}_b>l^{(i+1)}_b$, then the ratio

$$\frac{\Gamma(z_b-z_a+l^{(i)}_a-l^{(i+1)}_b+h)}{\Gamma(z_b-z_a+l^{(i)}_a-l^{(i)}_b+h+1)}$$

is a polynomial in z_a, z_b, h. Set

$$F_{\boldsymbol l}(\boldsymbol z;h) \,=\, \prod_{i=1}^{N-1}\,\prod_{a=1}^{\lambda^{(i)}}\,\prod_{\substack{b=1\\ b\neq a,\, l^{(i)}_b=l^{(i+1)}_b}}^{\lambda^{(i)}} (z_b-z_a+l^{(i)}_a-l^{(i)}_b+h)\,. \tag{4.36}$$

Then by formulae (4.32), (4.34), the product $\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,B_{\boldsymbol l}(\boldsymbol z;h)\,F_{\boldsymbol l}(\boldsymbol z;h)$ is an entire function of $\boldsymbol z, h$. Since by Lemma 4.14 below, the ratio $W_I(\Sigma_{I^{\min}_{\boldsymbol\lambda}}-\boldsymbol l;\boldsymbol z_{\sigma_J^{-1}};h)/F_{\boldsymbol l}(\boldsymbol z;h)$ is a polynomial in $\boldsymbol z, h$, the product $\widetilde{M}_{\boldsymbol\lambda}(\boldsymbol z;h)\,B_{\boldsymbol l}(\boldsymbol z;h)\,W_I(\Sigma_{I^{\min}_{\boldsymbol\lambda}}-\boldsymbol l;\boldsymbol z_{\sigma_J^{-1}};h)$ is an entire function of $\boldsymbol z, h$ too. Proposition 4.13 is proved. ☐

Lemma 4.14. *Let $F_{\boldsymbol l}(\boldsymbol z;h)$ be given by (4.36). Then the ratio $W_I(\Sigma_{I^{\min}_{\boldsymbol\lambda}}-\boldsymbol l;\boldsymbol z_{\sigma_J^{-1}};h)/F_{\boldsymbol l}(\boldsymbol z;h)$ is a polynomial in $\boldsymbol z, h$.*

Proof. The proof is by inspection, similarly to the proof of Lemma 4.4. ☐

Proof of Theorem 4.9. Recall that we assume $\kappa = 1$ and skip the corresponding argument in all functions. By formulae (4.19), (4.30)–(4.33), and Lemmas 4.7, 4.12,

$$\mathcal{M}_J(\Phi_{\boldsymbol{\lambda}} W_I)(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \mathcal{B}_{I,J}(\boldsymbol{z}_{\sigma_J};h;q_2/q_1,\dots,q_N/q_{N-1}) \,\times \\ \times \prod_{i=1}^{N}\bigl(e^{\pi\sqrt{-1}\,(n-\lambda_i)}\,q_i\bigr)^{\sum_{a\in J_i} z_a}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{b\in J_j}\frac{\pi}{\sin\bigl(\pi\,(z_a-z_b)\bigr)}\,. \tag{4.37}$$

Then by formulae (4.20), (4.22), (4.37), we have

$$\Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathcal{B}_{I,J}(\boldsymbol{z}_{\sigma_J};h;q_2/q_1,\dots,q_N/q_{N-1})\,v_I \,\times \\ \times\,\frac{\Omega_{\boldsymbol{\lambda}}(\boldsymbol{q})}{\bigl(\Gamma(h)\bigr)^{\lambda^{\{1\}}}}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\biggl(\prod_{b\in J_j}\frac{\pi}{\Gamma(z_b-z_a+h+1)}\prod_{\substack{c\in J_j\\ c<a}}(z_a-z_c-h)\biggr). \tag{4.38}$$

By Proposition 4.13, the function in the right-hand side of formula (4.38) is holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\mathbb{0}$, and may have poles in $\boldsymbol{z},h$ at most at the hyperplanes $z_a-z_b\in\mathbb{Z}_{\neq 0}$ for $a\neq b$, $z_a-z_b-h\in\mathbb{Z}_{\geqslant 0}$ for $a\in J_i$, $b\in J_j$, $i<j$, and $h\in\mathbb{Z}_{>0}$. To complete the proof of Theorem 4.9, it remains to show that those poles do not actually occur.

For the hyperplanes $z_a-z_b\in\mathbb{Z}_{\neq 0}$, $a\neq b$, we will show the regularity of the function $M_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})\,\Psi_J(\boldsymbol{z};h;\boldsymbol{q})$. Observe that $M_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})\,\Psi_J(\boldsymbol{z};h;\boldsymbol{q})$ is regular at the hyperplanes $z_a=z_b$ for all $a\neq b$, and solves qKZ difference equations (2.5). Since all the qKZ operators $K_1(\boldsymbol{z};h;\boldsymbol{q}),\dots,K_n(\boldsymbol{z};h;\boldsymbol{q})$ and their inverses are regular at the hyperplanes $z_a-z_b\in\mathbb{Z}$ for $a\neq b$, the function $M_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})\,\Psi_J(\boldsymbol{z};h;\boldsymbol{q})$ is regular at all hyperplanes $z_a-z_b\in\mathbb{Z}$ for $a\neq b$.

To deal with the hyperplanes $z_a-z_b-h\in\mathbb{Z}_{\geqslant 0}$ and $h\in\mathbb{Z}_{>0}$, we will show that for given $\boldsymbol{z}$, the function $\Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q})$ is entire in h, and it suffices to do it for generic $\boldsymbol{z}$.

To simplify writing, we will omit the argument $\boldsymbol{z}$ in all functions for a while. Let $\boldsymbol{r}=(r_1,\dots,r_{N-1})$ and $\boldsymbol{r}_*=(1,r_1,r_1r_2,\dots,r_1\dots r_{N-1})$. Denote $F(h;\boldsymbol{r})=\Psi^{\diamond}_J(h;\boldsymbol{r}_*)$, so that $\Psi^{\diamond}_J(h;\boldsymbol{q})=F(h;q_2/q_1,\dots,q_N/q_{N-1})$.

Recall the dynamical operators $X_1,\dots,X_N$, see (2.8). Set

$$X_{(i)}(h;\boldsymbol{r}) \,=\, \sum_{j=i+1}^{N}\Bigl(X_j(h;\boldsymbol{r}_*)-\sum_{a\in J_j}z_a\Bigr)\Big|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}\,,\qquad i=1,\dots,N-1\,.$$

The dynamical differential equations (2.7) for $\Psi_J(h;\boldsymbol{q})$ are equivalent to the following equations for $F(h;\boldsymbol{r})$,

$$r_i\,\frac{\partial}{\partial r_i}F(h;\boldsymbol{r}) \,=\, X_{(i)}(h;\boldsymbol{r})\,F(h;\boldsymbol{r})\,,\qquad i=1,\dots,N-1\,. \tag{4.39}$$

The operators $X_{(1)}(h;\boldsymbol{r}),\dots,X_{(N-1)}(h;\boldsymbol{r})$ are linear functions in h and rational functions in $\boldsymbol{r}$, regular provided $|r_i|<1$ for all $i=1,\dots,N-1$. The eigenvalues of the operator

$X_{(i)}(h;0)$ for a given i are $\sum_{j=i+1}^{N}\bigl(\sum_{a\in I_j} z_a - \sum_{a\in J_j} z_a\bigr)$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$. Hence, one of the eigenvalues of $X_{(i)}(h;0)$ equals zero and all other eigenvalues are not integers for generic $\boldsymbol{z}$. Therefore, a solution $F(h;\boldsymbol{r})$ of equations (4.39) holomorphic $|r_i|<1$ for all i, is uniquely determined by the value $F(h;0)$, and $F(h,\boldsymbol{r})$ is holomorphic in h whenever $F(h;0)$ is. The value $F(h;0)=\Psi^\diamond(h;\emptyset)$ can be found from formulae (4.20), (4.18), (4.32), (4.33), (4.37), (4.38). It is a polynomial in h by Lemmas 4.3, 4.4. Therefore, $F(h,\boldsymbol{r})$ is an entire function of h, and so is $\Psi^\diamond(h;\boldsymbol{q})$.

Formulae (4.23), (4.24), (4.25), and the rationality of $\Psi^\diamond_{J,\boldsymbol{m}}(\boldsymbol{z};h)$ follow from formulae (4.20), (4.30) – (4.33), (4.37), (4.38), and Lemma 4.3. The properties of poles of $\Psi^\diamond_{J,\boldsymbol{m}}(\boldsymbol{z};h)$ are determined by the analytic properties of $\Psi^\diamond_{J}(\boldsymbol{z};h;\boldsymbol{q})$. The fact that $\Psi^\diamond_{J,0}(\boldsymbol{z};h)$ is a polynomial in $\boldsymbol{z},h$ follows from Lemma 4.4. □

4.6. **Solutions of the dynamical and *qKZ* equations parametrized by Laurent polynomials.** In the sequel, we will use the following notation. For any variable x, there is the companion $\acute{x}$, and for any function $f(\acute{x})$, we set $\acute{f}(x;\kappa)=f(e^{2\pi\sqrt{-1}\,x/\kappa})$. The convention for collections of variables is similar. For instance, $\acute{\boldsymbol{z}}=(\acute{z}_1,\dots,\acute{z}_n)$ and $\acute{f}(\boldsymbol{z};\kappa)=f(e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa})$.

Introduce the variables $\gamma_{i,j}$, $i=1,\dots,N$, $j=1,\dots,\lambda_i$. Denote

$$\boldsymbol{\Gamma}=(\gamma_{1,1},\dots,\gamma_{1,\lambda_1},\dots,\gamma_{N,1},\dots,\gamma_{N,\lambda_N}),\qquad \acute{\boldsymbol{\Gamma}}^{\pm1}=(\acute{\gamma}^{\pm1}_{1,1},\dots,\acute{\gamma}^{\pm1}_{N,\lambda_N}). \tag{4.40}$$

Recall $\boldsymbol{z}_\sigma=(z_{\sigma(1)},\dots,z_{\sigma(n)})$, and the permutations σ_I, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, defined in Section 3.2. For a Laurent polynomial $P(\acute{\boldsymbol{\Gamma}};\acute{\boldsymbol{z}};\acute{h})$, set

$$\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)=\sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}\acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};h;\kappa)\,\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa), \tag{4.41}$$

where the functions $\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, are given by (4.20).

Proposition 4.15. *The function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is a solution of the joint system of dynamical differential equations (2.7) and qKZ difference equations (2.5).*

Proof. By Theorem 4.9, the function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ solves the system of equations (2.7) and (2.5) since $\acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};h;\kappa)$ does not depend on $\boldsymbol{q}$ and is κ-periodic function in each of the variables $z_1,\dots,z_n$. □

Let $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ be the space of Laurent polynomials in $\acute{\boldsymbol{\Gamma}}$ symmetric in $\acute{\gamma}_{i,1},\dots,\acute{\gamma}_{i,\lambda_i}$ for each $i=1,\dots,N$.

Proposition 4.16. *For any $P\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$, the function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is holomorphic in $\boldsymbol{z},h,\boldsymbol{q}$ provided $z_a-z_b+h\notin\kappa\mathbb{Z}_{\leqslant0}$ for all $a,b=1,\dots,n$, $a\neq b$, and $|q_{i+1}/q_i|<1$ for all $i=1,\dots,N-1$, with a branch of $\log q_i$ fixed for each $i=1,\dots,N$. The singularities of $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ at the hyperplanes $z_a-z_b+h\in\kappa\mathbb{Z}_{\leqslant0}$ are simple poles.*

Proof. By Theorem 4.9, we need only to show that $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is regular at the hyperplanes $z_a-z_b\in\kappa\mathbb{Z}$, $a\neq b$, where it might have simple poles.

The regularity of $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ at the hyperplanes $z_a=z_b$ for all $a\neq b$ follows from formula (4.25). Since $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ solves *qKZ* difference equations (2.5) and all the *qKZ* operators $K_1(\boldsymbol{z};h;\boldsymbol{q};\kappa),\dots,K_n(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ and their inverses are regular at the hyperplanes $z_a-z_b\in\kappa\mathbb{Z}$, the function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is regular at all hyperplanes $z_a-z_b\in\kappa\mathbb{Z}$, $a\neq b$. □

Denote by $\mathscr{S}_{\boldsymbol{\lambda}}$ the space of solutions of the system of dynamical differential equations (2.7) and qKZ difference equations (2.5) spanned over $\mathbb{C}$ by the functions $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, $P\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$. The space $\mathscr{S}_{\boldsymbol{\lambda}}$ is a $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$-module with $f(\acute{\boldsymbol{z}};\acute{h})$ acting as multiplication by $\acute{f}(\boldsymbol{z};h;\kappa)$.

Define the algebra

$$\mathcal{K}_{\boldsymbol{\lambda}}=\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]\Big/\Big\langle\prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\acute{\gamma}_{i,j})=\prod_{a=1}^{n}(u-\acute{z}_a)\Big\rangle,\tag{4.42}$$

where u is a formal variable. By (4.41), the assignment $P\mapsto\Psi_P$ defines a homomorphism

$$\mu^{\kappa}_{\boldsymbol{\lambda}}:\mathcal{K}_{\boldsymbol{\lambda}}\to\mathscr{S}_{\boldsymbol{\lambda}},\qquad Y\mapsto\Psi_Y,\tag{4.43}$$

of $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$-modules.

By Propositions A.2, A.3, the algebra $\mathcal{K}_{\boldsymbol{\lambda}}$ is a free $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$-module generated by the classes

$$Y_I(\boldsymbol{\Gamma})=V_I(\acute{\gamma}_{1,1}^{-1},\dots,\acute{\gamma}_{1,\lambda_1}^{-1},\dots,\acute{\gamma}_{N,1}^{-1},\dots,\acute{\gamma}_{N,\lambda_N}^{-1}),\qquad I\in\mathcal{I}_{\boldsymbol{\lambda}},\tag{4.44}$$

where the polynomials V_I are defined by formula (A.6). Introduce the coordinates of solutions Ψ_{Y_I}:

$$\Psi_{Y_I}(\boldsymbol{z};h;\boldsymbol{q};\kappa)=\sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}\bar{\Psi}_{I,J}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v_J.\tag{4.45}$$

Theorem 4.17. *Let $n\geqslant 2$. Then*

$$\det\big(\bar{\Psi}_{I,J}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\big)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}}=\Big(e^{\pi\sqrt{-1}\,(n-1)\,d^{(2)}_{\boldsymbol{\lambda}}}\prod_{i=1}^{N}q_i^{d^{(1)}_{\boldsymbol{\lambda},i}}\Big)^{\sum_{a=1}^{n}z_a/\kappa}\prod_{j=2}^{n-1}j^{(n-j)\,d^{(2)}_{\boldsymbol{\lambda}}}\times$$
$$\times\prod_{a=1}^{n-1}\prod_{b=a+1}^{n}\Big(2\pi\sqrt{-1}\,\kappa^2\,\Gamma\big((z_a-z_b+h)/\kappa\big)\,\Gamma\big(1+(z_b-z_a+h)/\kappa\big)\Big)^{d^{(2)}_{\boldsymbol{\lambda}}},\tag{4.46}$$

where $d^{(1)}_{\boldsymbol{\lambda},i}$, $d^{(2)}_{\boldsymbol{\lambda}}$ are given by formulae (4.27).

Proof. The statement follows from Theorem 4.10 and formula (A.7). □

Corollary 4.18. *The map $\mu^{\kappa}_{\boldsymbol{\lambda}}:\mathcal{K}_{\boldsymbol{\lambda}}\to\mathscr{S}_{\boldsymbol{\lambda}}$ is an isomorphism of $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1},\acute{h}^{\pm1}]$-modules.*

Remark 4.19. The algebra $\mathcal{K}_{\boldsymbol{\lambda}}$ is the equivariant K-theory algebra $K_{T\times\mathbb{C}^\times}(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ of the cotangent bundle of the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$, see the notation in Section 6.

4.7. **Levelt fundamental solution.** Recall Definition 4.8 of a function $f(\boldsymbol{q})$ holomorphic in the unit polydisk around $\mathbb{0}$. The dynamical Hamiltonians $X_1(\boldsymbol{z};h;\boldsymbol{q}),\dots,X_n(\boldsymbol{z};h;\boldsymbol{q})$ given by (2.8) are holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\mathbb{0}$ and

$$X_i(\boldsymbol{z};h;\mathbb{0}) \,=\, \sum_{a=1}^{n} z_a e^{(a)}_{i,i} - h \sum_{1\leqslant a<b\leqslant n} \biggl(\,\sum_{j=1}^{i-1} e^{(a)}_{j,i} e^{(b)}_{i,j} - \sum_{j=i+1}^{N} e^{(a)}_{i,j} e^{(b)}_{j,i}\biggr). \tag{4.47}$$

Notice that for $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$X_i(\boldsymbol{z};h;\mathbb{0})\, v_I \,=\, \sum_{a\in I_i} z_a v_I + \sum_{\substack{J\in\mathcal{I}_{\boldsymbol{\lambda}}\\ |\sigma_J|<|\sigma_I|}} \xi_{i,I,J}\, v_J\,,$$

where the coefficients $\xi_{i,I,J}$ take values $0,\pm h$. Therefore, the eigenvalues of the restriction of the operator $X_i(\boldsymbol{z};h;\mathbb{0})$ on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ are $\sum_{a\in I_i} z_a$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$. A more detailed statement is given by Proposition 4.20 below.

Recall the function $\Psi^{\diamond}_{I,0}(\boldsymbol{z};h)$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, given by (4.24).

Proposition 4.20. *Given $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, we have $X_i(\boldsymbol{z};h;\mathbb{0})\,\Psi^{\diamond}_{I,0}(\boldsymbol{z};h) = \sum_{a\in I_i} z_a\,\Psi^{\diamond}_{I,0}(\boldsymbol{z};h)$, and $\Psi^{\diamond}_{I,0}(\boldsymbol{z};h)\neq 0$ provided $z_a\neq z_b$ for all pairs a,b such that $a<b$ and $\sigma_I^{-1}(a)>\sigma_I^{-1}(b)$.*

Proof. The first part of the statement follows from Theorem 4.9. The nonvanishing of $\Psi^{\diamond}_{I,0}(\boldsymbol{z};h)$ is implied by formula (4.24) and Lemmas 4.3, 4.4. □

For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, set $\boldsymbol{E}_I(\boldsymbol{z}) = \bigl(E^{(1)}_I(\boldsymbol{z}),\dots,E^{(N-1)}_I(\boldsymbol{z})\bigr)$, where $E^{(i)}_I(\boldsymbol{z}) = \sum_{j=1}^{i}\sum_{a\in I_j} z_a$ is the eigenvalue of $X_1(\boldsymbol{z};h;\mathbb{0})+\ldots+X_i(\boldsymbol{z};h;\mathbb{0})$ on $\Psi^{\diamond}_{I,0}(\boldsymbol{z};h)$. For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, denote by $D_{I,J}$ the set of points $\boldsymbol{z}$ such that $\boldsymbol{E}_I(\boldsymbol{z})-\boldsymbol{E}_J(\boldsymbol{z})\in\mathbb{Z}^{N-1}_{\geqslant 0}$ and $\boldsymbol{E}_I(\boldsymbol{z})\neq\boldsymbol{E}_J(\boldsymbol{z})$. Set $D_{\boldsymbol{\lambda}} = \bigcup_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} D_{I,J}$.

Theorem 4.21. (i) *For any $\boldsymbol{z},h$ such that $z_a-z_b\notin\kappa\mathbb{Z}_{\neq 0}$ for all $1\leqslant a<b\leqslant n$, there exists an $\mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\bigr)$-valued function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$, holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\mathbb{0}$, such that $\Psi^{\#}(\boldsymbol{z};h;\mathbb{0})$ is the identity operator and the function*

$$\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \Psi^{\#}(\boldsymbol{z}/\kappa;h/\kappa;\boldsymbol{q}) \prod_{i=1}^{N} q_i^{X_i(\boldsymbol{z}/\kappa;h/\kappa;\mathbb{0})}, \tag{4.48}$$

solves dynamical differential equations (2.7). For given $\boldsymbol{z},h$, the function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ with the specified properties is unique if and only if $\boldsymbol{z}\notin\kappa D_{\boldsymbol{\lambda}}$. Furthermore, $\det\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})=1$ and

$$\det\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \prod_{i=1}^{N} q_i^{d^{(1)}_{\boldsymbol{\lambda},i}\sum_{a=1}^{n} z_a/\kappa}, \tag{4.49}$$

where $d^{(1)}_{\boldsymbol{\lambda},1},\dots,d^{(1)}_{\boldsymbol{\lambda},N}$ are given by formula (4.27).

(ii) *Define the function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ for generic $\boldsymbol{z},h$ as in item* (i). *Then $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is holomorphic in $\boldsymbol{z}$ if $z_a-z_b\notin\mathbb{Z}_{\neq 0}$ for all $1\leqslant a<b\leqslant n$, and is entire in h. The singularities of $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ at the hyperplanes $z_a-z_b\in\mathbb{Z}_{\neq 0}$ are simple poles.*

The theorem is proved in Section 4.8. An explicit expression for the function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is given by formula (4.60).

Following [AB, Chapter 2], we will call $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ the *Levelt fundamental solution* of dynamical differential equations (2.7) on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$, see also [CV, Section 6.2].

For a $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solution $\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ of dynamical differential equations (2.7), define its *principal term*

$$\Psi^{\mathbb{0}}(\boldsymbol{z};h;\kappa) \,=\, \widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)^{-1}\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,. \tag{4.50}$$

By Theorem 4.21, the principal term does not depend on $\boldsymbol{q}$.

Set

$$C_{\boldsymbol{\lambda}}(\boldsymbol{z};\kappa) \,=\, \prod_{i=1}^{N} e^{\pi\sqrt{-1}\,(n-\lambda_i)\sum_{a=\lambda^{(i-1)}+1}^{\lambda^{(i)}} z_a/\kappa} \tag{4.51}$$

and

$$G_{\boldsymbol{\lambda}}(\boldsymbol{z};h;\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \Gamma\big((z_a-z_b)/\kappa\big)\,\Gamma\big((z_b-z_a+h)/\kappa\big)\,. \tag{4.52}$$

Proposition 4.22. *For a Laurent polynomial* $P(\acute{\boldsymbol{\Gamma}};\acute{\boldsymbol{z}};\acute{h})$, *the principal term of the solution* $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, *given by* (4.41), *equals*

$$\Psi^{\mathbb{0}}_P(\boldsymbol{z};h;\kappa) \,=\, \sum_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} \acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};h;\kappa)\,C_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J};\kappa)\,G_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J};h;\kappa)\,\frac{W_I(\Sigma_J;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)}\,v_I\,. \tag{4.53}$$

Here $c_{\boldsymbol{\lambda}}(\Sigma_J;h)$ *is given by formula* (4.5).

The proposition is proved in Section 4.8.

4.8. **Proofs of Theorem 4.21 and Proposition 4.22.** To simplify writing, we assume in this section that $\kappa=1$, similarly to Section 4.5, and omit the corresponding argument in all functions. The general case can be recovered by the homogeneity properties (4.29).

Lemma 4.23. *Given* $A\in\operatorname{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big)$, *assume that the function*

$$F_A(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \prod_{i=1}^{N} q_i^{X_i(\boldsymbol{z};h;\mathbb{0})}\,A\,\prod_{i=1}^{N} q_i^{-X_i(\boldsymbol{z};h;\mathbb{0})}$$

is holomorphic in $\boldsymbol{q}$ *in the unit polydisk around* $\mathbb{0}$, *and* $F_A(\boldsymbol{z};h;\mathbb{0})=1$, *the identity operator. Then* $A=1$ *and* $F_A(\boldsymbol{z};h;\boldsymbol{q})=1$, *provided* $\boldsymbol{z}\notin D_{\boldsymbol{\lambda}}$. *On the other hand, if* $\boldsymbol{z}\in D_{\boldsymbol{\lambda}}$, *then there exists* $A\neq 1$ *such that the function* $F_A(\boldsymbol{z};h;\boldsymbol{q})$ *has the requested properties and is not a constant function of* $\boldsymbol{q}$.

Proof. The statement follows from a basic linear algebra reasoning. □

Proof of Theorem 4.21. For the uniqueness statement, assume that the function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is as required in Theorem 4.21. Then the function $\Psi^{\#}_1(\boldsymbol{z};h;\boldsymbol{q})$ has the same requested properties if and only if the product

$$E(\boldsymbol{z};h) \,=\, \prod_{i=1}^N q_i^{-X_i(\boldsymbol{z};h;\mathbb{0})}\, \Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})^{-1}\, \Psi^{\#}_1(\boldsymbol{z};h;\boldsymbol{q})\, \prod_{i=1}^N q_i^{X_i(\boldsymbol{z};h;\mathbb{0})}$$

does not depend on $\boldsymbol{q}$, whilst the function

$$F(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \prod_{i=1}^N q_i^{X_i(\boldsymbol{z};h;\mathbb{0})}\, E(\boldsymbol{z};h)\, \prod_{i=1}^N q_i^{-X_i(\boldsymbol{z};h;\mathbb{0})} \,=\, \Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})^{-1}\, \Psi^{\#}_1(\boldsymbol{z};h;\boldsymbol{q})\,.$$

is holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\mathbb{0}$, and $F(\boldsymbol{z};h;\mathbb{0})$ is the identity operator. Thus the uniqueness statement follows from Lemma 4.23.

Recall the functions $\Psi^\diamond_J(\boldsymbol{z};h;\boldsymbol{q})$ and $\Psi^\diamond_{J,0}(\boldsymbol{z};h)$, see (4.23), (4.24). By Lemma 4.3,

$$\Psi^\diamond_{J,0}(\boldsymbol{z};h) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^N \prod_{a\in J_i}\prod_{\substack{b\in J_j\\ b>a}} \frac{1}{z_a-z_b-h} \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_I(\Sigma_J;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)}\, v_I\,. \tag{4.54}$$

Let $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ to be the operator such that for any $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$\begin{aligned}\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q}) : v_I \,\mapsto\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \Psi^\diamond_J(\boldsymbol{z};h;\boldsymbol{q}) \prod_{i=1}^{N-1}\prod_{j=i+1}^N \prod_{a\in J_i}\prod_{b\in J_j} \frac{\Gamma(z_b-z_a)}{\pi}\,\times \\ \times\, \frac{\check W_I(\Sigma_J;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)} \prod_{i=1}^{N-1}\prod_{j=i+1}^N \prod_{a\in J_i}\prod_{\substack{b\in J_j\\ b<a}} \frac{1}{z_a-z_b-h}\end{aligned} \tag{4.55}$$

We will verify below that the function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is as required in Theorem 4.21.

By Theorem 4.9, the functions $\Psi^\diamond_J(\boldsymbol{z};h;\boldsymbol{q})$ are holomorphic in $\boldsymbol{q}$ in the unit polydisk around $\mathbb{0}$, hence so does $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$. Then $\Psi^{\#}(\boldsymbol{z};h;\mathbb{0})$ is the identity operator by formulae (4.23), (4.54), and orthogonality relation (4.9).

By formulae (4.9), (4.54), and Proposition 4.20,

$$\prod_{i=1}^N q_i^{X_i(\boldsymbol{z};h;\mathbb{0})} v_I \,=\, \sum_{J,K\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{\check W_I(\Sigma_J;\boldsymbol{z};h)\, W_K(\Sigma_J;\boldsymbol{z};h)}{c^2_{\boldsymbol{\lambda}}(\Sigma_J;h)\, R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})\, Q_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J};h)} \prod_{i=1}^N q_i^{\sum_{a\in J_i} z_a}\, v_K\,.$$

Then by formulae (4.55), (4.10), (4.22), the functions

$$\begin{aligned}\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q}) \prod_{i=1}^N q_i^{X_i(\boldsymbol{z};h;\mathbb{0})} v_I \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \Psi_J(\boldsymbol{z};h;\boldsymbol{q})\, \frac{\check W_I(\Sigma_J;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)} \prod_{i=1}^N e^{\pi\sqrt{-1}\,(\lambda_i-n)\sum_{a\in J_i} z_a}\,\times \\ \times \prod_{i=1}^{N-1}\prod_{j=i+1}^N \prod_{a\in J_i}\prod_{b\in J_j} \frac{-1}{\Gamma(1+z_a-z_b)\,\Gamma(1-z_a+z_b+h)}\end{aligned}$$

are linear combinations of the solutions $\Psi_J(\boldsymbol{z};h;\boldsymbol{q})$ of dynamical differential equations (2.7) with coefficients independent of $\boldsymbol{q}$. Hence, $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q}) = \Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q}) \prod_{i=1}^{N} q_i^{X_i(\boldsymbol{z};h;\mathbb{0})}$ solves dynamical differential equations (2.7).

The determinant $\det \widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q})$ satisfies the equations

$$\Big(q_i \frac{\partial}{\partial q_i} - \operatorname{tr} X_i(\boldsymbol{z};h;\boldsymbol{q})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}\Big) \det \widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q}) = 0\,, \qquad i=1,\dots,N\,,$$

Since $\operatorname{tr} X_i(\boldsymbol{z};h;\boldsymbol{q})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}} = \operatorname{tr} X_i(\boldsymbol{z};h;\mathbb{0})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}} = d^{(1)}_{\boldsymbol{\lambda},i}\,(z_1+\ldots+z_n)$, where $d^{(1)}_{\boldsymbol{\lambda},i}$ are given by (4.27), the determinant $\det \Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ does not depend on $\boldsymbol{q}$. Therefore,

$$\det \Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q}) = \det \Psi^{\#}(\boldsymbol{z};h;\mathbb{0}) = 1 \qquad \text{and} \qquad \det \widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q}) = \prod_{i=1}^{N} q_i^{d^{(1)}_{\boldsymbol{\lambda},i} \sum_{a=1}^{n} z_a}.$$

The functions $\Psi^{\diamond}_J(\boldsymbol{z};h;\boldsymbol{q})$ in formula (4.55) are entire in $\boldsymbol{z},h$ by Theorem 4.9, and the expressions

$$\frac{\check{W}_I(\Sigma_J;\boldsymbol{z};h)}{c_{\boldsymbol{\lambda}}(\Sigma_J;h)} \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{\substack{b\in J_j\\ b<a}} \frac{1}{z_a-z_b-h}$$

are polynomials by Lemma 4.4 applied to the functions $\check{W}_I(\boldsymbol{t};\boldsymbol{z};h)$. Hence, $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is holomorphic in $\boldsymbol{z},h$ provided $z_a-z_b\notin\mathbb{Z}$ for all $a\neq b$. Moreover, $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is regular for $z_a=z_b$ by formula (4.25), and the singularities at the hyperplanes $z_a-z_b\in\mathbb{Z}_{\neq 0}$ are simple poles. Theorem 4.21 is proved. □

Proof of Proposition 4.22. Denote by $\bar{\Psi}_P(\boldsymbol{z};h)$ the right-hand side of formula (4.53). Then formula (4.41), Proposition 4.20, and Theorem 4.9 yield $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q})\,\bar{\Psi}_P(\boldsymbol{z};h) = \Psi_P(\boldsymbol{z};h;\boldsymbol{q})$. Hence by definition (4.50) of the principal term, $\Psi^{\mathbb{0}}_P(\boldsymbol{z};h) = \bar{\Psi}_P(\boldsymbol{z};h)$. □

Consider the entries of $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ in the standard basis $\{v_I\,,\ I\in\mathcal{I}_{\boldsymbol{\lambda}}\}$ of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$,

$$\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q}) : v_J \mapsto \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \Psi^{\#}_{I,J}(\boldsymbol{z};h;\boldsymbol{q})\, v_I\,. \tag{4.56}$$

Recall the function $A(\boldsymbol{t};\boldsymbol{z};h;\kappa)$ at $\kappa=1$, see (4.16),

$$A(\boldsymbol{t};\boldsymbol{z};h) = \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}} \Bigg(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}} \frac{\Gamma(t^{(i)}_b - t^{(i)}_a + 1)}{\Gamma(t^{(i)}_b - t^{(i)}_a + h)} \prod_{c=1}^{\lambda^{(i+1)}} \frac{\Gamma(t^{(i+1)}_c - t^{(i)}_a + h)}{\Gamma(t^{(i+1)}_c - t^{(i)}_a + 1)} \Bigg), \tag{4.57}$$

where $\lambda^{(N)} = n$ and $t^{(N)}_a = z_a$, $a=1,\dots,n$. Notice that

$$A(\Sigma_I;\boldsymbol{z};h) = \big(\Gamma(h)\big)^{\lambda^{\{1\}}} \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in I_i}\prod_{b\in I_j} \frac{\Gamma(z_b-z_a+h)}{\Gamma(z_b-z_a+1)}\,,$$

where $\lambda^{\{1\}}=\sum_{i=1}^{N-1}\lambda^{(i)}$. Recall the function $\Omega_{\boldsymbol{\lambda}}(h;\boldsymbol{q};\kappa)$ at $\kappa=1$, see (4.21),

$$\Omega_{\boldsymbol{\lambda}}(h;\boldsymbol{q}) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-q_j/q_i)^{h\lambda_i}\,. \tag{4.58}$$

For $\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}$, set

$$\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};h) \,=\, \sum_{K\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{A(\Sigma_K-\boldsymbol{l};\boldsymbol{z};h)\,W_I(\Sigma_K-\boldsymbol{l};\boldsymbol{z};h)\,\check W_J(\Sigma_K;\boldsymbol{z};h)}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_K})\,Q_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_K};h)\,A(\Sigma_K;\boldsymbol{z};h)\,c_{\boldsymbol{\lambda}}(\Sigma_K-\boldsymbol{l};h)\,c_{\boldsymbol{\lambda}}(\Sigma_K;h)}\,. \tag{4.59}$$

Proposition 4.24. *We have*

$$\Psi^{\#}_{I,J}(\boldsymbol{z};h;\boldsymbol{q}) \,=\, \Omega_{\boldsymbol{\lambda}}(h;\boldsymbol{q})\sum_{\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}}\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};h)\,\prod_{i=1}^{N-1}(q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}}l_a^{(i)}}. \tag{4.60}$$

Proof. The statement follows from formula (4.55) and Theorem 4.9. □

4.9. **The map $\text{Б}_{\boldsymbol{\lambda}}$.** In Section 4.6, we introduced the space $\mathscr{S}_{\boldsymbol{\lambda}}$ of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of the joint system of dynamical differential equations (2.7) and qKZ difference equations (2.5) spanned over $\mathbb{C}$ by the functions $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ labeled by Laurent polynomials in $\acute{\boldsymbol{\Gamma}},\acute{\boldsymbol{z}},\acute h$; we also defined the map

$$\mu^{\mathcal{K}}_{\boldsymbol{\lambda}}:\mathcal{K}_{\boldsymbol{\lambda}}\to\mathscr{S}_{\boldsymbol{\lambda}}\,,\qquad Y\mapsto\Psi_Y\,,$$

see (4.43). In Section 4.7 we introduced the Levelt fundamental solution $\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ of dynamical differential equations (2.7), see (4.48), (4.55). Denote by $\mathscr{S}^{\widehat{\ }}_{\boldsymbol{\lambda}}$ the space of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of dynamical differential equations (2.7) spanned over $\mathbb{C}$ by the functions $\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v$, $v\in(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. Since $\det\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\neq 0$, see (4.49), there is an isomorphism

$$\mu^{\widehat{\ }}_{\boldsymbol{\lambda}}:(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\to\mathscr{S}^{\widehat{\ }}_{\boldsymbol{\lambda}}\,,\qquad v\mapsto\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v\,.$$

Let $L\subset\mathbb{C}^n\times\mathbb{C}$ be the complement of the union of the hyperplanes

$$z_a-z_b+h\in\kappa\mathbb{Z}_{\leqslant 0}\,,\qquad z_a-z_b\in\kappa\mathbb{Z}_{\neq 0}\,,\qquad a,b=1,\dots,n\,,\quad a\neq b\,. \tag{4.61}$$

Denote by $\mathcal{O}_L$ the ring of functions of $\boldsymbol{z},h$ holomorphic in L. Let $\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}}$ be the space of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of dynamical differential equations (2.7) holomorphic in $\boldsymbol{q}$ provided $|q_{i+1}/q_i|<1$ for all $i=1,\dots,N-1$, with a branch of $\log q_i$ fixed for each $i=1,\dots,N$, and holomorphic in $\boldsymbol{z},h$ in L. Both spaces $\mathscr{S}_{\boldsymbol{\lambda}}$ and $\mathscr{S}^{\widehat{\ }}_{\boldsymbol{\lambda}}$ are subspaces of $\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}}$, see Proposition 4.16 and Theorem 4.21. Let

$$\mu^{\mathcal{O}_L}_{\boldsymbol{\lambda}}:(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}_L\to\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}}\,,\qquad v\mapsto\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v\,, \tag{4.62}$$

be the $\mathcal{O}_L$-linear extension of the map $\mu^{\widehat{\ }}_{\boldsymbol{\lambda}}$.

Recall the functions

$$C_{\boldsymbol{\lambda}}(\boldsymbol{z};\kappa) \,=\, \prod_{i=1}^{N} e^{\pi\sqrt{-1}\,(n-\lambda_i)\sum_{a=\lambda^{(i-1)}+1}^{\lambda_i}z_a/\kappa}$$

and

$$G_{\boldsymbol\lambda}(\boldsymbol z;h;\kappa)\;=\;\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}\Gamma\big((z_a-z_b)/\kappa\big)\,\Gamma\big((z_b-z_a+h)/\kappa\big)\,,$$

see (4.51), (4.52). Define a map

$$\text{Б}_{\boldsymbol\lambda}\,:\,\mathcal K_{\boldsymbol\lambda}\to(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L\,,$$

$$[P]\,\mapsto\sum_{I,J\in\mathcal I_{\boldsymbol\lambda}}\acute P(\boldsymbol z_{\sigma_J};\boldsymbol z;h;\kappa)\,C_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};\kappa)\,G_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};h;\kappa)\,\frac{W_I(\Sigma_J;\boldsymbol z;h)}{c_{\boldsymbol\lambda}(\Sigma_J;h)}\,v_I\,, \tag{4.63}$$

where $[P]\in\mathcal K_{\boldsymbol\lambda}$ stands for the class of the Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z};\acute h)$. By Proposition 4.22, the map $\text{Б}_{\boldsymbol\lambda}$ sends the class $Y\in\mathcal K_{\boldsymbol\lambda}$ to the principal term of the solution Ψ_Y of the joint system of dynamical differential equations (2.7) and qKZ difference equations (2.5).

Proposition 4.25. *The map* $\text{Б}_{\boldsymbol\lambda}:\mathcal K_{\boldsymbol\lambda}\to(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L$ *is well-defined and the following diagram is commutative,*

$$\begin{array}{ccc}\mathcal K_{\boldsymbol\lambda} & \xrightarrow{\ \text{Б}_{\boldsymbol\lambda}\ } & (\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L\\ & {\scriptstyle\mu^{\mathcal K}_{\boldsymbol\lambda}}\searrow \quad \swarrow{\scriptstyle\mu^{\mathcal O_L}_{\boldsymbol\lambda}} & \\ & \mathscr S^{\mathcal O_L}_{\boldsymbol\lambda} & \end{array} \tag{4.64}$$

Proof. By Lemma 4.4, poles of the sum in the right-hand side of (4.63) are at most those of the function $G_{\boldsymbol\lambda}(\boldsymbol z;h;\kappa)$ and, therefore, in addition to hyperplanes (4.61) can occur only at the hyperplanes $z_a=z_b$, $a\ne b$. However, the sums

$$\sum_{J\in\mathcal I_{\boldsymbol\lambda}}\acute P(\boldsymbol z_{\sigma_J};\boldsymbol z;h;\kappa)\,C_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};\kappa)\,G_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};h;\kappa)\,\frac{W_I(\Sigma_J;\boldsymbol z;h)}{c_{\boldsymbol\lambda}(\Sigma_J;h)}$$

are regular at the hyperplanes $z_a=z_b$ for all a,b, by the standard reasoning. Hence, the map $\text{Б}_{\boldsymbol\lambda}$ is well-defined.

The commutativity of diagram (4.64) follows from Proposition 4.22. □

Remark 4.26. Since $\widehat\Psi(\boldsymbol z;h;\boldsymbol q;\kappa)$ is holomorphic in $\boldsymbol z,h$ in L, and $\big(\det\widehat\Psi(\boldsymbol z;h;\boldsymbol q;\kappa)\big)^{-1}$, is entire in $\boldsymbol z,h$, see (4.49), the inverse matrix $\widehat\Psi(\boldsymbol z;h;\boldsymbol q;\kappa)^{-1}$ is holomorphic in $\boldsymbol z,h$ in L. Therefore, for every $\Psi\in\mathscr S^{\mathcal O_L}_{\boldsymbol\lambda}$, its principal term $\Psi^{\boldsymbol\emptyset}=\widehat\Psi^{-1}\Psi$, defined by (4.50), belongs to $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L$, and there is an isomorphism $\mathscr S^{\mathcal O_L}_{\boldsymbol\lambda}\to(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L$, $\Psi\mapsto\Psi^{\boldsymbol\emptyset}$. The inverse map equals $\mu^{\mathcal O_L}_{\boldsymbol\lambda}$, see (4.62), so that, $\mathscr S^{\mathcal O_L}_{\boldsymbol\lambda}=\mathscr S^{\hat{}}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O_L$.

4.10. **Example** $\boldsymbol\lambda=(1,n-1)$**.** Throughout this section, let $N=2$ and $\boldsymbol\lambda=(1,n-1)$. Denote by $[a]$ the element $\big(\{a\},\{1,\dots,a-1,a+1,\dots,n\}\big)\in\mathcal I_{\boldsymbol\lambda}$. The space $(\mathbb C^2)^{\otimes n}_{\boldsymbol\lambda}$ has a basis $v_{[1]},\dots,v_{[n]}$, where $v_{[a]}=v_2^{\otimes(a-1)}\otimes v_1\otimes v_2^{\otimes(n-a)}$. Clearly $e^{(a)}_{1,1}v_{[b]}=\delta_{a,b}\,v_{[b]}$ and $e^{(a)}_{2,2}v_{[b]}=(1-\delta_{a,b})\,v_{[b]}$.

The qKZ operators $K_1,\ldots,K_n$, see (2.3), are

$$K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, R^{(a,a-1)}(z_a-z_{a-1}+\kappa;h)\,\ldots\,R^{(a,1)}(z_a-z_1+\kappa;h)\,\times \\ \times\, q_1^{e_{1,1}^{(a)}} q_2^{e_{2,2}^{(a)}}\,R^{(a,n)}(z_a-z_n;h)\,\ldots\,R^{(a,a+1)}(z_i-z_{i+1};h)\,. \tag{4.65}$$

The R-matrices in the right-hand side preserve the subspace $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol{\lambda}}\subset(\mathbb{C}^2)^{\otimes n}$, acting there as follows,

$$R^{(a,b)}(z;h)\,v_{[a]} \,=\, \frac{z}{z-h}\,v_{[a]}-\frac{h}{z-h}\,v_{[b]}\,,\qquad R^{(a,b)}(z;h)\,v_{[b]} \,=\, \frac{z}{z-h}\,v_{[b]}-\frac{h}{z-h}\,v_{[a]}\,,$$

$$R^{(a,b)}(z;h)\,v_{[c]} \,=\, v_{[c]}\,,\qquad c\neq a,b\,.$$

The qKZ difference equations (2.5) are

$$f(z_1,\ldots,z_a+\kappa,\ldots,z_n;h;\boldsymbol{q};\kappa) \,=\, K_a(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,f(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,,\qquad a=1,\ldots,n\,. \tag{4.66}$$

The dynamical Hamiltonians X_1,X_2, see (2.8), act on $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol{\lambda}}$ as follows

$$X_1(\boldsymbol{z};h;\boldsymbol{q})\,v_{[a]} \,=\, z_a\,v_{[a]}-h\sum_{b=1}^{a-1}v_{[b]}-\frac{hq_2}{q_1-q_2}\sum_{\substack{b=1\\ b\neq a}}^{n}v_{[b]}\,, \\ X_2(\boldsymbol{z};h;\boldsymbol{q})\,v_{[a]} \,=\, \Bigl(-X_1(\boldsymbol{z};h;\boldsymbol{q})+\sum_{b=1}^{n}z_b\Bigr)\,v_{[a]}\,, \tag{4.67}$$

and the dynamical differential equations (2.7) are

$$\kappa\,q_1\frac{\partial}{\partial q_1}\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, X_1(\boldsymbol{z};h;\boldsymbol{q})\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,, \\ \kappa\,q_2\frac{\partial}{\partial q_2}\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, X_2(\boldsymbol{z};h;\boldsymbol{q})\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,. \tag{4.68}$$

In this section, we use the variable $t=t^{(1)}_1$. The substitution $\boldsymbol{t}=\Sigma_{[a]}$ reads as $t=z_a$. The weight functions are

$$W_{[a]}(t;\boldsymbol{z};h) \,=\, \prod_{b=1}^{a-1}(t-z_b)\prod_{b=a+1}^{n}(t-z_b-h)\,,\qquad a=1,\ldots,n\,. \tag{4.69}$$

We have $c_{\boldsymbol{\lambda}}(t;h)=1$, see (4.5). The permutations $\sigma_{[1]},\ldots,\sigma_{[n]}$ are

$$\sigma_{[a]}(1)=a\,,\qquad \sigma_{[a]}(b)=b-1\,,\quad b=1,\ldots,a-1\,,\qquad \sigma_{[a]}(b)=b\,,\quad b=a+1,\ldots,n\,,$$

and $|\sigma_{[a]}| = a-1$. We have

$$W_{[a]}(z_a;\boldsymbol{z};h) \,=\, \prod_{b=1}^{a-1}(z_a-z_b)\prod_{b=a+1}^{n}(z_a-z_b-h)\,, \tag{4.70}$$

$$W_{[a]}(z_b;\boldsymbol{z};h) \,=\, 0\,, \qquad b=1,\dots,a-1\,,$$

and $W_{[a]}(z_b;\boldsymbol{z};h)$ is divisible by the product $\prod_{b=a+1}^{n}(z_a-z_b-h)$ for any $b=1,\dots,n$, cf. Lemmas 4.3, 4.4

The functions $\check W_{[a]}(t;\boldsymbol{z};h)$, see (4.7), are

$$\check W_{[a]}(t;\boldsymbol{z};h) \,=\, \prod_{b=1}^{a-1}(t-z_b-h)\prod_{c=a+1}^{n}(t-z_b)\,, \qquad a=1,\dots,n\,. \tag{4.71}$$

Set

$$\bar R_a(\boldsymbol{z}) \,=\, \prod_{\substack{b=1\\ b\neq a}}^{n}(z_a-z_b)\,, \qquad \bar Q_a(\boldsymbol{z};h) \,=\, \prod_{\substack{b=1\\ b\neq a}}^{n}(z_a-z_b-h)\,, \qquad a=1,\dots,n\,.$$

Then $\bar R_a(\boldsymbol{z}) = R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_{[a]}})$, and $\bar Q_a(\boldsymbol{z};h) = Q_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_{[a]}};h)$, where the functions $R_{\boldsymbol{\lambda}}(\boldsymbol{z})$, $Q_{\boldsymbol{\lambda}}(\boldsymbol{z};h)$, are given by (4.8). Biorthogonality relations (4.9), (4.10) become

$$\sum_{c=1}^{n}\frac{W_{[a]}(z_c;\boldsymbol{z};h)\,\check W_{[b]}(z_c;\boldsymbol{z};h)}{\bar R_c(\boldsymbol{z})\,\bar Q_c(\boldsymbol{z};h)} \,=\, \delta_{a,b}\,,$$

$$\sum_{c=1}^{n} W_{[c]}(z_a;\boldsymbol{z};h)\,\check W_{[c]}(z_b;\boldsymbol{z};h) \,=\, \delta_{a,b}\,\bar R_a(\boldsymbol{z})\,\bar Q_a(\boldsymbol{z};h)\,.$$

The master function, see (4.11), is

$$\Phi_{\boldsymbol{\lambda}}(t;\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=$$

$$=\,(e^{\pi\sqrt{-1}}q_2)^{\sum_{a=1}^{n}z_a/\kappa}\,(e^{\pi\sqrt{-1}\,(n-2)}q_1/q_2)^{t/\kappa}\prod_{a=1}^{n}\Gamma\big((t-z_a)/\kappa\big)\,\Gamma\big((z_a-t+h)/\kappa\big)\,.$$

The hypergeometric solutions (4.20) of the joint system of differential equations (4.68) and difference equations (4.66) have the form

$$\Psi_{[a]}(\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, \frac{(1-q_2/q_1)^{h/\kappa}}{\kappa\,\Gamma(h/\kappa)}\sum_{b=1}^{n}\sum_{l=0}^{\infty}\mathrm{Res}_{t=\,z_a-l\kappa}\,\Phi_{\boldsymbol{\lambda}}(t;\boldsymbol{z};h;\boldsymbol{q};\kappa)\;W_{[b]}(z_a-l\kappa;h)\,v_{[b]}\,,$$

where the residues of the master function are

$$\mathrm{Res}_{t=\,z_a-l\kappa}\,\Phi_{\boldsymbol{\lambda}}(t;\boldsymbol{z};h;\boldsymbol{q};\kappa) \,=\, q_1^{z_a/\kappa}\,q_2^{\sum_{c=1,\,c\neq a}^{n}z_c/\kappa}\prod_{\substack{c=1\\ c\neq a}}^{n}\frac{\pi\,e^{\pi\sqrt{-1}\,(z_a+z_c)/\kappa}}{\sin\big(\pi\,(z_a-z_c)/\kappa\big)}\,\times$$

$$\times\,\frac{\kappa\,(q_2/q_1)^l\,\Gamma(l+h/\kappa)}{l!}\prod_{\substack{c=1\\ c\neq a}}^{n}\frac{\Gamma\big(l+(z_c-z_a+h)/\kappa\big)}{\Gamma\big(l+1+(z_c-z_a)/\kappa\big)}\,.$$

Determinant formula for coordinates of the hypergeometric solutions, see (4.26), is

$$\det\Big(\sum_{l=0}^{\infty}\operatorname{Res}_{t=\,z_a-l\kappa}\Phi_{\boldsymbol\lambda}(t;\boldsymbol z;h;\boldsymbol q;\kappa)\,W_{[b]}(z_a-l\kappa;h)\Big)_{a,b=1}^{n}= \tag{4.72}$$

$$=\frac{\big(q_1q_2^{n-1}\big)^{\sum_{a=1}^n z_a/\kappa}\big(\kappa\,\Gamma(h/\kappa)\big)^n}{(1-q_2/q_1)^{nh/\kappa}}\prod_{a=1}^{n-1}\prod_{b=a+1}^{n}\frac{\pi\kappa^2\,\Gamma\big((z_a-z_b+h)/\kappa\big)\,\Gamma\big(1+(z_b-z_a+h)/\kappa\big)}{e^{-\pi\sqrt{-1}\,(z_a+z_b)/\kappa}\,\sin\big(\pi\,(z_a-z_b)/\kappa\big)}\,.$$

By formulae (4.69), (4.70), and the Vandermonde determinant formula, equality (4.72) transforms to

$$\det\Big(\sum_{l=0}^{\infty}\sum_{c=1}^{n}\operatorname{Res}_{t=\,z_c-l\kappa}\big(t^{a-1}e^{-2\pi\sqrt{-1}\,(b-1)\,t/\kappa}\,\Phi_{\boldsymbol\lambda}(t;\boldsymbol z;h;\boldsymbol q;\kappa)\big)\Big)_{a,b=1}^{n}= \tag{4.73}$$

$$=\frac{\big(q_1q_2^{n-1}\big)^{\sum_{a=1}^n z_a/\kappa}\big(-2\pi\sqrt{-1}\,\big)^{n(n-1)/2}\kappa^{n(n+1)/2}\big(\Gamma(h/\kappa)\big)^n}{(1-q_2/q_1)^{nh/\kappa}}\times$$

$$\times\prod_{a=1}^{n-1}\prod_{b=a+1}^{n}\Gamma\big((z_a-z_b+h)/\kappa\big)\,\Gamma\big((z_b-z_a+h)/\kappa\big)\,.$$

Let $\acute\gamma=\acute\gamma_{1,1}$. For a Laurent polynomial $P(\acute\gamma;\acute{\boldsymbol z};\acute h)$, we have

$$\acute P(\gamma;\boldsymbol z;h;\kappa)\,=\,P(e^{2\pi\sqrt{-1}\,\gamma/\kappa};e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa};e^{2\pi\sqrt{-1}\,h/\kappa})\,.$$

The solution Ψ_P of differential equations (4.68) and difference equations (4.66) corresponding to $P(\acute\gamma;\acute{\boldsymbol z};\acute h)$ is

$$\Psi_P(\boldsymbol z;h;\boldsymbol q;\kappa)\,=\,\sum_{a=1}^{n}\acute P(z_a;\boldsymbol z;h;\kappa)\,\Psi_{[a]}(\boldsymbol z;h;\boldsymbol q;\kappa)\,. \tag{4.74}$$

By Proposition 4.16, $\Psi_P(\boldsymbol z;h;\boldsymbol q;\kappa)$ is holomorphic in $\boldsymbol z,h,\boldsymbol q$ provided $z_a-z_b+h\notin\kappa\mathbb Z_{\leqslant 0}$ for all $a,b=1,\dots,n$, $a\neq b$, and $|q_2/q_1|<1$ with branches of $\log q_1$ and $\log q_2$ fixed. The singularities of $\Psi_P(\boldsymbol z;h;\boldsymbol q;\kappa)$ at the hyperplanes $z_a-z_b+h\in\kappa\mathbb Z_{\leqslant 0}$ are simple poles.

The solution $\Psi_P(\boldsymbol z;h;\boldsymbol q;\kappa)$ can be written as an integral over a suitable contour C encircling the poles of the product $\prod_{a=1}^n\Gamma\big((t-z_a)/\kappa\big)$ counterclockwise and separating them from the poles of the product $\prod_{a=1}^n\Gamma\big((z_a-t+h)/\kappa\big)$,

$$\Psi_P(\boldsymbol z;h;\boldsymbol q;\kappa)\,= \tag{4.75}$$

$$=\frac{(1-q_2/q_1)^{h/\kappa}}{2\pi\sqrt{-1}\,\kappa\,\Gamma(h/\kappa)}\int_C\acute P(t;\boldsymbol z;h;\kappa)\,\Phi_{\boldsymbol\lambda}(t;\boldsymbol z;h;\boldsymbol q;\kappa)\sum_{a=1}^{n}W_{[a]}(t;\boldsymbol z;h)\,v_{[a]}\,dt\,.$$

For instance, if h/κ is sufficiently large positive real, the integral can be taken over the parabola

$$C\,=\,\{\,h/2-\kappa\big(s^2-s\sqrt{-1}\,\big)\;\mid\;s\in\mathbb R\,\}\,.$$

Formula (4.75) can be used to give an alternative proof of analytic properties of the function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$.

Let $\mathscr{S}_{\boldsymbol{\lambda}}$ be the space of solutions of the joint system of dynamical differential equations (4.68) and qKZ difference equations (4.66) spanned over $\mathbb{C}$ by the functions $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ corresponding to Laurent polynomials $P(\acute{\gamma};\acute{\boldsymbol{z}};\acute{h})$. The space $\mathscr{S}_{\boldsymbol{\lambda}}$ is a $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm 1},\acute{h}^{\pm 1}]$-module with $f(\acute{\boldsymbol{z}};\acute{h})$ acting as multiplication by $f(e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa};e^{2\pi\sqrt{-1}\,h/\kappa})$.

For $\boldsymbol{\lambda}=(1,n-1)$, the algebra $\mathcal{K}_{\boldsymbol{\lambda}}$, see (4.42), can be presented as follows

$$\mathcal{K}_{\boldsymbol{\lambda}} \,=\, \mathbb{C}[\acute{\gamma}^{\pm 1},\acute{\boldsymbol{z}}^{\pm 1},\acute{h}^{\pm 1}]\Big/\Big\langle \prod_{a=1}^{n}(\acute{\gamma}-\acute{z}_a)\,=\,0\Big\rangle . \tag{4.76}$$

The function $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ depends only on the class of the Laurent polynomial P in $\mathcal{K}_{\boldsymbol{\lambda}}$. The assignment $P\mapsto\Psi_P$ defines the homomorphism

$$\mu^{\kappa}_{\boldsymbol{\lambda}}:\mathcal{K}_{\boldsymbol{\lambda}}\to\mathscr{S}_{\boldsymbol{\lambda}}\,,\qquad Y\mapsto\Psi_Y\,,$$

of $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm 1},\acute{h}^{\pm 1}]$-modules. Formula (4.73) implies that the homomorphism $\mu^{\kappa}_{\boldsymbol{\lambda}}$ is an isomorphism.

The algebra $\mathcal{K}_{\boldsymbol{\lambda}}$ is the equivariant K-theory algebra $K_{T\times\mathbb{C}^\times}(T^*\mathbb{CP}^{n-1};\mathbb{C})$ of the cotangent bundle of the projective space $\mathbb{CP}^{n-1}$, see the notation in Section 6.

Recall Definition 4.8 of a function $f(\boldsymbol{q})$ holomorphic in the unit polydisk around $\boldsymbol{q}=\emptyset$. For example, the dynamical Hamiltonians X_1,X_2, see (4.67), are holomorphic in the unit polydisk around $\boldsymbol{q}=\emptyset$, and $X_1(\boldsymbol{z};h;\emptyset)$, $X_2(\boldsymbol{z};h;\emptyset)$ act on $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol{\lambda}}$ as follows

$$X_1(\boldsymbol{z};h;\emptyset)\,v_{[a]}\,=\,z_a\,v_{[a]}-h\sum_{b=1}^{a-1}v_{[b]}\,,\qquad X_2(\boldsymbol{z};h;\emptyset)\,v_{[a]}\,=\,\Big(-X_1(\boldsymbol{z};h;\emptyset)+\sum_{b=1}^{n}z_b\Big)\,v_{[a]}\,,$$

The Levelt fundamental solution $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, see (4.48), of differential equations (4.68) is

$$\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,=\,\Psi^{\#}(\boldsymbol{z}/\kappa;h/\kappa;\boldsymbol{q})\;q_1^{X_1(\boldsymbol{z};h;\emptyset)/\kappa}q_2^{X_2(\boldsymbol{z};h;\emptyset)/\kappa},$$

where the $\operatorname{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big)$-valued function $\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})$ is as follows,

$$\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})\,:\,v_{[b]}\;\mapsto\;\sum_{a=1}^{n}\Psi^{\#}_{a,b}(\boldsymbol{z};h;\boldsymbol{q})\,v_{[a]}\,,$$

$$\Psi^{\#}_{a,b}(\boldsymbol{z};h;\boldsymbol{q})\,=$$
$$=\,(1-q_2/q_1)^h\sum_{l=0}^{\infty}(q_2/q_1)^l\sum_{c=1}^{n}\frac{W_{[a]}(z_c-l;\boldsymbol{z};h)\,\check{W}_{[b]}(z_c;\boldsymbol{z};h)}{\bar{R}_c(\boldsymbol{z})\,\bar{Q}_c(\boldsymbol{z};h)}\prod_{d=1}^{n}\prod_{m=0}^{l-1}\frac{z_d-z_c+h+m}{z_d-z_c+1+m}\,.$$

Furthermore, one has $\det\Psi^{\#}(\boldsymbol{z};h;\boldsymbol{q})=1$ and $\det\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,=\,\big(q_1\,q_2^{n-1}\big)^{\sum_{a=1}^{n}z_a/\kappa}$.

For a solution $\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ of dynamical differential equations (4.68), its principal term, see (4.50), is

$$\Psi^{\varnothing}(\boldsymbol{z};h;\kappa) \,=\, \widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)^{-1}\,\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,.$$

The principal term of the solution $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, corresponding to a Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}};\acute{h})$, see (4.53), equals

$$\Psi^{\varnothing}_P(\boldsymbol{z};h;\kappa) \,=\, \sum_{a=1}^{n} v_{[a]} \sum_{b=1}^{n} W_{[a]}(z_b;\boldsymbol{z};h)\,\acute{P}(z_b;\boldsymbol{z};h;\kappa)\,e^{\pi\sqrt{-1}\left((n-2)z_b+\sum_{c=1}^{n}z_c\right)}\,\times \\ \times\prod_{\substack{c=1\\ c\neq b}}^{n}\Gamma\bigl((z_b-z_c)/\kappa\bigr)\,\Gamma\bigl((z_c-z_b+h)/\kappa\bigr)\,. \tag{4.77}$$

Let L be the complement of the union of the hyperplanes

$$z_a-z_b\in\kappa\mathbb{Z}_{\neq 0}\,,\qquad z_a-z_b+h\in\kappa\mathbb{Z}_{\leqslant 0}\,,\qquad a,b=1,\dots,n\,,\quad a\neq b\,,$$

cf. (4.61). Denote by $\mathcal{O}_L$ the ring of functions of $\boldsymbol{z},h$ holomorphic in L. The map

$$\text{Б}_{\boldsymbol{\lambda}}:\mathcal{K}_{\boldsymbol{\lambda}}\to(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}_L\,, \tag{4.78}$$

$$[P]\,\mapsto\sum_{a=1}^{n} v_{[a]} \sum_{b=1}^{n} W_{[a]}(z_b;\boldsymbol{z};h)\,\acute{P}(z_b;\boldsymbol{z};h;\kappa)\,e^{\pi\sqrt{-1}\,\left((n-2)z_b+\sum_{c=1}^{n}z_c\right)/\kappa}\,\times \\ \times\prod_{\substack{c=1\\ c\neq b}}^{n}\Gamma\bigl((z_b-z_c)/\kappa\bigr)\,\Gamma\bigl((z_c-z_b+h)/\kappa\bigr)\,,$$

sends the class $[P]\in\mathcal{K}_{\boldsymbol{\lambda}}$ of the Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}};\acute{h})$ to the principal term of the solution Ψ_P of the joint system of differential equations (4.68) and difference equations (4.66).

Let $\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}}$ be the space of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of dynamical differential equations (4.68) holomorphic in $\boldsymbol{q}$ provided $|q_2/q_1|<1$ with branches of $\log q_1$ and $\log q_2$ fixed, and holomorphic in $\boldsymbol{z},h$ in L. The space $\mathscr{S}_{\boldsymbol{\lambda}}$ is a subspace of $\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}}$.

Since the matrix $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ is holomorphic in $\boldsymbol{z},h$ in L, and $\bigl(\det\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\bigr)^{-1}$ is entire in $\boldsymbol{z},h$, the inverse matrix $\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)^{-1}$ is also holomorphic in $\boldsymbol{z},h$ in L. Thus the map

$$\mu^{\mathcal{O}_L}_{\boldsymbol{\lambda}}:(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}_L\to\mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}},\qquad v\mapsto\widehat{\Psi}(\boldsymbol{z};h;\boldsymbol{q};\kappa)\,v\,,$$

gives an isomorphism of $\mathcal{O}_L$-modules. Furthermore, the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{K}_{\boldsymbol{\lambda}} & \xrightarrow{\ \text{Б}_{\boldsymbol{\lambda}}\ } & (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}_L \\ & {\scriptstyle\mu^{\mathcal{K}}_{\boldsymbol{\lambda}}}\searrow \quad \swarrow{\scriptstyle\mu^{\mathcal{O}_L}_{\boldsymbol{\lambda}}} & \\ & \mathscr{S}^{\mathcal{O}_L}_{\boldsymbol{\lambda}} & \end{array}$$

see Proposition 4.25.

5. Limit $h\to\infty$ for solutions of the dynamical and qKZ equations

5.1. Limiting weight functions $W^\circ_I(\boldsymbol t;\boldsymbol z)$. For $I\in\mathcal I_{\boldsymbol\lambda}$, define the *limiting weight functions*

$$W^\circ_I(\boldsymbol t;\boldsymbol z)\,=\,\mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}}\,\dots\,\mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\,U^\circ_I(\boldsymbol t;\boldsymbol z)\,,\tag{5.1}$$

$$U^\circ_I(\boldsymbol t;\boldsymbol z)\,=\,\prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\Bigg(\prod_{\substack{c=1\\ i^{(j+1)}_c<i^{(j)}_a}}^{\lambda^{(j+1)}}(t^{(j)}_a-t^{(j+1)}_c)\prod_{b=a+1}^{\lambda^{(j)}}\frac{1}{t^{(j)}_b-t^{(j)}_a}\Bigg)\,.$$

Recall the element $I^{\min}_{\boldsymbol\lambda}$ and the permutation σ_I. Set

$$d_I\,=\,\sum_{j=1}^{N-1}\lambda^{(j)}(\lambda^{(j+1)}-1)-|\sigma_I|\,.$$

Lemma 5.1. *For $I\in\mathcal I_{\boldsymbol\lambda}$, we have*

$$W^\circ_I(\boldsymbol t;\boldsymbol z)\,=\,\lim_{h\to\infty}\,(-h)^{-d_I}\,W_I(\boldsymbol t;\boldsymbol z;h)\,.$$

Proof. The statement follows from formulae (4.2), (5.1) by induction on the length of σ_I. □

Example. Let $N=2$, $n=2$, $\boldsymbol\lambda=(1,1)$, $I=(\{1\},\{2\})$, $J=(\{2\},\{1\})$. Then

$$W^\circ_I(\boldsymbol t;\boldsymbol z)\,=\,1\,,\qquad W^\circ_J(\boldsymbol t;\boldsymbol z)\,=\,t^{(1)}_1-z_1\,.$$

Lemma 5.2. *We have $W^\circ_{I^{\min}_{\boldsymbol\lambda}}(\boldsymbol t;\boldsymbol z)=1$ and*

$$W^\circ_{\sigma_0(I^{\min}_{\boldsymbol\lambda})}(\boldsymbol t;\boldsymbol z)\,=\,\prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\prod_{b=\lambda^{(j)}+1}^{\lambda^{(j+1)}}(t^{(j)}_a-z_{n-b+1})\,,\tag{5.2}$$

where $\sigma_0\in S_n$ is the longest permutation, $\sigma_0(a)=n+1-a$, $a=1,\dots,n$.

Let $\Upsilon_{\boldsymbol\lambda}=\{\,i\mid\lambda_{i+1}\neq0\,\}\subset\{1,\dots,N-1\}$.

Lemma 5.3. *For the transpositions $s_{\lambda^{(i)},\,\lambda^{(i)}+1}$, $i\in\Upsilon_{\boldsymbol\lambda}$, we have*

$$W^\circ_{s_{\lambda^{(i)},\,\lambda^{(i)}+1}(I^{\min}_{\boldsymbol\lambda})}(\boldsymbol t;\boldsymbol z)\,=\,\sum_{j=1}^{\lambda^{(i)}}(t^{(i)}_j-z_j)\,.\tag{5.3}$$

Lemmas 5.2 and 5.3 are proved in Appendix C

The following lemma describes the $h\to\infty$ limit of the three-term relations of Lemma 4.1 for functions $W^\circ_I(\boldsymbol t;\boldsymbol z;h)$.

Lemma 5.4. *For any $I\in\mathcal I_{\boldsymbol\lambda}$, $i=1,\dots,N-1$, and $a=1,\dots,n-1$, we have*

$$W^\circ_{s_{a,a+1}(I)}(\boldsymbol t;\boldsymbol z)\,=\,\frac{W^\circ_I(\boldsymbol t;\boldsymbol z)-W^\circ_I(\boldsymbol t;z_1,\dots,z_{a+1},z_a,\dots,z_n)}{z_{a+1}-z_a}\,,$$

if $|s_{a,a+1}\sigma_I|<|\sigma_I|$, and $W^\circ_I(\boldsymbol t;z_1,\dots,z_{a+1},z_a,\dots,z_n)\,=\,W^\circ_I(\boldsymbol t;\boldsymbol z)$, otherwise.

Proof. Notice that $\sigma_{s_{a,a+1}(I)} = s_{a,a+1}\sigma_I$ unless $s_{a,a+1}(I) = I$. Now the statement follows from Lemmas 5.1 and 4.1. □

Remark 5.5. Define the operators $\Delta_1, \dots, \Delta_{n-1}$ acting on functions of $z_1, \dots, z_n$:

$$\Delta_a f(\boldsymbol{z}) = \frac{f(\boldsymbol{z}) - f(z_1, \dots, z_{a+1}, z_a, \dots, z_n)}{z_a - z_{a+1}}.$$

They satisfy the nil-Coxeter relations,

$$\Delta_a^2 = 0, \qquad \Delta_a \Delta_b = \Delta_b \Delta_a, \quad |a-b| > 1, \qquad \Delta_a \Delta_{a+1} \Delta_a = \Delta_{a+1} \Delta_a \Delta_{a+1},$$

for any a, b.

Denote $\boldsymbol{y} = (y_1, \dots, y_n)$. Let the functions $\widetilde{W}_I^\circ(\boldsymbol{y};\boldsymbol{z})$ be obtained from $W_I^\circ(\boldsymbol{t};\boldsymbol{z})$ by the substitution $t_j^{(i)} = y_j$, $i = 1, \dots, N-1$, $j = 1, \dots, \lambda^{(i)}$. By Lemmas 5.2, 5.4, B.4, and Proposition B.3, the functions $\widetilde{W}_I^\circ(\boldsymbol{y};\boldsymbol{z})$ coincide with the A-type double Schubert polynomials $\mathfrak{S}_\sigma(\boldsymbol{y};\boldsymbol{z})$,

$$\widetilde{W}_I^\circ(\boldsymbol{y};\boldsymbol{z}) = \mathfrak{S}_{\sigma_I}(\boldsymbol{y};\boldsymbol{z}). \tag{5.4}$$

For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, define

$$\check{W}_I^\circ(\boldsymbol{t};\boldsymbol{z}) = W_{\sigma_0(I)}^\circ(\boldsymbol{t}; z_n, \dots, z_1), \tag{5.5}$$

where σ_0 is the longest permutation. Recall $R_{\boldsymbol{\lambda}}(\boldsymbol{z})$, see (4.8), and $\boldsymbol{z}_\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$.

Lemmas 5.6 and 5.7 below follow by Lemma 5.1 from Lemma 4.3, Proposition 4.5, and Corollary 4.6, respectively. Lemmas 5.6 and 5.7 are equivalent to the vanishing and orthogonality properties of the double Schubert polynomials.

Lemma 5.6. *For $I, J \in \mathcal{I}_{\boldsymbol{\lambda}}$, we have $W_J^\circ(\Sigma_I;\boldsymbol{z};h) = 0$ unless $I = J$ or $|\sigma_I| > |\sigma_J|$, and*

$$W_I^\circ(\Sigma_I;\boldsymbol{z}) = \prod_{j=1}^{N-1} \prod_{k=j+1}^{N} \prod_{a\in I_j} \prod_{\substack{b\in I_k \\ b<a}} (z_a - z_b). \tag{5.6}$$

Lemma 5.7. *The functions $W_I^\circ(\boldsymbol{t};\boldsymbol{z})$ and $\check{W}_J^\circ(\boldsymbol{t};\boldsymbol{z})$ are biorthogonal,*

$$\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W_J^\circ(\Sigma_I;\boldsymbol{z})\,\check{W}_K^\circ(\Sigma_I;\boldsymbol{z})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_I})} = \delta_{J,K}, \qquad \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} W_I^\circ(\Sigma_J;\boldsymbol{z})\,\check{W}_I^\circ(\Sigma_K;\boldsymbol{z}) = \delta_{J,K}\, R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J}). \tag{5.7}$$

5.2. **Limiting master function.** Define the *limiting master function*:

$$\Phi_{\boldsymbol{\lambda}}^\circ(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa) = (\kappa^{\lambda_N-n}p_N)^{\sum_{a=1}^n z_a/\kappa} \prod_{i=1}^{N-1} \bigl(\kappa^{\lambda_i+\lambda_{i+1}}p_i/p_{i+1}\bigr)^{\sum_{a=1}^{\lambda^{(i)}} t_a^{(i)}/\kappa} \times$$
$$\times \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}} \Biggl(\prod_{\substack{b=1 \\ b\neq a}}^{\lambda^{(i)}} \frac{1}{\Gamma\bigl((t_a^{(i)} - t_b^{(i)})/\kappa\bigr)} \prod_{c=1}^{\lambda^{(i+1)}} \Gamma\bigl((t_a^{(i)} - t_c^{(i+1)})/\kappa\bigr) \Biggr), \tag{5.8}$$

where $\lambda^{(N)} = n$ and $t_a^{(N)} = z_a$, $a = 1, \dots, n$. The last formula can be also written as

$$\Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa) \,=\, (\kappa^{\lambda_N - n} p_N)^{\sum_{a=1}^n z_a/\kappa} \prod_{i=1}^{N-1} \bigl(\kappa^{\lambda_i+\lambda_{i+1}} p_i/p_{i+1}\bigr)^{\sum_{a=1}^{\lambda^{(i)}} t_a^{(i)}/\kappa} \times$$

$$\times \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \Biggl(\prod_{b=1}^{a-1} \frac{(t_a^{(i)} - t_b^{(i)}) \sin\bigl(\pi\,(t_b^{(i)} - t_a^{(i)})/\kappa\bigr)}{\pi\kappa} \prod_{c=1}^{\lambda^{(i+1)}} \Gamma\bigl((t_a^{(i)} - t_c^{(i+1)})/\kappa\bigr) \Biggr),$$

Denote by $\widetilde\Phi_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; h; \boldsymbol p; \kappa)$ the function obtained from the master function $\Phi_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; h; \boldsymbol q; \kappa)$ by the substitution

$$q_i \,=\, p_i\, h^{\sum_{j=i+1}^N \lambda_j \,-\, \sum_{j=1}^{i-1} \lambda_j}\, e^{\pi\sqrt{-1}\,(\lambda_i - n)}, \qquad i = 1, \dots, N, \tag{5.9}$$

cf. (3.9).

Recall $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)}$, $\lambda^{\{2\}} = \sum_{i=1}^{N-1} \bigl(\lambda^{(i)}\bigr)^2$, $\lambda_{\{2\}} = \sum_{1 \leqslant i < j \leqslant N} \lambda_i \lambda_j$.

Lemma 5.8. *For* $|\arg(h/\kappa)| < \pi$, *we have*

$$\Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa) \,=\, \lim_{h\to\infty} (-h)^{\lambda^{\{2\}} - \lambda^{\{1\}}} \bigl(\Gamma(h/\kappa)\bigr)^{-\lambda^{\{1\}} - \lambda_{\{2\}}}\, \widetilde\Phi_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; h; \boldsymbol p; \kappa).$$

Proof. The statement follows from formulae (4.11), (5.8) by Stirling's formula

$$\frac{\Gamma(\alpha + h/\kappa)}{\Gamma(\beta + h/\kappa)} \,\sim\, (h/\kappa)^{\alpha-\beta}, \qquad h \to \infty, \quad |\arg(h/\kappa)| < \pi. \tag{5.10}$$

□

5.3. **Solutions of the limiting dynamical and *qKZ* equations.** For a polynomial $f(\boldsymbol t; \boldsymbol z)$, define the Jackson integral

$$\mathcal M_J(\Phi^\circ_{\boldsymbol\lambda} f)(\boldsymbol z; \boldsymbol p; \kappa) \,= \sum_{\boldsymbol l \in \mathbb Z^{\lambda^{\{1\}}}} \mathrm{Res}_{\boldsymbol t = \Sigma_J - \boldsymbol l\kappa} \bigl(\Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa)\, f(\boldsymbol t; \boldsymbol z)\bigr), \qquad J \in \mathcal I_{\boldsymbol\lambda}. \tag{5.11}$$

Notice that the master function $\Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa)$ has only simple poles, and

$$\mathrm{Res}_{\boldsymbol t = \Sigma_J - \boldsymbol l\kappa} \bigl(\Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa)\, f(\boldsymbol t; \boldsymbol z)\bigr) \,=\, f(\Sigma_J - \boldsymbol l\kappa; \boldsymbol z)\ \mathrm{Res}_{\boldsymbol t = \Sigma_J - \boldsymbol l\kappa}\, \Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa).$$

A closed expression for the residue $\mathrm{Res}_{\boldsymbol t = \Sigma_J - \boldsymbol l\kappa}\, \Phi^\circ_{\boldsymbol\lambda}(\boldsymbol t; \boldsymbol z; \boldsymbol p; \kappa)$ is given by Lemma 5.9 below.

Set

$$M^\circ_{\boldsymbol\lambda}(\boldsymbol z; \kappa) \,=\, \pi^{-\lambda_{\{2\}}} \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}} \sin\bigl(\pi\,(z_a - z_b)/\kappa\bigr), \tag{5.12}$$

cf. (4.15), and

$$A^\circ(\boldsymbol t; \boldsymbol z; \kappa) \,=\, \prod_{i=1}^{N-1} \prod_{a=1}^{\lambda^{(i)}} \Biggl(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}} \Gamma\bigl(1 + (t_b^{(i)} - t_a^{(i)})/\kappa\bigr) \prod_{c=1}^{\lambda^{(i+1)}} \frac{1}{\Gamma\bigl(1 + (t_c^{(i+1)} - t_a^{(i)})/\kappa\bigr)} \Biggr). \tag{5.13}$$

Lemma 5.9. *If* $\boldsymbol{l}\notin\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}$, *then* $\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\,\Phi^\circ_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa)=0$. *For* $\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}$,

$$\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\,\Phi^\circ_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa)\;=\;\frac{A^\circ_{\boldsymbol{\lambda}}(\Sigma_J-\boldsymbol{l}\kappa;\boldsymbol{z};\kappa)}{M^\circ_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J};\kappa)}\;\times \tag{5.14}$$

$$\times\;\kappa^{\lambda^{\{1\}}}\prod_{i=1}^{N}\bigl(\kappa^{\sum_{j=i+1}^N\lambda_j-\sum_{j=1}^{i-1}\lambda_j}\,p_i\bigr)^{\sum_{a\in J_i}z_a/\kappa}\,\prod_{i=1}^{N-1}\bigl((-\kappa)^{-\lambda_i-\lambda_{i+1}}\,p_{i+1}/p_i\bigr)^{\sum_{a=1}^{\lambda^{(i)}}l_a^{(i)}}.$$

In particular,

$$\mathrm{Res}_{\boldsymbol{t}=\Sigma_J}\,\Phi^\circ_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa)\;= \tag{5.15}$$

$$=\;\kappa^{\lambda^{\{1\}}}\prod_{i=1}^{N}\bigl(\kappa^{\sum_{j=i+1}^N\lambda_j-\sum_{j=1}^{i-1}\lambda_j}\,p_i\bigr)^{\sum_{a\in J_i}z_a/\kappa}\,\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{b\in J_j}\Gamma\bigl((z_a-z_b)/\kappa\bigr)\,.$$

By Lemma 5.9, the actual summation in formula (5.11) is only over the positive cone of the lattice,

$$\mathcal{M}_J(\Phi_{\boldsymbol{\lambda}}f)(\boldsymbol{z};\boldsymbol{p};\kappa)\;=\sum_{\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}} f(\Sigma_J-\boldsymbol{l}\kappa;\boldsymbol{z};\boldsymbol{p})\;\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\,\Phi^\circ_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa)\,. \tag{5.16}$$

Lemma 5.10. *For* $|\arg(h/\kappa)|<\pi$, *we have*

$$\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\,\Phi^\circ_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\kappa)\;=$$

$$=\lim_{h\to\infty}\,(-h)^{\lambda^{\{2\}}-\lambda^{\{1\}}}\bigl(\Gamma(h/\kappa)\bigr)^{-\lambda^{\{1\}}-\lambda_{\{2\}}}\;\mathrm{Res}_{\boldsymbol{t}=\Sigma_J-\boldsymbol{l}\kappa}\,\widetilde{\Phi}_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{z};h;\boldsymbol{p};\kappa)\,.$$

The convergence is uniform in $\boldsymbol{l}$ *and locally uniform in* $\boldsymbol{z},\boldsymbol{p}$.

Proof. The claim follows from formulae (4.17), (5.9), (5.14) by Lemmas D.1, D.2 that are detalization of Stirling's formula (5.10). □

Define

$$\Psi^\circ_J(\boldsymbol{z};\boldsymbol{p};\kappa)\;=\;\kappa^{-\lambda^{\{1\}}}\,\Omega^\circ_{\boldsymbol{\lambda}}(\boldsymbol{p};\kappa)\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathcal{M}_J(\Phi^\circ_{\boldsymbol{\lambda}}W^\circ_I)(\boldsymbol{z};\boldsymbol{p};\kappa)\,v_I\,, \tag{5.17}$$

where

$$\Omega^\circ_{\boldsymbol{\lambda}}(\boldsymbol{p};\kappa)\;=\;e^{\sum^\circ_{i<j}p_j/(\kappa p_i)} \tag{5.18}$$

the sum taken over all pairs $1\leqslant i<j\leqslant N$ such that $\lambda_i=1$ and $\lambda_s=0$ for all $s=i+1,\dots,j$.

Recall the function $\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, see (4.20). Let $\widetilde{\Psi}_J(\boldsymbol{z};h;\boldsymbol{p};\kappa)$ be the function obtained from $\Psi_J(\boldsymbol{z};h;\boldsymbol{q};\kappa)$ by substitution (5.9).

Proposition 5.11. *For* $|\arg(h/\kappa)|<\pi$, *we have*

$$\Psi^\circ_J(\boldsymbol{z};\boldsymbol{p};\kappa)\;=\;\lim_{h\to\infty}\bigl(-h\Gamma(h/\kappa)\bigr)^{-\lambda_{\{2\}}}(-h)^{\sum_{b<c,\,j<k}e^{(b)}_{k,k}e^{(c)}_{j,j}}\;\widetilde{\Psi}_J(\boldsymbol{z};h;\boldsymbol{p};\kappa)\,. \tag{5.19}$$

The convergence is locally uniform in $\boldsymbol{z},\boldsymbol{p}$.

Proof. The statement follows from formulae (4.19), (4.20), (4.21), (5.16), (5.17), (5.18), and Lemmas 5.1, 5.10. □

Definition 5.12. *Say that a function* $f(\boldsymbol{p})$ *is entire in* $\boldsymbol{p}_{\div}$ *if* $f(\boldsymbol{p}) = g(p_2/p_1, \dots, p_N/p_{N-1})$ *for an entire function* $g(s_1, \dots, s_{N-1})$. *Denote* $f(\mathbb{0}) = g(0, \dots, 0)$.

Theorem 5.13. *The* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$*-valued function* $\Psi^{\circ}_J(\boldsymbol{z};\boldsymbol{p};\kappa)$ *is a solution of the joint system of dynamical differential equations* (3.13) *and qKZ difference equations* (3.12). *It has the form*

$$\Psi^{\circ}_J(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \Psi^{\vartriangle}_J(\boldsymbol{z};\boldsymbol{p};\kappa) \prod_{i=1}^{N} \bigl(\kappa^{\sum_{j=i+1}^{N}\lambda_j - \sum_{j=1}^{i-1}\lambda_j}\, p_i\bigr)^{\sum_{a\in J_i} z_a/\kappa} \times \\ \times \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{b\in J_j} \frac{1}{\sin\bigl(\pi\,(z_a - z_b)/\kappa\bigr)}\,, \tag{5.20}$$

where the function $\Psi^{\vartriangle}_J(\boldsymbol{z};\boldsymbol{p};\kappa)$ *is entire in* $\boldsymbol{z}$ *and is entire in* $\boldsymbol{p}_{\div}$. *In more detail,*

$$\Psi^{\vartriangle}_J(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in J_i}\prod_{b\in J_j} \frac{\pi}{\Gamma\bigl(1+(z_b - z_a)/\kappa\bigr)} \times \\ \times \biggl(\Psi^{\vartriangle}_{J,0}(\boldsymbol{z}) + \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^{N-1}_{\geqslant 0}\\ \boldsymbol{m}\neq 0}} \Psi^{\vartriangle}_{J,\boldsymbol{m}}(\boldsymbol{z};\kappa) \prod_{i=1}^{N-1} (p_{i+1}/p_i)^{m_i}\biggr), \tag{5.21}$$

where

$$\Psi^{\vartriangle}_{J,0}(\boldsymbol{z}) \,=\, W^{\circ}_J(\Sigma_J;\boldsymbol{z})\, v_J \,+ \sum_{\substack{I\in\mathcal{I}_{\boldsymbol{\lambda}}\\ |\sigma_I|<|\sigma_J|}} W^{\circ}_I(\Sigma_J;\boldsymbol{z})\, v_I \tag{5.22}$$

is a polynomial in $\boldsymbol{z}$, *and* $\Psi^{\vartriangle}_{J,\boldsymbol{m}}(\boldsymbol{z};\kappa)$ *for* $\boldsymbol{m}\neq 0$ *are rational functions of* $\boldsymbol{z},\kappa$ *with at most simple poles on the hyperplanes* $z_a - z_b \in \kappa\mathbb{Z}_{>0}$ *for* $a\in J_i$, $b\in J_j$, $1\leqslant i<j\leqslant N$. *Furthermore, for any transposition* $s_{a,b}\in S_n$,

$$\Psi^{\vartriangle}_J(\boldsymbol{z};\boldsymbol{p};\kappa)\big|_{z_a=z_b} \,=\, \Psi^{\vartriangle}_{s_{a,b}(J)}(\boldsymbol{z};\boldsymbol{p};\kappa)\big|_{z_a=z_b}. \tag{5.23}$$

Proof. The statement follows from Theorem 4.9, Lemmas 3.2, 3.10, 5.1, 5.9, 5.10, and Proposition 5.11. See also [TV6, Section 11]. □

The functions $\Psi^{\circ}_J(\boldsymbol{z};\boldsymbol{p};\kappa)$ are called the *multidimensional hypergeometric solutions* of the dynamical equations 3.13 and *qKZ* difference equations (3.12).

The next theorem computes the determinant of coordinates of solutions $\Psi^{\circ}_J(\boldsymbol{z};\boldsymbol{p};\kappa)$ and is analogous to [TV6, formula (11.23)].

Theorem 5.14. *Let* $n \geqslant 2$ *. Then*

$$\det\big(\Omega^\circ_{\boldsymbol{\lambda}}(\boldsymbol{p},\kappa)\,\mathcal{M}_J(\Phi^\circ_{\boldsymbol{\lambda}} W^\circ_I)(\boldsymbol{z};\boldsymbol{p};\kappa)\big)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} = \tag{5.24}$$

$$= \kappa^{\lambda^{\{1\}} d_{\boldsymbol{\lambda}}} \prod_{i=1}^{N} p_i^{d^{(1)}_{\boldsymbol{\lambda},i} \sum_{a=1}^n z_a/\kappa} \prod_{a=1}^{n-1} \prod_{b=a+1}^{n} \bigg(\frac{\pi\kappa}{\sin\big(\pi\,(z_a-z_b)/\kappa\big)}\bigg)^{d^{(2)}_{\boldsymbol{\lambda}}},$$

where $\lambda^{\{1\}} = \sum_{i=1}^{N-1}(N-i)\,\lambda_i$ *and* $d_{\boldsymbol{\lambda}}$, $d^{(1)}_{\boldsymbol{\lambda},i}$, $d^{(2)}_{\boldsymbol{\lambda}}$ *are given by formulae* (4.27).

Proof. The statement follows from Lemma 5.1, Proposition 5.11, and formula (4.26).

Alternatively, denote by $F(\boldsymbol{z};\boldsymbol{p})$ the determinant in the left-hand side of formula (5.24). By Theorem 5.13, it solves the differential equations

$$\Big(\kappa p_i \frac{\partial}{\partial p_i} - \operatorname{tr} X^\circ_i(\boldsymbol{z};\boldsymbol{p})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}\Big) F(\boldsymbol{z};\boldsymbol{p}) \,=\, 0\,, \qquad i=1,\dots,N,$$

where $X^\circ_i(\boldsymbol{z};\boldsymbol{p})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}$ are the restrictions of dynamical Hamiltonians (3.8) to the invariant subspace $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. Since $\operatorname{tr} X^\circ_i(\boldsymbol{z};h;\boldsymbol{q})|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}} = d^{(1)}_{\boldsymbol{\lambda},i}\sum_{a=1}^n z_a$, the function $F(\boldsymbol{z};\boldsymbol{p})$ equals the product of powers of $p_1,\dots,p_n$ in the right-hand side of formula (4.26) multiplied by a factor that does not depend on $\boldsymbol{p}$. This factor can be found by taking the limit $p_{i+1}/p_i \to 0$ for all $i=1,\dots,N-1$, using Theorem 5.13. □

Remark 5.15. By Theorem 5.13, the determinant $F(\boldsymbol{z};\boldsymbol{p})$ in Theorem 5.14 solves the difference equations

$$F(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{p}) \,=\, \det K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}\, F(\boldsymbol{z};\boldsymbol{p})\,, \qquad a=1,\dots,n, \tag{5.25}$$

where $K^\circ_a(\boldsymbol{z};h;\boldsymbol{q};\kappa)|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}}$ are the restrictions of *qKZ* operators (3.4) to the invariant subspace $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. Since $\det K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa) = (-1)^{d^{(2)}_{\boldsymbol{\lambda}}}$, equations (5.25) determine the product of sines in the right-hand side of formula (5.24) up to a κ-periodic function of $z_1,\dots,z_n$.

5.4. **Solutions parametrized by Laurent polynomials.** Recall the notation from Section 4.6. For a Laurent polynomial $P(\acute{\boldsymbol{\Gamma}};\acute{\boldsymbol{z}})$, set

$$\Psi^\circ_P(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};\kappa)\,\Psi^\circ_J(\boldsymbol{z};\boldsymbol{p};\kappa)\,. \tag{5.26}$$

Let $\widetilde{\Psi}_P(\boldsymbol{z};h;\boldsymbol{p};\kappa)$ be the function obtained from $\Psi_P(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, see (4.41), by substitution (5.9).

Lemma 5.16. *For* $|\arg(h/\kappa)| < \pi,$ *we have*

$$\Psi^\circ_P(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \lim_{h\to\infty} \big(-h\Gamma(h/\kappa)\big)^{-\lambda^{\{2\}}} (-h)^{\sum_{b<c,\, j<k} e^{(b)}_{k,k} e^{(c)}_{j,j}}\, \widetilde{\Psi}_P(\boldsymbol{z};h;\boldsymbol{p};\kappa)\,. \tag{5.27}$$

The convergence is locally uniform in $\boldsymbol{z},\boldsymbol{p}$.

Proof. The statement follows from formulae (4.41), (5.26), and Proposition 5.11. □

Proposition 5.17. *The function* $\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ *is a solution of the joint system of limiting dynamical differential equations* (3.13) *and qKZ difference equations* (3.12)*. Furthermore, for* $P\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$*, the function* $\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ *is entire in* $\boldsymbol{z}$ *and is holomorphic in* $\boldsymbol{p}$ *provided a branch of* $\log p_i$ *is fixed for each* $i=1,\dots,N$.

Proof. The statement follows from Propositions 4.15, 4.16, and Lemma 5.16. □

Denote by $\mathscr{S}^{\circ}_{\boldsymbol{\lambda}}$ the space of solutions of the system of dynamical differential equations (3.13) and *qKZ* difference equations (3.12) spanned over $\mathbb{C}$ by the functions $\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$, $P\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$. The space $\mathscr{S}^{\circ}_{\boldsymbol{\lambda}}$ is a $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$-module with $f(\acute{\boldsymbol{z}})$ acting as multiplication by $\acute{f}(\boldsymbol{z})$.

Define the algebra

$$\mathcal{K}^{\circ}_{\boldsymbol{\lambda}}\,=\,\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]\Big/\Big\langle\prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\acute{\gamma}_{i,j})\,=\,\prod_{a=1}^{n}(u-\acute{z}_a)\Big\rangle\,, \tag{5.28}$$

where u is a formal variable. By (5.26), the assignment $P\mapsto\Psi^{\circ}_P$ defines a homomorphism

$$\mu^{\mathcal{K}^{\circ}}_{\boldsymbol{\lambda}}:\mathcal{K}^{\circ}_{\boldsymbol{\lambda}}\to\mathscr{S}^{\circ}_{\boldsymbol{\lambda}}\,,\qquad Y\mapsto\Psi^{\circ}_Y\,, \tag{5.29}$$

of $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$-modules.

By Proposition A.2, A.3, the algebra $\mathcal{K}^{\circ}_{\boldsymbol{\lambda}}$ is a free $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$-module generated by the classes

$$Y_I(\boldsymbol{\Gamma})\,=\,V_I(\acute{\gamma}^{-1}_{1,1},\dots,\acute{\gamma}^{-1}_{1,\lambda_1},\dots,\acute{\gamma}^{-1}_{N,1},\dots,\acute{\gamma}^{-1}_{N,\lambda_N})\,,\qquad I\in\mathcal{I}_{\boldsymbol{\lambda}}\,, \tag{5.30}$$

where the polynomials V_I are defined by formula (A.6). Introduce the coordinates of solutions $\Psi^{\circ}_{Y_I}$:

$$\Psi^{\circ}_{Y_I}(\boldsymbol{z};\boldsymbol{p};\kappa)\,=\,\sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}\bar{\Psi}^{\circ}_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa)\,v_J\,.$$

Theorem 5.18. *Let* $n\geqslant 2$*. Then*

$$\det\big(\bar{\Psi}^{\circ}_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa)\big)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}}= \tag{5.31}$$

$$=\,\big(2\pi\sqrt{-1}\,\kappa\big)^{n(n-1)\,d^{(2)}_{\boldsymbol{\lambda}}/2}\,\kappa^{\lambda\{1\}\,d_{\boldsymbol{\lambda}}}\Big(e^{-\pi\sqrt{-1}\,(n-1)\,d^{(2)}_{\boldsymbol{\lambda}}}\prod_{i=1}^{N}p_i^{d^{(1)}_{\boldsymbol{\lambda},i}}\Big)^{\sum_{a=1}^{n}z_a/\kappa}\prod_{j=2}^{n-1}j^{(n-j)\,d^{(2)}_{\boldsymbol{\lambda}}}\,,$$

where $d_{\boldsymbol{\lambda}}$, $d^{(1)}_{\boldsymbol{\lambda},i}$, $d^{(2)}_{\boldsymbol{\lambda}}$ *are given by formulae* (4.27)*.*

Proof. The statement follows from Theorem 5.14 and formula (A.7). Alternatively, the statement follows from Theorem 4.17 and Lemma 5.16. □

Corollary 5.19. *The map* $\mu^{\mathcal{K}^{\circ}}_{\boldsymbol{\lambda}}:\mathcal{K}^{\circ}_{\boldsymbol{\lambda}}\to\mathscr{S}^{\circ}_{\boldsymbol{\lambda}}$ *is an isomorphism of* $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$*-modules.*

Remark 5.20. The algebra $\mathcal{K}^{\circ}_{\boldsymbol{\lambda}}$ is the equivariant K-theory algebra $K_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ of the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$, see Section 6.5.

5.5. **Levelt fundamental solution.** Recall Definition 5.12 of a function $f(\boldsymbol{p})$ entire in $\boldsymbol{p}_{\div}$. The dynamical Hamiltonians $X^\circ_1(\boldsymbol{z};\boldsymbol{p}),\dots,X^\circ_n(\boldsymbol{z};\boldsymbol{p})$ given by (3.8) are entire in $\boldsymbol{p}_{\div}$ and

$$X^\circ_i(\boldsymbol{z};\mathbb{0}) \;=\; \sum_{a=1}^n z_a\, e^{(a)}_{i,i} - \sum_{1\leqslant a<b\leqslant n}\biggl(\,\sum_{j=1}^{i-1} Q^{a,b}_{i,j} - \sum_{j=i+1}^{N} Q^{b,a}_{i,j}\biggr). \tag{5.32}$$

Notice that for $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$X^\circ_i(\boldsymbol{z};\mathbb{0})\,v_I \;=\; \sum_{a\in I_i} z_a v_I + \sum_{\substack{J\in\mathcal{I}_{\boldsymbol{\lambda}}\\ |\sigma_J|<|\sigma_I|}} \xi_{i,I,J}\,v_J\,,$$

where the coefficients $\xi_{i,I,J}$ take values $0,\pm1$. Therefore, the eigenvalues of the restriction of the operator $X^\circ_i(\boldsymbol{z};\mathbb{0})$ on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ are $\sum_{a\in I_i} z_a$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$. A more detailed statement is given by Proposition 5.21 below.

Recall the function $\Psi^{\vartriangle}_{I,0}(\boldsymbol{z})$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, given by (5.22).

Proposition 5.21. *Given $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, we have $X^\circ_i(\boldsymbol{z};\mathbb{0})\,\Psi^{\vartriangle}_{I,0}(\boldsymbol{z}) = \sum_{a\in I_i} z_a\,\Psi^{\vartriangle}_{I,0}(\boldsymbol{z})$, and $\Psi^{\vartriangle}_{I,0}(\boldsymbol{z})\neq 0$ provided $z_a\neq z_b$ for all pairs a,b such that $a<b$ and $\sigma_I^{-1}(a)>\sigma_I^{-1}(b)$.*

Proof. The first part of the statement follows from Theorem 5.13. The nonvanishing of $\Psi^{\vartriangle}_{I,0}(\boldsymbol{z})$ is implied by formula (5.22) and Lemma 5.6. □

For $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, set $\boldsymbol{E}_I(\boldsymbol{z}) = \bigl(E^{(1)}_I(\boldsymbol{z}),\dots,E^{(N-1)}_I(\boldsymbol{z})\bigr)$, where $E^{(i)}_I(\boldsymbol{z}) = \sum_{j=1}^i\sum_{a\in I_j} z_a$ is the eigenvalue of $X^\circ_1(\boldsymbol{z};\mathbb{0})+\ldots+X^\circ_i(\boldsymbol{z};\mathbb{0})$ on $\Psi^{\vartriangle}_{I,0}(\boldsymbol{z})$. For $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$, denote by $D_{I,J}$ the set of points $\boldsymbol{z}$ such that $\boldsymbol{E}_I(\boldsymbol{z})-\boldsymbol{E}_J(\boldsymbol{z})\in\mathbb{Z}^{N-1}_{\geqslant0}$ and $\boldsymbol{E}_I(\boldsymbol{z})\neq\boldsymbol{E}_J(\boldsymbol{z})$. Set $D_{\boldsymbol{\lambda}} = \bigcup_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} D_{I,J}$.

Theorem 5.22. (i) *For any $\boldsymbol{z}$ such that $z_a-z_b\notin\kappa\mathbb{Z}_{\neq0}$ for all $1\leqslant a<b\leqslant n$, there exists an $\mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\bigr)$-valued function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$, entire in $\boldsymbol{p}_{\div}$ such that $\Psi^\bullet(\boldsymbol{z};\mathbb{0};\kappa)$ is the identity operator and the function*

$$\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa) \;=\; \Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)\,\prod_{i=1}^N p_i^{X^\circ_i(\boldsymbol{z};\mathbb{0})/\kappa}, \tag{5.33}$$

solves dynamical differential equations (3.13). For given $\boldsymbol{z}$, the function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ with the specified properties is unique if and only if $\boldsymbol{z}\notin\kappa D_{\boldsymbol{\lambda}}$. Furthermore, $\det\Psi^\#(\boldsymbol{z};\boldsymbol{p};\kappa)=1$ and

$$\det\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa) \;=\; \prod_{i=1}^N p_i^{d^{(1)}_{\boldsymbol{\lambda},i}\sum_{a=1}^n z_a/\kappa}, \tag{5.34}$$

where $d^{(1)}_{\boldsymbol{\lambda},1},\dots,d^{(1)}_{\boldsymbol{\lambda},N}$ are given by formula (4.27).

(ii) *Define the function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ for generic $\boldsymbol{z}$ as in item* (i)*. Then $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ is holomorphic in $\boldsymbol{z}$ if $z_a-z_b\notin\kappa\mathbb{Z}_{\neq0}$ for all $1\leqslant a<b\leqslant n$. The singularities of $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ at the hyperplanes $z_a-z_b\in\kappa\mathbb{Z}_{\neq0}$ are simple poles.*

Proof. The proof of the uniqueness statement is similar to that in Theorem 4.21. To prove the existence part, we give an explicit expression for the function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$, see formulae (5.37), (5.38).

Recall the function $A^\circ(\boldsymbol{t};\boldsymbol{z};\kappa)$ at $\kappa=1$, see (5.13),

$$A^\circ(\boldsymbol{t};\boldsymbol{z}) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Bigg(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\Gamma(1+t^{(i)}_b-t^{(i)}_a)\prod_{c=1}^{\lambda^{(i+1)}}\frac{1}{\Gamma(1+t^{(i+1)}_c-t^{(i)}_a)}\Bigg), \tag{5.35}$$

where $\lambda^{(N)}=n$ and $t^{(N)}_a=z_a$, $a=1,\dots,n$. Notice that

$$A^\circ(\Sigma_I;\boldsymbol{z}) \,=\, \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in I_i}\prod_{b\in I_j}\frac{1}{\Gamma(z_b-z_a+1)}\,.$$

For $\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}$, set

$$\mathcal{J}^\circ_{I,J,\boldsymbol{l}}(\boldsymbol{z}) \,=\, \sum_{K\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{A^\circ(\Sigma_K-\boldsymbol{l};\boldsymbol{z})\,W^\circ_I(\Sigma_K-\boldsymbol{l};\boldsymbol{z})\,\check W^\circ_J(\Sigma_K;\boldsymbol{z})}{A^\circ(\Sigma_K;\boldsymbol{z})\,R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_K})}\,. \tag{5.36}$$

Recall the function $\Omega^\circ_{\boldsymbol{\lambda}}(\boldsymbol{p};\kappa)=e^{\sum^\circ_{i<j}p_j/(\kappa p_i)}$, where the sum is taken over all pairs $1\leqslant i<j\leqslant N$ such that $\lambda_i=1$ and $\lambda_s=0$ for all $s=i+1,\dots,j$, see (5.18). Set

$$\Psi^\bullet_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \Omega^\circ_{\boldsymbol{\lambda}}(\boldsymbol{p};\kappa)\sum_{\boldsymbol{l}\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}}\kappa^{|\sigma_I|-|\sigma_J|}\,\mathcal{J}^\circ_{I,J,\boldsymbol{l}}(\boldsymbol{z}/\kappa)\prod_{i=1}^{N-1}\big((-\kappa)^{-\lambda_i-\lambda_{i+1}}\,p_{i+1}/p_i\big)^{\sum_{a=1}^{\lambda^{(i)}}l^{(i)}_a}. \tag{5.37}$$

Let $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ be the linear operator with the entries $\Psi^\bullet_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa)$ in the standard basis $\{v_I\,,\,I\in\mathcal{I}_{\boldsymbol{\lambda}}\}$ of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$,

$$\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)\,:\,v_J\,\mapsto\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\Psi^\bullet_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa)\,v_I\,. \tag{5.38}$$

To verify that the function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ is as required in Theorem 5.22, recall the function $\Psi^\#(\boldsymbol{z};h;\boldsymbol{q})$, introduced in Theorem 4.21 and its entries $\Psi^\#_{I,J}(\boldsymbol{z};h;\boldsymbol{q})$. Let the functions $\widetilde\Psi^\#_{I,J}(\boldsymbol{z};h;\boldsymbol{p};\kappa)$ be obtained from $\Psi^\#_{I,J}(\boldsymbol{z}/\kappa;h/\kappa;\boldsymbol{q})$ by substitution (5.9). Formulae (4.59), (4.60), (5.36), (5.37), and Lemma D.2 imply that

$$\Psi^\bullet_{I,J}(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \lim_{h\to\infty}\widetilde\Psi^\#_{I,J}(\boldsymbol{z};h;\boldsymbol{p};\kappa) \tag{5.39}$$

locally uniformly in $\boldsymbol{z},\boldsymbol{p}$. Therefore, the properties of $\Psi^\#(\boldsymbol{z};h;\boldsymbol{q})$ established in Theorem 4.21 yield the properties of $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ required in Theorem 5.22. □

Following [AB, Chapter 2], we will call $\widehat\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ the *Levelt fundamental solution* of dynamical differential equations (3.13) on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$, see also [CV, Section 6.2].

For a solution $\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ of dynamical differential equations (3.13), define its *principal term*

$$\Psi^\varnothing(\boldsymbol{z};\kappa) \,=\, \widehat\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)^{-1}\,\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\,. \tag{5.40}$$

By Theorem 5.22, the principal term does not depend on $\boldsymbol{p}$.

Set

$$C^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)\,=\,\prod_{i=1}^N \kappa^{\left(\sum_{j=i+1}^N\lambda_j-\sum_{j=1}^{i-1}\lambda_j\right)\sum_{a=\lambda^{(i-1)}+1}^{\lambda^{(i)}} z_a/\kappa} \tag{5.41}$$

and

$$G^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)\,=\,\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}\Gamma\bigl((z_a-z_b)/\kappa\bigr)\,. \tag{5.42}$$

Proposition 5.23. *For a Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$, the principal term of the solution $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$, given by (5.26), equals*

$$\Psi^\varnothing_P(\boldsymbol z;\kappa)\,=\sum_{I,J\in\mathcal I_{\boldsymbol\lambda}}\acute P(\boldsymbol z_{\sigma_J};\boldsymbol z;\kappa)\,C^\circ_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};\kappa)\,G^\circ_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J};\kappa)\,W^\circ_I(\Sigma_J;\boldsymbol z)\,v_I\,. \tag{5.43}$$

Proof. Denote by $\bar\Psi^\circ_P(\boldsymbol z;\kappa)$ the right-hand side of formula (5.43). Then formula (5.26), Proposition 5.21, and Theorem 5.13 yield $\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\,\bar\Psi^\circ_P(\boldsymbol z;\kappa)\,=\,\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$. Hence by definition (5.40) of the principal term, $\Psi^\varnothing_P(\boldsymbol z;\kappa)=\bar\Psi^\circ_P(\boldsymbol z;\kappa)$. □

5.6. **The map $\mathrm{E}^\circ_{\boldsymbol\lambda}$.** In Section 5.4, we introduced the space $\mathscr S^\circ_{\boldsymbol\lambda}$ of $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$-valued solutions of the joint system of dynamical differential equations (3.13) and qKZ difference equations (3.12) spanned over $\mathbb C$ by the functions $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ labeled by Laurent polynomials in $\acute{\boldsymbol\Gamma},\acute{\boldsymbol z}$; we also defined a map

$$\mu^{\mathcal K^\circ}_{\boldsymbol\lambda}:\mathcal K^\circ_{\boldsymbol\lambda}\to\mathscr S^\circ_{\boldsymbol\lambda}\,,\qquad Y\mapsto\Psi^\circ_Y\,, \tag{5.44}$$

see (5.29). In Section 5.5 we introduced the Levelt fundamental solution $\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)$ of dynamical differential equations (3.13), see (5.33), (5.38). Denote by $\mathscr S^{\hat\circ}_{\boldsymbol\lambda}$ the space of $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$-valued solutions of dynamical differential equations (3.13) spanned over $\mathbb C$ by the functions $\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\,v$, $v\in(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$. Since $\det\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\neq 0$, see (5.34), there is an isomorphism

$$\mu^{\hat\circ}_{\boldsymbol\lambda}:(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\to\mathscr S^{\hat\circ}_{\boldsymbol\lambda},\qquad v\mapsto\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\,v\,. \tag{5.45}$$

Let $L^\circ\subset\mathbb C^n$ be the complement of the union of the hyperplanes

$$z_a-z_b\in\kappa\,\mathbb Z_{\neq0}\,,\qquad a,b=1,\dots,n\,,\quad a\neq b\,. \tag{5.46}$$

Denote by $\mathcal O$ the ring of functions of $\boldsymbol z$ holomorphic in L°. Let $\mathscr S^{\mathcal O}_{\boldsymbol\lambda}$ be the space of $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$-valued solutions of dynamical differential equations (3.13) holomorphic in $\boldsymbol z$ in L°. Both spaces $\mathscr S^\circ_{\boldsymbol\lambda}$ and $\mathscr S^{\hat\circ}_{\boldsymbol\lambda}$ are subspaces of $\mathscr S^{\mathcal O}_{\boldsymbol\lambda}$, see Proposition 5.17 and Theorem 5.22. Let

$$\mu^{\mathcal O}_{\boldsymbol\lambda}:(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb C}\mathcal O\,\to\,\mathscr S^{\mathcal O}_{\boldsymbol\lambda},\qquad v\mapsto\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\,v\,, \tag{5.47}$$

be the $\mathcal O$-linear extension of the map $\mu^{\hat\circ}_{\boldsymbol\lambda}$.

Define a map

$$\text{Б}^\circ_{\boldsymbol\lambda} : \mathcal{K}^\circ_{\boldsymbol\lambda} \to (\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda} \otimes_{\mathbb{C}} \mathcal{O}\,, \tag{5.48}$$

$$[P] \mapsto \sum_{I,J\in\mathcal{I}_{\boldsymbol\lambda}} \acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};\kappa)\, C^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)\, G^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)\, W_I(\Sigma_J;\boldsymbol{z})\, v_I\,, \tag{5.49}$$

where $[P]\in\mathcal{K}^\circ_{\boldsymbol\lambda}$ stands for the class of the Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$ and the functions $C^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)$, $G^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)$ are given by (5.41), (5.42), respectively. By Proposition 5.23, the map $\text{Б}^\circ_{\boldsymbol\lambda}$ sends the class $Y\in\mathcal{K}^\circ_{\boldsymbol\lambda}$ to the principal term of the solution Ψ°_Y of the joint system of dynamical differential equations (3.13) and *qKZ* difference equations (3.12).

Proposition 5.24. *The map* $\text{Б}^\circ_{\boldsymbol\lambda} : \mathcal{K}^\circ_{\boldsymbol\lambda} \to (\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda} \otimes_{\mathbb{C}} \mathcal{O}$ *is well-defined and the following diagram is commutative,*

$$\begin{array}{ccc} \mathcal{K}^\circ_{\boldsymbol\lambda} & \xrightarrow{\ \text{Б}^\circ_{\boldsymbol\lambda}\ } & (\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb{C}}\mathcal{O} \\ & {\scriptstyle\mu^K_{H_{\boldsymbol\lambda}}}\searrow \quad \swarrow{\scriptstyle\mu^{\mathcal{O}}_{H_{\boldsymbol\lambda}}} & \\ & \mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol\lambda}} & \end{array} \tag{5.50}$$

Proof. Poles of the sum in the right-hand side of (5.49) are at most those of the function $G^\circ_{\boldsymbol\lambda}(\boldsymbol{z};h;\kappa)$ and, therefore, in addition to hyperplanes (5.46) can occur only at the hyperplanes $z_a=z_b$, $a\neq b$. However, the sums

$$\sum_{I,J\in\mathcal{I}_{\boldsymbol\lambda}} \acute{P}(\boldsymbol{z}_{\sigma_J};\boldsymbol{z};\kappa)\, C^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)\, G^\circ_{\boldsymbol\lambda}(\boldsymbol{z}_{\sigma_J};\kappa)\, W_I(\Sigma_J;\boldsymbol{z})$$

are regular at the hyperplanes $z_a=z_b$ for all $a\neq b$ by the standard reasoning. Hence, the map $\text{Б}^\circ_{\boldsymbol\lambda}$ is well-defined.

The commutativity of diagram (5.50) follows from Proposition 5.23. □

Remark 5.25. Since $\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ is holomorphic in $\boldsymbol z$ in L°, and $\bigl(\det\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\bigr)^{-1}$ is entire in $\boldsymbol z$, see (5.34), the inverse matrix $\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)^{-1}$ is holomorphic in $\boldsymbol z$ in L°. Therefore, for every $\Psi\in\mathscr{S}^{\mathcal{O}}_{\boldsymbol\lambda}$, its principal term $\Psi^\varnothing=\widehat{\Psi}^{\circ\,-1}\Psi$, defined by (5.40), belongs to $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb{C}}\mathcal{O}$, and we have an isomorphism $\mathscr{S}^{\mathcal{O}}_{\boldsymbol\lambda}\to(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}\otimes_{\mathbb{C}}\mathcal{O}$, $\Psi\mapsto\Psi^\varnothing$. The inverse map, equals $\mu^{\mathcal{O}}_{\boldsymbol\lambda}$, see (5.47), so that $\mathscr{S}^{\mathcal{O}}_{\boldsymbol\lambda}=\mathscr{S}^{\hat\circ}_{\boldsymbol\lambda}\otimes_{\mathbb{C}}\mathcal{O}$.

5.7. **Example $\boldsymbol\lambda=(1,n-1)$.** Throughout this section, let $N=2$, $n\geqslant 2$, $\boldsymbol\lambda=(1,n-1)$. Like in Section 4.10, denote by $[a]$ the element $\bigl(\{a\},\{1,\dots,a-1,a+1,\dots,n\}\bigr)\in\mathcal{I}_{\boldsymbol\lambda}$, The space $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol\lambda}$ has a basis $v_{[1]},\dots,v_{[n]}$, where $v_{[a]}=v_2^{\otimes(a-1)}\otimes v_1\otimes v_2^{\otimes(n-a)}$. Clearly $e^{(a)}_{1,1}v_{[b]}=\delta_{a,b}\,v_{[b]}$ and $e^{(a)}_{2,2}v_{[b]}=(1-\delta_{a,b})\,v_{[b]}$.

The *qKZ* operators $K^\circ_1,\dots,K^\circ_n$, see (3.4), are

$$\begin{aligned} K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, & \bigl(R^\circ(z_a-z_{a-1}+\kappa)\bigr)^{(a,a-1)}\dots\bigl(R^\circ(z_a-z_1+\kappa)\bigr)^{(a,1)}\times \\ & \times\, p_1^{e^{(a)}_{1,1}}p_2^{e^{(a)}_{2,2}}\bigl(R^\circ(z_i-z_n)\bigr)^{(a,n)}\dots\bigl(R^\circ(z_a-z_{a+1})\bigr)^{(a,a+1)}\,. \end{aligned} \tag{5.51}$$

The R-matrices in the right-hand side preserve the subspace $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol{\lambda}} \subset (\mathbb{C}^2)^{\otimes n}$, acting there as follows,

$$\bigl(R^\circ(z)\bigr)^{(a,b)} v_{[a]} \,=\, v_{[b]} - z\, v_{[a]}\,, \qquad \bigl(R^\circ(z)\bigr)^{(a,b)} v_{[b]} \,=\, v_{[a]}\,,$$

$$\bigl(R^\circ(z)\bigr)^{(a,b)} v_{[c]} \,=\, v_{[c]}\,, \qquad c \neq a, b\,.$$

The qKZ difference equations (3.12) are

$$f(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{p};\kappa) \,=\, K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)\, f(\boldsymbol{z};\boldsymbol{p};\kappa)\,, \qquad a=1,\dots,n. \tag{5.52}$$

The dynamical Hamiltonians X°_1, X°_2, see (3.8), act on $(\mathbb{C}^2)^{\otimes n}_{\boldsymbol{\lambda}}$ as follows

$$\begin{aligned} X^\circ_1(\boldsymbol{z};\boldsymbol{p})\, v_{[1]} \,&=\, z_1\, v_{[1]} + \frac{p_2}{p_1}\, v_{[n]}\,, \\ X^\circ_1(\boldsymbol{z};\boldsymbol{p})\, v_{[a]} \,&=\, z_a\, v_{[a]} + v_{[a-1]}\,, && a=2,\dots,n\,, \\ X^\circ_2(\boldsymbol{z};\boldsymbol{p})\, v_{[b]} \,&=\, \Bigl(-X^\circ_1(\boldsymbol{z};\boldsymbol{p}) + \sum_{c=1}^n z_c\Bigr)\, v_{[b]}\,, && b=1,\dots,n\,, \end{aligned} \tag{5.53}$$

and the dynamical differential equations (2.7) are

$$\begin{aligned} \kappa\, p_1 \frac{\partial}{\partial p_1}\, \Psi(\boldsymbol{z};\boldsymbol{p};\kappa) \,&=\, X^\circ_1(\boldsymbol{z};\boldsymbol{p})\, \Psi(\boldsymbol{z};\boldsymbol{p};\kappa)\,, \\ \kappa\, p_2 \frac{\partial}{\partial p_2}\, \Psi(\boldsymbol{z};\boldsymbol{p};\kappa) \,&=\, X^\circ_2(\boldsymbol{z};\boldsymbol{p})\, \Psi(\boldsymbol{z};\boldsymbol{p};\kappa)\,. \end{aligned} \tag{5.54}$$

In this section, we use the variable $t = t^{(1)}_1$. The substitution $\boldsymbol{t} = \Sigma_{[a]}$ reads as $t = z_a$. The weight functions are

$$W^\circ_{[a]}(t;\boldsymbol{z}) \,=\, \prod_{b=1}^{a-1}(t-z_b)\,, \qquad a=1,\dots,n\,. \tag{5.55}$$

The permutations $\sigma_{[1]},\dots,\sigma_{[n]}$ are

$$\sigma_{[a]}(1)=a\,, \qquad \sigma_{[a]}(b)=b-1\,, \quad b=1,\dots,a-1\,, \qquad \sigma_{[a]}(b)=b\,, \quad b=a+1,\dots,n\,,$$

and $|\sigma_{[a]}| = a-1$. We have

$$\begin{aligned} W^\circ_{[a]}(z_a;\boldsymbol{z}) \,&=\, \prod_{b=1}^{a-1}(z_a-z_b)\,, \\ W^\circ_{[a]}(z_b;\boldsymbol{z}) \,&=\, 0\,, \qquad b=1,\dots,a-1\,, \end{aligned} \tag{5.56}$$

cf. Lemma 5.6.

The functions $\check W^\circ_{[a]}(t;\boldsymbol{z})$, see (5.5), are

$$\check W^\circ_{[a]}(t;\boldsymbol{z}) \,=\, \prod_{c=a+1}^{n}(t-z_b)\,, \qquad a=1,\dots,n\,. \tag{5.57}$$

Set
$$\bar R_a(\boldsymbol z)\,=\,\prod_{\substack{b=1\\ b\neq a}}^{n}(z_a-z_b)\,,\qquad a=1,\dots,n\,.$$

Then $\bar R_a(\boldsymbol z)\,=\,R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_{[a]}})$,, where the function $R_{\boldsymbol\lambda}(\boldsymbol z)$ is given by (4.8). Biorthogonality relations (5.7) become
$$\sum_{c=1}^{n}\frac{W^\circ_{[a]}(z_c;\boldsymbol z)\,\check W^\circ_{[b]}(z_c;\boldsymbol z)}{\bar R_c(\boldsymbol z)}\,=\,\delta_{a,b}\,,\qquad \sum_{c=1}^{n}W^\circ_{[c]}(z_a;\boldsymbol z)\,\check W^\circ_{[c]}(z_b;\boldsymbol z)\,=\,\delta_{a,b}\,\bar R_a(\boldsymbol z)\,.$$

The master function, see (5.8), is

$$\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\,=\,(p_2/\kappa)^{\sum_{a=1}^n z_a/\kappa}\,(\kappa^n p_1/p_2)^{t/\kappa}\prod_{a=1}^{n}\Gamma\bigl((t-z_a)/\kappa\bigr)\,.\tag{5.58}$$

The hypergeometric solutions (4.20) of the joint system of differential equations (4.68) and difference equations (4.66) have the form
$$\Psi^\circ_{[a]}(\boldsymbol z;\boldsymbol p;\kappa)\,=\,\frac1\kappa\sum_{b=1}^{n}\sum_{l=0}^{\infty}\operatorname{Res}_{t=\,z_a-l\kappa}\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\,W^\circ_{[b]}(z_a-l\kappa)\,v_{[b]}\,,$$

where the residues of the master function are
$$\operatorname{Res}_{t=\,z_a-l\kappa}\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\,=$$
$$=\,\kappa\,(\kappa^{n-1}p_1)^{z_a/\kappa}(p_2/\kappa)^{\sum_{c=1,\,c\neq a}^n z_c/\kappa}\,\frac{(-\kappa^{-n}p_2/p_1)^l}{l!}\prod_{\substack{c=1\\ c\neq a}}^{n}\Gamma\bigl(-l+(z_a-z_c)/\kappa\bigr)\,.$$

Determinant formula for coordinates of the hypergeometric solutions, see (5.24), is

$$\det\Biggl(\sum_{l=0}^{\infty}\operatorname{Res}_{t=\,z_a-l\kappa}\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\,W^\circ_{[b]}(z_a-l\kappa)\Biggr)_{a,b=1}^{n}\,=\tag{5.59}$$
$$=\,\pi^{n(n-1)/2}\kappa^{n(n+1)/2}\bigl(p_1p_2^{n-1}\bigr)^{\sum_{a=1}^n z_a/\kappa}\prod_{a=1}^{n-1}\prod_{b=a+1}^{n}\frac{1}{\sin\bigl(\pi\,(z_a-z_b)/\kappa\bigr)}\,.$$

By formula (5.55) and the Vandermonde determinant formula, equality (4.72) transforms to

$$\det\Biggl(\sum_{l=0}^{\infty}\sum_{c=1}^{n}\operatorname{Res}_{t=\,z_c-l\kappa}\bigl(t^{a-1}e^{-2\pi\sqrt{-1}\,(b-1)\,t/\kappa}\,\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\bigr)\Biggr)_{a,b=1}^{n}\,=\tag{5.60}$$
$$=\,\bigl(2\pi\sqrt{-1}\,\bigr)^{n(n-1)/2}\kappa^{n(n+1)/2}\bigl(e^{-\pi\sqrt{-1}\,(n-1)}p_1p_2^{n-1}\bigr)^{\sum_{a=1}^n z_a/\kappa}.$$

Let $\acute\gamma=\acute\gamma_{1,1}$. For a Laurent polynomial $P(\acute\gamma;\acute{\boldsymbol z})$, we have
$$\acute P(\gamma;\boldsymbol z;\kappa)\,=\,P(e^{2\pi\sqrt{-1}\,\gamma/\kappa};e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa})\,.$$

The solution Ψ°_P of differential equations (5.54) and difference equations (5.52) corresponding to $P(\acute{\gamma};\acute{\boldsymbol z})$ is

$$\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)\,=\,\sum_{a=1}^n \acute P(z_a;\boldsymbol z;\kappa)\,\Psi^\circ_{[a]}(\boldsymbol z;\boldsymbol p;\kappa)\,. \tag{5.61}$$

By Proposition 5.17, it is entire in $\boldsymbol z$ and is holomorphic in p_1,p_2 provided branches of $\log p_1$ and $\log p_2$ are fixed.

The solution $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ can be written as an integral over a suitable contour C encircling the poles of the product $\prod_{a=1}^n\Gamma\bigl((t-z_a)/\kappa\bigr)$ counterclockwise,

$$\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)\,=\,\frac{1}{2\pi\sqrt{-1}\,\kappa}\int\limits_C \acute P(t;\boldsymbol z;\kappa)\,\Phi^\circ_{\boldsymbol\lambda}(t;\boldsymbol z;\boldsymbol p;\kappa)\,\sum_{a=1}^n W^\circ_{[a]}(t;\boldsymbol z)\,v_{[a]}\,dt\,. \tag{5.62}$$

For instance, the integral can be taken over the parabola

$$C\,=\,\{\,\kappa\bigl(A-s^2+s\sqrt{-1}\,\bigr)\ \mid\ s\in\mathbb R\,\}\,,$$

where A is a sufficiently large positive real number. Formula (4.75) can be used to give an alternative proof of analytic properties of the function $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$.

Let $\mathscr S^\circ_{\boldsymbol\lambda}$ be the space of solutions of the joint system of dynamical differential equations (5.54) and qKZ difference equations (5.52) spanned over $\mathbb C$ by the functions $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ corresponding to Laurent polynomials $P(\acute\gamma;\acute{\boldsymbol z})$. The space $\mathscr S^\circ_{\boldsymbol\lambda}$ is a $\mathbb C[\acute{\boldsymbol z}^{\pm1}]$-module with $f(\acute{\boldsymbol z})$ acting as multiplication by $f(e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa})$.

For $\boldsymbol\lambda=(1,n-1)$, the algebra $\mathcal K^\circ_{\boldsymbol\lambda}$, see (5.28), can be presented as follows

$$\mathcal K^\circ_{\boldsymbol\lambda}\,=\,\mathbb C[\acute\gamma^{\pm1},\acute{\boldsymbol z}^{\pm1}]\Bigm/\Bigl\langle\,\prod_{a=1}^n(\acute\gamma-\acute z_a)\,=\,0\,\Bigr\rangle\,. \tag{5.63}$$

The function $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ depends only on the class of the Laurent polynomial P in $\mathcal K^\circ_{\boldsymbol\lambda}$. The assignment $P\mapsto\Psi^\circ_P$ defines the homomorphism

$$\mu^{\mathcal K^\circ}_{\boldsymbol\lambda}:\mathcal K^\circ_{\boldsymbol\lambda}\to\mathscr S^\circ_{\boldsymbol\lambda}\,,\qquad Y\mapsto\Psi^\circ_Y\,,$$

of $\mathbb C[\acute{\boldsymbol z}^{\pm1}]$-modules. Formula (5.60) implies that the homomorphism $\mu^{\mathcal K^\circ}_{\boldsymbol\lambda}$ is an isomorphism.

The algebra $\mathcal K^\circ_{\boldsymbol\lambda}$ is the equivariant K-theory algebra $K_T(\mathbb{CP}^{n-1};\mathbb C)$ of the projective space $\mathbb{CP}^{n-1}$, see the notation in Section 6.

Recall Definition 5.12 of a function $f(\boldsymbol p)$ entire in $\boldsymbol p_\div$. For example, the dynamical Hamiltonians X°_1,X°_2, see (5.53), are entire in $\boldsymbol p_\div$ and $X^\circ_1(\boldsymbol z;\emptyset)$, $X^\circ_2(\boldsymbol z;\emptyset)$ act on $(\mathbb C^2)^{\otimes n}_{\boldsymbol\lambda}$ as follows

$$X^\circ_1(\boldsymbol z;\emptyset)\,v_{[a]}\,=\,z_a\,v_{[a]}+v_{[a-1]}\,,\qquad X^\circ_2(\boldsymbol z;\emptyset)\,v_{[a]}\,=\,\Bigl(-X^\circ_1(\boldsymbol z;\emptyset)+\sum_{b=1}^n z_b\Bigr)\,v_{[a]}\,,$$

where $v_{[0]}=0$.

The Levelt fundamental solution $\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$, see (5.33), of differential equations (5.54) is

$$\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)\, p_1^{X_1^\circ(\boldsymbol{z};\mathbb{0})/\kappa} p_2^{X_2^\circ(\boldsymbol{z};\mathbb{0})/\kappa},$$

where the $\mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big)$-valued function $\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)$ is as follows,

$$\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa) : v_{[b]} \,\mapsto\, \sum_{a=1}^n \Psi^\bullet_{a,b}(\boldsymbol{z};\boldsymbol{p};\kappa)\, v_{[a]}\,, \tag{5.64}$$

$$\Psi^\bullet_{a,b}(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \sum_{l=0}^\infty \kappa^{a-b}\mathcal{J}^\circ_{a,b,l}(\boldsymbol{z}/\kappa)\,\big((-\kappa)^{-n}p_2/p_1\big)^l,$$

$$\mathcal{J}^\circ_{a,b,l}(\boldsymbol{z}) \,=\, \sum_{c=1}^n W^\circ_{[a]}(z_c-l;\boldsymbol{z})\,\check{W}^\circ_{[b]}(z_c;\boldsymbol{z})\,\frac{(-1)^{n-1}}{l!}\prod_{\substack{d=1\\ d\neq c}}^n\prod_{m=0}^l \frac{1}{z_d-z_c+m}\,.$$

Furthermore, one has $\det\Psi^\bullet(\boldsymbol{z};\boldsymbol{p};\kappa)=1$ and $\det\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \big(p_1p_2^{n-1}\big)^{\sum_{a=1}^n z_a/\kappa}$.

For a solution $\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ of dynamical differential equations (5.54), its principal term is

$$\Psi^\varnothing(\boldsymbol{z};\kappa) \,=\, \widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)^{-1}\,\Psi^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\,,$$

see (5.40). The principal term of the solution $\Psi^\circ_P(\boldsymbol{z};\boldsymbol{p};\kappa)$, corresponding to a Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}})$, see (5.43), equals

$$\Psi^\varnothing_P(\boldsymbol{z};\kappa) \,=\, \sum_{a=1}^n v_{[a]}\sum_{b=1}^n W^\circ_{[a]}(z_b;\boldsymbol{z})\,\acute{P}(z_b;\boldsymbol{z};\kappa)\,\kappa^{\left(nz_b-\sum_{c=1}^n z_c\right)/\kappa}\prod_{\substack{c=1\\ c\neq b}}^n\Gamma\big((z_b-z_c)/\kappa\big)\,. \tag{5.65}$$

Let L° be the complement of the union of the hyperplanes $z_a-z_b\in\kappa\mathbb{Z}_{\neq 0}$, $a,b=1,\dots,n$, $a\neq b$, cf. (5.46). Denote by $\mathcal{O}$ the ring of functions of $\boldsymbol{z}$ holomorphic in L°. The map

$$\text{Б}^\circ_{\boldsymbol{\lambda}} : \mathcal{K}^\circ_{\boldsymbol{\lambda}} \to (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}\,, \tag{5.66}$$

$$[P] \,\mapsto\, \sum_{a=1}^n v_{[a]}\sum_{b=1}^n W^\circ_{[a]}(z_b;\boldsymbol{z})\,\acute{P}(z_b;\boldsymbol{z};\kappa)\,\kappa^{\left(nz_b-\sum_{c=1}^n z_c\right)/\kappa}\prod_{\substack{c=1\\ c\neq b}}^n\Gamma\big((z_b-z_c)/\kappa\big)$$

sends the class $[P]\in\mathcal{K}^\circ_{\boldsymbol{\lambda}}$ of the Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}})$ to the principal term of the solution Ψ°_P of the joint system of differential equations (5.54) and difference equations (5.52).

Let $\mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}}$ be the space of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of dynamical differential equations (5.54) holomorphic in $\boldsymbol{z}$ in L°. The space $\mathscr{S}^\circ_{\boldsymbol{\lambda}}$ is a subspace of $\mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}}$.

Since the matrix $\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ is holomorphic in $\boldsymbol{z}$ in L°, and $\big(\det\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\big)^{-1}$ is entire in $\boldsymbol{z}$, the inverse matrix $\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)^{-1}$ is also holomorphic in $\boldsymbol{z}$ in L°. Thus the map

$$\mu^{\mathcal{O}}_{\boldsymbol{\lambda}} : (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O} \to \mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}}, \qquad v\mapsto\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\,v\,,$$

gives an isomorphism of $\mathcal{O}$-modules. Furthermore, the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{K}^\circ_{\boldsymbol{\lambda}} & \xrightarrow{\;\text{Б}^\circ_{\boldsymbol{\lambda}}\;} & (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes_{\mathbb{C}} \mathcal{O} \\ & {\scriptstyle \mu^{\mathcal{K}^\circ}_{\boldsymbol{\lambda}}} \searrow \quad \swarrow {\scriptstyle \mu^{\mathcal{O}}_{\boldsymbol{\lambda}}} & \\ & \mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}} & \end{array}$$

see Proposition 5.24.

6. Equations for partial flag varieties

6.1. **Equivariant cohomology of partial flag varieties.** Consider the partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \ldots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1, \ldots, N$.

Given a basis of $\mathbb{C}^n$, the group $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$. Let $T \subset GL_n(\mathbb{C})$ be the torus of diagonal matrices. Let $z_1, \ldots, z_n$ the Chern roots corresponding to the factors of T, and for $i = 1, \ldots, N$, let $\gamma_{i,1}, \ldots, \gamma_{i,\lambda_i}$ be the Chern roots of the bundle over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber F_i/F_{i-1}. Denote $\boldsymbol{\Gamma} = (\gamma_{1,1}, \ldots, \gamma_{1,\lambda_1}, \ldots, \gamma_{N,1}, \ldots, \gamma_{N,\lambda_N})$ and $\boldsymbol{z} = (z_1, \ldots, z_n)$, cf. (4.40). Let $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$ be the space of polynomials in $\boldsymbol{\Gamma}$ symmetric in $\gamma_{i,1}, \ldots, \gamma_{i,\lambda_i}$ for each $i = 1, \ldots, N$.

Consider the equivariant cohomology algebra $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$. Then

$$H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C}) \,=\, \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}] \Big/ \Big\langle \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) \,=\, \prod_{a=1}^{n} (u - z_a) \Big\rangle, \tag{6.1}$$

where u is a formal variable. For a polynomial $f(\boldsymbol{\Gamma}; \boldsymbol{z}) \in \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}]$, denote by $[f]$ its class in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$. Notice that for each $i = 1, \ldots, N$,

$$c_1(E_i) \,=\, [\gamma_{i,1} + \ldots + \gamma_{i,\lambda_i}] \tag{6.2}$$

is the equivariant first Chern class of the vector bundle E_i over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fiber F_i/F_{i-1}.

Given a point $\boldsymbol{z}^0 \in \mathbb{C}^n$, consider the algebra

$$H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})_{\boldsymbol{z}^0} \,=\, \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \Big/ \Big\langle \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} (u - \gamma_{i,j}) \,=\, \prod_{a=1}^{n} (u - z^0_a) \Big\rangle. \tag{6.3}$$

The evaluation map $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\boldsymbol{z}] \to \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$, $f(\boldsymbol{\Gamma}; \boldsymbol{z}) \mapsto f(\boldsymbol{\Gamma}; \boldsymbol{z}^0)$, identifies $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})_{\boldsymbol{z}^0}$ with the quotient $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C}) / \langle \boldsymbol{z} = \boldsymbol{z}^0 \rangle$. Denote by $[f]_{\boldsymbol{z}^0}$ the class of $f(\boldsymbol{\Gamma}) \in \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$ in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})_{\boldsymbol{z}^0}$.

Recall the polynomials $V_I(\boldsymbol{\Gamma})$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, given by formula (A.6).

Lemma 6.1. *The algebra $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$ is a free module over $H^*_T(pt; \mathbb{C}) = \mathbb{C}[\boldsymbol{z}]$ generated by the classes $[V_I]$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$. For every $\boldsymbol{z}^0 \in \mathbb{C}^n$, the classes $[V_I]_{\boldsymbol{z}^0}$, $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, give a basis of $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})_{\boldsymbol{z}^0}$. In particular, $\dim H^*_T(\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})_{\boldsymbol{z}^0} = n!/(\lambda_1! \ldots \lambda_N!)$ for every $\boldsymbol{z}^0$.*

Proof. The statement follows from Propositions A.2 and A.3. □

Recall the polynomial $R_{\boldsymbol{\lambda}}(\boldsymbol{z})$, see (4.8). Let

$$V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \prod_{i=1}^{N-1}\prod_{j=1}^{\lambda_i}\prod_{a=\lambda^{(i)}+1}^{n}(\gamma_{i,j}-z_a)\,, \tag{6.4}$$

cf. (A.1). Then $V_{\boldsymbol{\lambda}}(\boldsymbol{z};\boldsymbol{z})=R_{\boldsymbol{\lambda}}(\boldsymbol{z})$ and $V_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_I};\boldsymbol{z})=0$ for $I\neq I^{\min}_{\boldsymbol{\lambda}}$.

For a polynomial $f(\boldsymbol{\Gamma};\boldsymbol{z})$, consider the restrictions $f(\boldsymbol{z}_{\sigma_I};\boldsymbol{z})$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$.

Lemma 6.2. *The map* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\to\bigoplus_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathbb{C}[z]$, $[f]\mapsto\big(f(\boldsymbol{z}_{\sigma_I};\boldsymbol{z}),\,I\in\mathcal{I}_{\boldsymbol{\lambda}}\big)$ *is well-defined and injective. A collection* $\big(F_I(\boldsymbol{z}),\,I\in\mathcal{I}_{\boldsymbol{\lambda}}\big)$ *belongs to the image of this map if and only if for any* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$ *and any transposition* $s_{a,b}$,

$$F_I(\boldsymbol{z})\big|_{z_a=z_b} \,=\, F_{s_{a,b}(I)}(\boldsymbol{z})\big|_{z_a=z_b}. \tag{6.5}$$

For a collection $\big(F_I(\boldsymbol{z}),\,I\in\mathcal{I}_{\boldsymbol{\lambda}}\big)$ *obeying* (6.5), *the function*

$$f(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{F_I(\boldsymbol{z})\,V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_I})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_I})}\,. \tag{6.6}$$

is a polynomial and $f(\boldsymbol{z}_{\sigma_I};\boldsymbol{z})=F_I(\boldsymbol{z})$ *for every* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$.

Proof. Straightforward. □

Lemma 6.3. *For any* $f(\boldsymbol{\Gamma};\boldsymbol{z})\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}]$, *the function*

$$\mathcal{E}\langle f\rangle(\boldsymbol{z}) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{f(\boldsymbol{z}_{\sigma_I},\boldsymbol{z})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_I})} \tag{6.7}$$

is a polynomial depending only on the class of f *in* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$.

Proof. Straightforward. □

The induced map

$$\mathcal{E}\,:\,H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\,\to\,\mathbb{C}[\boldsymbol{z}]\,,\qquad [f]\mapsto\mathcal{E}\langle f\rangle\,, \tag{6.8}$$

is the *equivariant integration map on* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$.

Identify $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$ with the subspace of $\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}]$ of polynomials not depending on $\boldsymbol{z}$. For $f(\boldsymbol{\Gamma})\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$, denote $\mathcal{E}_{\boldsymbol{z}^0}\langle f\rangle=\mathcal{E}\langle f\rangle(\boldsymbol{z}^0)$.

Lemma 6.4. *For any* $f(\boldsymbol{\Gamma};\boldsymbol{z})\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}]$ *and* $\boldsymbol{z}^0\in\mathbb{C}^n$, *we have*

$$\mathcal{E}\langle f(\boldsymbol{\Gamma};\boldsymbol{z})\rangle(\boldsymbol{z}^0) \,=\, \mathcal{E}_{\boldsymbol{z}^0}\langle f(\boldsymbol{\Gamma};\boldsymbol{z}^0)\rangle\,.$$

Proof. Straightforward. □

By Lemmas 6.3, 6.4, for every $\boldsymbol{z}^0 \in \mathbb{C}^n$, there is a well-defined map

$$\mathcal{E}_{\boldsymbol{z}^0} : H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0} \to \mathbb{C}\,, \qquad \mathcal{E}_{\boldsymbol{z}^0} : [f]_{\boldsymbol{z}^0} \mapsto \mathcal{E}_{\boldsymbol{z}^0}\langle f\rangle\,, \tag{6.9}$$

called the *integration map on* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$.

6.2. **Stable envelope map.** Recall the functions $W^\circ_I(\boldsymbol{t};\boldsymbol{z})$, $\check W^\circ_I(\boldsymbol{t};\boldsymbol{z})$, see (5.1), (5.5). Let $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ be the polynomials respectively obtained from $\check W^\circ_I(\boldsymbol{t};\boldsymbol{z})$, $W^\circ_I(\boldsymbol{t};\boldsymbol{z})$ by the substitution

$$(t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}}) \,=\, (\gamma_{1,1},\dots,\gamma_{1,\lambda_1},\dots,\gamma_{i,1},\dots,\gamma_{i,\lambda_i})\,, \qquad i=1,\dots,N-1\,. \tag{6.10}$$

Notice that

$$\mathrm{Stab}_I(\boldsymbol{z}_{\sigma_J};\boldsymbol{z}) \,=\, \check W^\circ_I(\Sigma_J;\boldsymbol{z})\,, \qquad \mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{z}_{\sigma_J};\boldsymbol{z}) \,=\, W^\circ_I(\Sigma_J;\boldsymbol{z})\,. \tag{6.11}$$

By Lemma 5.2 and formula (6.4), we have

$$\begin{aligned} &\mathrm{Stab}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z})\,, \qquad &&\mathrm{Stab}_{\sigma_0(I^{\min}_{\boldsymbol{\lambda}})}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, 1\,,\\ &\mathrm{Stab}^{\mathrm{op}}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, 1\,, &&\mathrm{Stab}^{\mathrm{op}}_{\sigma_0(I^{\min}_{\boldsymbol{\lambda}})}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0})\,, \end{aligned} \tag{6.12}$$

where σ_0 is the longest permutation, $\sigma_0(a) = n+1-a$, $a=1,\dots,n$. By Proposition B.5, the polynomials $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ coincide with the A-type double Schubert polynomials $\mathfrak{S}_\sigma(\boldsymbol{\Gamma};\boldsymbol{z})$,

$$\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0})\,, \qquad \mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \mathfrak{S}_{\sigma_I}(\boldsymbol{\Gamma};\boldsymbol{z})\,. \tag{6.13}$$

Lemma 6.5. *For any* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{\check W^\circ_I(\Sigma_J;\boldsymbol{z})\,V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_J})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})}\,,$$

and

$$\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{W^\circ_I(\Sigma_J;\boldsymbol{z})\,V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_J})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})}\,. \tag{6.14}$$

Proof. Fix a generic $\boldsymbol{z}$ and consider both sides of each formula as functions of $\boldsymbol{\Gamma}$. Formulae (6.12) and Lemma 5.4 imply that $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ are in the span of the polynomials $V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_J})$, $J\in\mathcal{I}_{\boldsymbol{\lambda}}$. Then the statement follows from formulae (6.6), (6.13). □

Lemma 6.6. *The polynomials* $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ *and* $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ *are biorthogonal,*

$$\mathcal{E}\langle \mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})\,\mathrm{Stab}^{\mathrm{op}}_J(\boldsymbol{\Gamma};\boldsymbol{z})\rangle \,=\, \delta_{I,J}\,. \tag{6.15}$$

Proof. The statement follows from formulae (6.7), (6.13), and Lemma 5.7. □

Lemma 6.15 is equivalent to the orthogonality relation for the Schubert polynomials.

Let Stab_I, $\mathrm{Stab}^{\mathrm{op}}_I$ be the classes of $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$ in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. Define the *stable envelope map* by the rule

$$\mathrm{Stab}_{\boldsymbol{\lambda}} : (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes\mathbb{C}[\boldsymbol{z}] \,\to\, H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\,, \qquad v_I \mapsto \mathrm{Stab}_I\,, \quad I\in\mathcal{I}_{\boldsymbol{\lambda}}\,. \tag{6.16}$$

Lemma 6.7. *The map* $\mathrm{Stab}_{\boldsymbol{\lambda}}$ *is an isomorphism of free* $\mathbb{C}[\boldsymbol{z}]$*-modules.*

Proof. The statement follows from Lemma 6.6. For any $f\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{z}]$, one has

$$\bigl(\mathrm{Stab}_{\boldsymbol{\lambda}}\bigr)^{-1}:\,[f]\,\mapsto\,\sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathcal{E}\langle f(\boldsymbol{\Gamma};\boldsymbol{z})\,\mathrm{Stab}^{\mathrm{op}}_J(\boldsymbol{\Gamma};\boldsymbol{z})\rangle\,v_J\,.$$

□

Let $\mathrm{Stab}_{I,\boldsymbol{z}^0}$, $\mathrm{Stab}^{\mathrm{op}}_{I,\boldsymbol{z}^0}$ be the classes of $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z}^0)$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}^0)$ in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$. Consider the map

$$\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0}:(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\to H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}\,,\qquad v_I\mapsto\mathrm{Stab}_{I,\boldsymbol{z}^0}\,,\quad I\in\mathcal{I}_{\boldsymbol{\lambda}}\,.\tag{6.17}$$

Lemma 6.8. *For every* $\boldsymbol{z}^0\in\mathbb{C}^n$*, the classes* $\mathrm{Stab}_{I,\boldsymbol{z}^0}$*,* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$*, give a basis of* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$*. That is, the map* $\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0}$ *is an isomorphism.*

Proof. The statement follows from Lemmas 6.6, 6.4. For any $f\in\mathbb{C}[\boldsymbol{\Gamma}]^{S_{\boldsymbol{\lambda}}}$, one has

$$\bigl(\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0}\bigr)^{-1}:\,[f]_{\boldsymbol{z}^0}\,\mapsto\,\sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}}\mathcal{E}_{\boldsymbol{z}^0}\langle f(\boldsymbol{\Gamma})\,\mathrm{Stab}^{\mathrm{op}}_J(\boldsymbol{\Gamma};\boldsymbol{z}^0)\rangle\,v_J\,.$$

□

Consider a polynomial

$$\widehat{W}^{\circ}_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{\Gamma})\,=\,\prod_{i=1}^{N-1}\prod_{j=1}^{\lambda^{(i)}}\prod_{k=1}^{\lambda_{i+1}}(t^{(i)}_j-\gamma_{i+1,k})\,.\tag{6.18}$$

and its image $\bigl[\widehat{W}^{\circ}_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{\Gamma})\bigr]$ in $\mathbb{C}[\boldsymbol{t}]\otimes H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. Clearly, for any $I,J\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$\widehat{W}^{\circ}_{\boldsymbol{\lambda}}(\Sigma_I;\boldsymbol{z}_J)\,=\,\delta_{I,J}\,R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_I})\,.\tag{6.19}$$

Theorem 6.9. *We have*

$$\bigl[\widehat{W}^{\circ}_{\boldsymbol{\lambda}}(\boldsymbol{t};\boldsymbol{\Gamma})\bigr]\,=\,\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}W^{\circ}_I(\boldsymbol{t};\boldsymbol{z})\,\mathrm{Stab}_I\,.$$

Proof. The statement follows from [TV6, Proposition 9.2] and Lemmas 5.1, 5.2. □

Remark 6.10. According to formulae (6.13), the classes Stab_I, $\mathrm{Stab}^{\mathrm{op}}_I$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, are the equivariant fundamental classes of the Schubert subvarieties in $\mathcal{F}_{\boldsymbol{\lambda}}$ defined for the opposite orderings of the chosen basis of $\mathbb{C}^n$ in the standard way, see [O, Section 4.1], cf. [RV, Section 6]. On the equivariant fundamental classes of Schubert varieties see, in particular, [R, FRW].

Remark 6.11. Stable envelope maps for Nakajima quiver varieties were introduced in [MO]. They were defined in [MO] geometrically in terms of the associated torus action. The map $\mathrm{Stab}_{\boldsymbol{\lambda}}$ given by formula (6.16) is the limit as $h\to\infty$ of the stable envelope map of [MO] associated with the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ of the partial flag variety. In [RTV1], the stable envelope map for $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is described in terms of the Chern roots of the bundles $\boldsymbol{F}_1,\dots,$ $\boldsymbol{F}_{N-1}$ over $\mathcal{F}_{\boldsymbol{\lambda}}$ with fibers $F_1,\dots,F_{N-1}$, respectively.

Remark 6.12. Unlike the stable envelope map $\mathrm{Stab}_{\boldsymbol{\lambda}}$ defined by (6.16), the stable envelope map for the cotangent bundle $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ is not an isomorphism of free $\mathbb{C}[\boldsymbol{z}]$-modules $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes$ $\mathbb{C}[\boldsymbol{z}]$ and $H^*_{T\times\mathbb{C}^\times}(T^*\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, but only an embedding.

6.3. **Quantum multiplication.** The quantum multiplication in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, see for example [M, BM], is a deformation of the multiplication in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ depending on quantum parameters. In the notation of [M], the quantum parameters are q^{α_i}, $i =, \dots, N-1$. In the notation of this paper, we have $q^{\alpha_i} = p_{i+1}/p_i$.

For $Y \in H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, denote by $Y *_{\boldsymbol{p}}$ the operator of quantum multiplication by Y. Recall that the operators $Y *_{\boldsymbol{p}}$ are $\mathbb{C}[\boldsymbol{z}]$-module endomorphisms of $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$.

The dynamical operators $X^\circ_1(\boldsymbol{z};\boldsymbol{p}), \dots, X^\circ_N(\boldsymbol{z};\boldsymbol{p})$, see (3.8), are linear functions of $\boldsymbol{z}$. Thus their action on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ extends to the $\mathbb{C}[\boldsymbol{z}]$-linear action on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes \mathbb{C}[\boldsymbol{z}]$.

For $i = 1, \dots, N$, denote $D_i = \gamma_{i,1} + \dots + \gamma_{i,\lambda_i}$, cf. (6.2).

Theorem 6.13. *The isomorphism* $\mathrm{Stab}_{\boldsymbol{\lambda}}$ *intertwines the dynamical operators* $X^\circ_1(\boldsymbol{z};\boldsymbol{p})$, $\dots, X^\circ_N(\boldsymbol{z};\boldsymbol{p})$ *acting on* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}} \otimes \mathbb{C}[\boldsymbol{z}]$ *and the operators of quantum multiplication* $[D_1] *_{\boldsymbol{p}}$, $\dots, [D_N] *_{\boldsymbol{p}}$ *acting on* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$,

$$\mathrm{Stab}_{\boldsymbol{\lambda}} \circ X^\circ_i(\boldsymbol{z};\boldsymbol{p}) \,=\, [D_i] *_{\boldsymbol{p}} \circ \mathrm{Stab}_{\boldsymbol{\lambda}}\,. \tag{6.20}$$

Proof. Theorem 6.4 in [M] describes the quantum multiplication in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ by the equivariant divisor classes. In the notation of this paper those classes equal $\mathrm{Stab}_{\sigma_0 s_{\lambda^{(i)},\lambda^{(i)}+1}(I^{\min}_{\boldsymbol{\lambda}})}$, $i = 1, \dots, N-1$, where σ_0 is the longest permutation. By formulae (6.16), (5.5), (5.3),

$$\mathrm{Stab}_{\sigma_0 s_{\lambda^{(i)},\lambda^{(i)}+1}(I^{\min}_{\boldsymbol{\lambda}})}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, D_1 + \dots + D_i - z_{n-\lambda^{(i)}+1} - \dots - z_n\,. \tag{6.21}$$

Comparing formula (3.8) term by term with formula (6.1) in [M] yields formula (6.20) for $i = 1, \dots, N-1$. Formula (6.20) for $i = N$ holds since $[D_1 + \dots + D_N] = z_1 + \dots + z_n$ in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ and $X_1(\boldsymbol{z};\boldsymbol{p}) + \dots + X_N(\boldsymbol{z};\boldsymbol{p}) = z_1 + \dots + z_n$. Recall that for each $i = 1, \dots,$ N, the operator $z_i *_{\boldsymbol{p}}$ is the ordinary multiplication by z_i, □

Since the operators of quantum multiplication act $\mathbb{C}[\boldsymbol{z}]$-linearly on $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, the quantum multiplication in $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ can be restricted to $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$. Denote by $Y *_{\boldsymbol{p},\boldsymbol{z}^0}$ the operator of quantum multiplication by $Y \in H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$.

Corollary 6.14. *For every* $\boldsymbol{z}^0 \in \mathbb{C}^n$, *the isomorphism* $\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0}$ *intertwines the dynamical operators* $X^\circ_1(\boldsymbol{z}^0;\boldsymbol{p}), \dots, X^\circ_N(\boldsymbol{z}^0;\boldsymbol{p})$ *acting on* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ *and the operators of quantum multiplication* $[D_1]_{\boldsymbol{z}^0} *_{\boldsymbol{p},\boldsymbol{z}^0}, \dots, [D_N]_{\boldsymbol{z}^0} *_{\boldsymbol{p},\boldsymbol{z}^0}$ *acting on* $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$,

$$\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0} \circ X^\circ_i(\boldsymbol{z}^0;\boldsymbol{p}) \,=\, [D_i]_{\boldsymbol{z}^0} *_{\boldsymbol{p},\boldsymbol{z}^0} \circ \mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0}\,. \tag{6.22}$$

6.4. **Differential and difference equations.** Consider the space $\mathbb{C}^n \times \mathbb{C}^N$ with coordinates $\boldsymbol{z}, \boldsymbol{p}$. By Lemma 6.1, we have a trivial vector bundle $H_{\boldsymbol{\lambda}} \to \mathbb{C}^n \times \mathbb{C}^N$ with fiber over a point $(\boldsymbol{z}^0, \boldsymbol{p}^0)$ given by $H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}^0}$. Let $U_{\boldsymbol{\lambda}} \to \mathbb{C}^n \times \mathbb{C}^N$ be the trivial vector bundle with fiber $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$.

Lemma 6.15. *The map* $\mathrm{Stab}^\diamond_{\boldsymbol{\lambda}} : U_{\boldsymbol{\lambda}} \to H_{\boldsymbol{\lambda}}$, $(\boldsymbol{z}^0, \boldsymbol{p}^0, v) \mapsto (\boldsymbol{z}^0, \boldsymbol{p}^0, \mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}^0} v)$, *is an isomorphism of vector bundles.*

Proof. The statement follows from Lemma 6.8. □

The *equivariant quantum differential equations* for sections of $H_{\boldsymbol\lambda}$ is a system of differential equations

$$\kappa p_i \frac{\partial f}{\partial p_i} = [D_i]_{\boldsymbol z} *_{\boldsymbol p,\boldsymbol z} f\,, \qquad i=1,\dots,N\,, \tag{6.23}$$

where κ is the parameter of the equations. By Corollary 6.14, the isomorphism $\mathrm{Stab}^{\diamond}_{\boldsymbol\lambda}$ identifies equations (6.23) with the limiting dynamical differential equation (3.13) for sections of $U_{\boldsymbol\lambda}$,

$$\kappa p_i \frac{\partial f}{\partial p_i} = X^{\circ}_i(\boldsymbol z;\boldsymbol p)\, f\,, \qquad i=1,\dots,N\,. \tag{6.24}$$

Furthermore, the isomorphism $\mathrm{Stab}^{\diamond}_{\boldsymbol\lambda}$ and the limiting *qKZ* equations (3.12) for sections of $U_{\boldsymbol\lambda}$ define the *qKZ difference equations in cohomology*

$$f(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol p;\kappa) = K^H_a(\boldsymbol z;\boldsymbol p;\kappa)\, f(\boldsymbol z;\boldsymbol p;\kappa)\,, \qquad a=1,\dots,n\,, \tag{6.25}$$

where for each $a=1,\dots,n$, and fixed $\boldsymbol z,\boldsymbol p$, the operator $K^H_a(\boldsymbol z;\boldsymbol p;\kappa)$ is a map of fibers,

$$K^H_a(\boldsymbol z;\boldsymbol p;\kappa)\,:\, H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z} \to H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{(z_1,\dots,z_a+\kappa,\dots,z_n)}\,,$$

$$K^H_a(\boldsymbol z;\boldsymbol p;\kappa) = \mathrm{Stab}_{\boldsymbol\lambda,(z_1,\dots,z_a+\kappa,\dots,z_n)} \circ K^{\circ}_a(\boldsymbol z;\boldsymbol p;\kappa)\circ(\mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z})^{-1},$$

and the operator $K^{\circ}_a(\boldsymbol z;\boldsymbol p;\kappa)$ is given by (3.4).

Theorem 6.16. *Quantum differential equations* (6.23) *and qKZ difference equations* (6.25) *define a compatible system of differential and difference equations for sections of* $H_{\boldsymbol\lambda}$.

Proof. The statement follows from Theorem 3.12. □

6.5. **Solutions with values in cohomology.** Recall the isomorphism $\mathrm{Stab}^{\diamond}_{\boldsymbol\lambda} : U_{\boldsymbol\lambda} \to H_{\boldsymbol\lambda}$ of vector bundles, see Lemma 6.15. For any function $f(\boldsymbol z;\boldsymbol p;\kappa)$ with values in $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$, that is, a section of $U_{\boldsymbol\lambda}$, denote by $\mathrm{Stab}_{\boldsymbol\lambda} f$ the corresponding section of $H_{\boldsymbol\lambda}$ with values

$$\mathrm{Stab}_{\boldsymbol\lambda} f(\boldsymbol z;\boldsymbol p;\kappa) = \mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z}\big(f(\boldsymbol z;\boldsymbol p;\kappa)\big)\,.$$

Recall the solutions $\Psi^{\circ}_P(\boldsymbol z;\boldsymbol p;\kappa)$ of the joint system of limiting dynamical differential equations (3.13) and *qKZ* difference equations (3.12) labeled by Laurent polynomials $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$, see formula (5.26) and Proposition 5.17.

Theorem 6.17. *The isomorphism* $\mathrm{Stab}^{\diamond}_{\boldsymbol\lambda}$ *transforms the solutions* $\Psi^{\circ}_P(\boldsymbol z;\boldsymbol p;\kappa)$ *of equations* (3.13), (3.12) *to solutions of the joint system of quantum differential equations and qKZ difference equations* (6.23), (6.25). *Namely, for each* $P\in\mathbb C[\acute{\boldsymbol\Gamma}^{\pm1}]^{S_{\boldsymbol\lambda}}\otimes\mathbb C[\acute{\boldsymbol z}^{\pm1}]$, *the function* $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^{\circ}_P(\boldsymbol z;\boldsymbol p;\kappa)$ *is a solution of equations* (3.13), (3.12).

Proof. The theorem follows from Theorem 6.13 and the definition of the *qKZ* difference equations in (6.25). □

In this section we work out the geometric interpretation of the obtained solutions. Notice that it would be interesting to relate our solutions with constructions in [C], where Mellin-Barnes integral representations of solutions of the quantum differential equations were constructed for a class of smooth projective varieties.

For $i = 1, \dots, N$, let $\acute\gamma_{1,1}, \dots, \acute\gamma_{1,\lambda_i}$ be virtual line bundles such that $\bigoplus_{j=1}^{\lambda_i} \acute\gamma_{i,j} = F_i/F_{i-1}$, and $\acute z_1, \dots, \acute z_n$ correspond to the factors of the torus T. Denote $\acute{\boldsymbol\Gamma}^{\pm 1} = (\acute\gamma_{1,1}^{\pm 1}, \dots, \acute\gamma_{N,\lambda_N}^{\pm 1})$ and $\acute{\boldsymbol z}^{\pm 1} = (\acute z_1^{\pm 1}, \dots, \acute z_n^{\pm 1})$, cf. (4.40). Let $\mathbb C[\acute{\boldsymbol\Gamma}^{\pm 1}]^{S_{\boldsymbol\lambda}}$ be the space of Laurent polynomials in $\acute{\boldsymbol\Gamma}$ symmetric in $\acute\gamma_{i,1}, \dots, \acute\gamma_{i,\lambda_i}$ for each $i = 1, \dots, N$.

Consider the equivariant K-theory algebra $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$. Then

$$K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C) \,=\, \mathbb C[\acute{\boldsymbol\Gamma}^{\pm 1}]^{S_{\boldsymbol\lambda}} \otimes \mathbb C[\acute{\boldsymbol z}^{\pm 1}] \Big/ \Big\langle \prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u - \acute\gamma_{i,j}) \,=\, \prod_{a=1}^{n}(u - \acute z_a)\Big\rangle , \tag{6.26}$$

where u is a formal variable. That is, $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ coincides with the algebra $\mathcal K^\circ_{\boldsymbol\lambda}$, see (5.28). For a Laurent polynomial $P(\acute{\boldsymbol\Gamma}^{\pm 1};\acute{\boldsymbol z}^{\pm 1}) \in \mathbb C[\acute{\boldsymbol\Gamma}^{\pm 1}]^{S_{\boldsymbol\lambda}} \otimes \mathbb C[\boldsymbol z^{\pm 1}]$, denote by $[P]$ its class in $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$.

Lemma 6.18. *The algebra $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ is a free module over $\mathbb C[\acute{\boldsymbol z}^{\pm 1}]$.*

Proof. The statement follows from Propositions A.2 and A.3. □

Denote by $\mathscr S^K_{H_{\boldsymbol\lambda}}$ the space spanned over $\mathbb C$ by the solutions $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ of equations (6.23), (6.25). By Proposition 5.17, each element of $\mathscr S^K_{H_{\boldsymbol\lambda}}$ is a section of $H_{\boldsymbol\lambda}$ holomorphic in $\boldsymbol p$ provided a branch of $\log p_i$ is fixed for each $i = 1, \dots, N$, and entire in $\boldsymbol z$.

Since the function $\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ depends only on the class of P in $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$, then so does the section $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ of $H_{\boldsymbol\lambda}$. Thus the map

$$\mu^K_{H_{\boldsymbol\lambda}} : K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C) \to \mathscr S^K_{H_{\boldsymbol\lambda}}, \qquad [P] \mapsto \mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P , \tag{6.27}$$

is well-defined, cf. (5.29). By Corollary 5.19, the map $\mu^K_{H_{\boldsymbol\lambda}}$ is an isomorphism of $\mathbb C[\acute{\boldsymbol z}^{\pm 1}]$-modules.

In Section 5.5 we introduced the Levelt fundamental solution $\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)$ of dynamical differential equations (3.13). Consider the vector bundle $E H_{\boldsymbol\lambda} \to \mathbb C^n \times \mathbb C^N$ with fiber over a point $(\boldsymbol z^0, \boldsymbol p^0)$ given by $\mathrm{End}\big(H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z^0}\big)$, and its section $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ$ with values

$$\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa) \,=\, \mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z} \circ \widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa) \circ \big(\mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z}\big)^{-1}. \tag{6.28}$$

By Theorem 4.21 and Corollary 6.14, for any section f of $H_{\boldsymbol\lambda}$ not depending on $\boldsymbol p$, the section $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ f$ of $H_{\boldsymbol\lambda}$ with values $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa)\, f(\boldsymbol z)$ is a solution of quantum differential equations (6.23). We will call the section $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ$ of $E H_{\boldsymbol\lambda}$ the *Levelt fundamental solution* of quantum differential equations (6.23).

Recall the function $\Psi^\bullet(\boldsymbol z;\boldsymbol p;\kappa)$ with values in $\mathrm{End}\big((\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\big)$, see (5.33). Consider the section and $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\bullet$ of $E H_{\boldsymbol\lambda}$ with values

$$\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\bullet(\boldsymbol z;\boldsymbol p;\kappa) \,=\, \mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z} \circ \Psi^\bullet(\boldsymbol z;\boldsymbol p;\kappa) \circ \big(\mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z}\big)^{-1}.$$

Then by formulae (5.33) and (6.22),

$$\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(\boldsymbol z;\boldsymbol p;\kappa) \,=\, \big(\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\bullet(\boldsymbol z;\boldsymbol p;\kappa)\big) \prod_{i=1}^{N} p_i^{[D_i]_{\boldsymbol z}/\kappa}, \tag{6.29}$$

where $\prod_{i=1}^{N} p_i^{[D_i]_{\boldsymbol z}/\kappa}$ acts on the fiber $H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z}$ as multiplication by itself. The expression for $\operatorname{Stab}_{\boldsymbol\lambda}\Psi^{\bullet}(\boldsymbol z;\boldsymbol p;\kappa)$ is given by formula (6.33) below.

Recall the function $A^{\circ}(\boldsymbol t;\boldsymbol z;\kappa)$, see (5.13), (5.35),

$$A^{\circ}(\boldsymbol t;\boldsymbol z;\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\biggl(\,\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\Gamma\bigl(1+(t^{(i)}_b-t^{(i)}_a)/\kappa\bigr)\prod_{c=1}^{\lambda^{(i+1)}}\frac{1}{\Gamma\bigl(1+(t^{(i+1)}_c-t^{(i)}_a)/\kappa\bigr)}\biggr),$$

where $\lambda^{(N)}=n$ and $t^{(N)}_a=z_a$, $a=1,\dots,n$. For $I\in\mathcal I_{\boldsymbol\lambda}$ and $\boldsymbol m=(m_1,\dots,m_{N-1})\in\mathbb Z^{N-1}_{\geqslant 0}$, set

$$\hat{\mathcal J}^{H}_{I,\boldsymbol m}(\boldsymbol\Gamma;\boldsymbol z;\kappa) \,=\, \sum_{i=1}^{N-1}\sum_{\substack{\boldsymbol l\in\mathbb Z^{\lambda\{1\}}\\ |\boldsymbol l^{(i)}|=m_i}}\sum_{J\in\mathcal I_{\boldsymbol\lambda}}\frac{A^{\circ}(\Sigma_J-\boldsymbol l\kappa;\boldsymbol z;\kappa)\,W^{\circ}_I(\Sigma_J-\boldsymbol l\kappa;\boldsymbol z)\,V_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z_{\sigma_J})}{A^{\circ}(\Sigma_J;\boldsymbol z)\,R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J})}\,, \tag{6.30}$$

where $|\boldsymbol l^{(i)}|=l^{(i)}_j+\ldots+l^{(i)}_{\lambda^{(i)}}$, $i=1,\dots,N-1$. cf. (5.36), (6.6). The functions $\mathcal J^{H}_{I,\boldsymbol l}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$ are polynomials in $\boldsymbol\Gamma$ and rational functions in $\boldsymbol z,\kappa$.

Proposition 6.19. *The functions $\hat{\mathcal J}^{H}_{I,\boldsymbol m}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$ are regular if $z_a-z_b\notin\kappa\mathbb Z_{\neq 0}$ for all $a\neq b$, and have at most simple poles at the hyperplanes $z_a-z_b\in\kappa\mathbb Z_{\neq 0}$.*

Proof. Recall the functions $\Psi^{\bullet}_{I,J}(\boldsymbol z;\boldsymbol p;\kappa)$ and $\Omega^{\circ}_{\boldsymbol\lambda}(\boldsymbol p;\kappa)$, see (5.37), (5.18), respectively. By formulae (5.36), (5.37), (6.14), (5.7), (6.30), we have

$$\begin{aligned}\sum_{J\in\mathcal I_{\boldsymbol\lambda}}\Psi^{\bullet}_{I,J}(\boldsymbol z;\boldsymbol p;\kappa)\operatorname{Stab}^{\mathrm{op}}_J(\boldsymbol\Gamma;\boldsymbol z) \,&=\\ =\,\Omega^{\circ}_{\boldsymbol\lambda}(\boldsymbol p;\kappa)&\sum_{\boldsymbol m\in\mathbb Z^{N-1}_{\geqslant 0}}\hat{\mathcal J}^{H}_{I,\boldsymbol m}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\prod_{i=1}^{N-1}\bigl((-\kappa)^{-\lambda_i-\lambda_{i+1}}p_{i+1}/p_i\bigr)^{m_i}.\end{aligned} \tag{6.31}$$

Formula (5.38) and Theorem 5.22 imply that the functions $\Psi^{\bullet}_{I,J}(\boldsymbol z;\boldsymbol p;\kappa)$ are regular if $z_a-z_b\notin\kappa\mathbb Z_{\neq 0}$ for all $a\neq b$, and have at most simple poles at the hyperplanes $z_a-z_b\in\kappa\mathbb Z_{\neq 0}$. Thus the statement of Proposition 6.19 follows from formula (6.31). □

Define the section $\hat{\mathcal J}^{H}_{\boldsymbol m}$ of the bundle $EH_{\boldsymbol\lambda}$ with values

$$\begin{gathered}\hat{\mathcal J}^{H}_{\boldsymbol m}(\boldsymbol z;\kappa)\,:\,H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z}\,\to\,H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z}\,,\\ \hat{\mathcal J}^{H}_{\boldsymbol m}(\boldsymbol z;\kappa)\,:\,Y\mapsto\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\mathcal E_{\boldsymbol z}\bigl\langle Y\,[\hat{\mathcal J}^{H}_{I,\boldsymbol m}(\boldsymbol\Gamma;\boldsymbol z;\kappa)]_{\boldsymbol z}\bigr\rangle\operatorname{Stab}_{I,\boldsymbol z}\,,\end{gathered} \tag{6.32}$$

where $\mathcal E_{\boldsymbol z}$ is the integration map on $H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z}$, see (6.9).

Proposition 6.20. *We have*

$$\operatorname{Stab}_{\boldsymbol\lambda}\Psi^{\bullet}(\boldsymbol z;\boldsymbol p;\kappa)\,=\,\Omega^{\circ}_{\boldsymbol\lambda}(\boldsymbol p;\kappa)\sum_{\boldsymbol m\in\mathbb Z^{N-1}_{\geqslant 0}}\hat{\mathcal J}^{H}_{\boldsymbol m}(\boldsymbol z;\kappa)\prod_{i=1}^{N-1}\bigl((-\kappa)^{-\lambda_i-\lambda_{i+1}}p_{i+1}/p_i\bigr)^{m_i}. \tag{6.33}$$

Proof. The statement follows from formulae (5.38), (6.15), (6.31), (6.32). □

Proposition 6.21. *For any* $Y \in H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}}$,

$$\mathcal{E}_{\boldsymbol{z}}\big\langle(\hat{\mathcal{J}}^H_{\boldsymbol{m}}(\boldsymbol{z};\kappa)\,Y\,)\,\mathrm{Stab}^{\mathrm{op}}_{I,\boldsymbol{z}}\big\rangle \,=\, \mathcal{E}_{\boldsymbol{z}}\big\langle Y\,[\hat{\mathcal{J}}^H_{I,\boldsymbol{m}}(\boldsymbol{\Gamma};\boldsymbol{z};\kappa)]_{\boldsymbol{z}}\big\rangle\,. \tag{6.34}$$

In particular,

$$\mathcal{E}_{\boldsymbol{z}}\big\langle\hat{\mathcal{J}}^H_{\boldsymbol{m}}(\boldsymbol{z};\kappa)\,Y\,\big\rangle \,=\, \mathcal{E}_{\boldsymbol{z}}\big\langle Y\,[\hat{\mathcal{J}}^H_{I^{\min}_{\boldsymbol{\lambda}},\boldsymbol{m}}(\boldsymbol{\Gamma};\boldsymbol{z};\kappa)]_{\boldsymbol{z}}\big\rangle\,. \tag{6.35}$$

Proof. The statement follows from formulae (6.32), (6.15), (6.12). □

6.6. **The map** $\text{Б}^H_{\boldsymbol{\lambda}}$ **.** Let $L^\circ\subset\mathbb{C}^n$ be the complement of the union of the hyperplanes

$$z_a - z_b \in \kappa\mathbb{Z}_{\neq 0}\,, \qquad a,b=1,\dots,n\,,\quad a\neq b\,. \tag{6.36}$$

Denote by $\mathcal{O}$ the ring of functions of $\boldsymbol{z}$ holomorphic in L°, and by $\mathcal{O}_{H_{\boldsymbol{\lambda}}}$ the space of sections of the bundle $H_{\boldsymbol{\lambda}}$ holomorphic in $\boldsymbol{z}$ for $\boldsymbol{z}\in L^\circ$ and not depending on $\boldsymbol{p}$. The stable envelope map $\mathrm{Stab}_{\boldsymbol{\lambda}}$ induces an isomorphism $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\otimes_{\mathbb{C}}\mathcal{O}\to\mathcal{O}_{H_{\boldsymbol{\lambda}}}$, $f\mapsto\mathrm{Stab}_{\boldsymbol{\lambda}}f$.

Let $\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$ be the space of sections of $H_{\boldsymbol{\lambda}}$ that are solutions of quantum differential equations (6.24) holomorphic in $\boldsymbol{z}$ for $\boldsymbol{z}\in L^\circ$. The space $\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$ is a counterpart of the space $\mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}}$ of solutions of limiting dynamical differential equations (3.13) introduced in Section 5.6. The map $\mathrm{Stab}_{\boldsymbol{\lambda}}$ induces an isomorphism $\mathscr{S}^{\mathcal{O}}_{\boldsymbol{\lambda}}\to\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$.

Recall the Levelt solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ$ of quantum differential equations (6.23).

Proposition 6.22. *For any* $f\in\mathcal{O}_{H_{\boldsymbol{\lambda}}}$ *, the section* $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ f$ *with values*

$$(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ f\,)(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \big(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\big)\,f(\boldsymbol{z})$$

belongs to $\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$. *Moreover, the map*

$$\mu^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}:\mathcal{O}_{H_{\boldsymbol{\lambda}}}\to\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}, \qquad f\,\mapsto\,\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ f\,, \tag{6.37}$$

is an isomorphism.

Proof. For any $f\in\mathcal{O}_{H_{\boldsymbol{\lambda}}}$, the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ f$ solves quantum differential equations (6.23). Since by formula (6.28), Lemma 6.15, and Theorem 5.22, the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)$ is holomorphic in $\boldsymbol{z}$ for $\boldsymbol{z}\in L^\circ$, the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ f$ belongs to $\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$. Backwards, for any $f\in\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$, the section $f^{\varnothing}$ with values

$$f^{\varnothing}(\boldsymbol{z}) \,=\, \big(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\big)^{-1}f(\boldsymbol{z};\boldsymbol{p}) \tag{6.38}$$

does not depend on $\boldsymbol{p}$ because $f(\boldsymbol{z};\boldsymbol{p})$ is a solution of quantum differential equations (6.24). Moreover, $\big(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\big)^{-1}$ is holomorphic in $\boldsymbol{z}$ for $\boldsymbol{z}\in L^\circ$ since $\big(\det\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^\circ(\boldsymbol{z};\boldsymbol{p};\kappa)\big)^{-1}$ is entire in $\boldsymbol{z}$. Thus $f^{\varnothing}\in\mathcal{O}_{H_{\boldsymbol{\lambda}}}$ and $f=\mu^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}f^{\varnothing}$. Hence, the map $\mu^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$ is an isomorphism. □

For a solution $f\in\mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$ of quantum differential equations (6.23), we call the section $f^{\varnothing}$ defined by (6.38) the *principal term* of f. For the solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^\circ_P$ corresponding to a Laurent polynomial $P(\acute{\boldsymbol{\Gamma}};\acute{\boldsymbol{z}})$, see (6.27), the principal term is described in Proposition 6.23 below.

Recall the functions $C^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)$ and $G^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)$, see (5.41), (5.42),

$$C^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)\,=\,\prod_{i=1}^N \kappa^{\left(\sum_{j=i+1}^N \lambda_j\,-\,\sum_{j=1}^{i-1}\lambda_j\right)\sum_{a=\lambda^{(i-1)}+1}^{\lambda^{(i)}} z_a/\kappa},$$

$$G^\circ_{\boldsymbol\lambda}(\boldsymbol z;\kappa)\,=\,\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}\Gamma\bigl((z_a-z_b)/\kappa\bigr)\,,$$

and the function $\acute P(\boldsymbol\Gamma;\boldsymbol z;\kappa)$ obtained from a Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$ by substituting the variables $\acute\gamma_{i,j}$ and $\acute z_a$ with the exponentials $e^{2\pi\sqrt{-1}\,\gamma_{i,j}/\kappa}$ and $e^{2\pi\sqrt{-1}\,z_a/\kappa}$, respectively. Set

$$C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{C^\circ_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_I};\kappa)\,V_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z_{\sigma_I})}{R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J})}\,,\qquad G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}G^\circ_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_I};\kappa)\,V_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z_{\sigma_I})\,,$$

and

$$\acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{\acute P(\boldsymbol z_{\sigma_I};\boldsymbol z;\kappa)\,V_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z_{\sigma_I})}{R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_J})}\,,$$

cf. (6.6).

Proposition 6.23. *For a Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$, the values of the principal term $\Psi^{\varnothing}_P$ of the solution $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P$ are*

$$\Psi^{\varnothing}_P(\boldsymbol z;\kappa)\,=\,\bigl[\acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\bigr]_{\boldsymbol z}\,.\tag{6.39}$$

Proof. By Proposition 5.23,

$$\Psi^{\varnothing}_P(\boldsymbol z;\kappa)\,=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\mathcal E_{\boldsymbol z}\bigl\langle\acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol\Gamma;\boldsymbol z)\bigr\rangle\,\mathrm{Stab}_{I,\boldsymbol z}\,.\tag{6.40}$$

By Lemmas 6.6, 6.8, for any section f of $H_{\boldsymbol\lambda}$,

$$[f(\boldsymbol\Gamma;\boldsymbol z)]_{\boldsymbol z}=\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\mathcal E_{\boldsymbol z}\bigl\langle f(\boldsymbol\Gamma;\boldsymbol z)\,\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol\Gamma;\boldsymbol z)\bigr\rangle\,\mathrm{Stab}_{I,\boldsymbol z}\,,$$

Hence, the right-hand sides of formulae (6.39) and (6.40) coincide. □

The functions $C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$, $G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$, and $\acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$ are polynomials in $\boldsymbol\Gamma$ analytically depending on $\boldsymbol z$. The sections of $H_{\boldsymbol\lambda}$ defined by their cohomology classes can be thought of as the cohomology classes of the respective analytic functions of $\boldsymbol\Gamma$,

$$C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\,=\,\prod_{i=1}^N \kappa^{\left(\sum_{j=i+1}^N \lambda_j\,-\,\sum_{j=1}^{i-1}\lambda_j\right)\sum_{j=1}^{\lambda_i}\gamma_{i,j}/\kappa},\tag{6.41}$$

$$G^{+}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\,=\,\kappa^{\lambda_{\{2\}}}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{k=1}^{\lambda_i}\prod_{l=1}^{\lambda_j}\Gamma\bigl(1+(\gamma_{i,k}-\gamma_{j,l})/\kappa\bigr)\,,$$

and $\acute P(\boldsymbol\Gamma;\boldsymbol z;\kappa)$. The class of $C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$ is a product of exponentials of the equivariant first Chern classes $c_1(E_i)\,=\,[\gamma_{i,1}+\ldots+\gamma_{i,\lambda_i}]$ of the vector bundles E_i over $\mathcal F_{\boldsymbol\lambda}$ with fibers F_i/F_{i-1}, the class of $G^{+}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$ is the equivariant Gamma-class of $\mathcal F_{\boldsymbol\lambda}$, and the class of $\acute P(\boldsymbol\Gamma;\boldsymbol z;\kappa)$ is the equivariant Chern character of the class of the Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$ in $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$.

Define a map

$$\text{Б}^{H}_{\boldsymbol\lambda} : K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C) \to \mathcal O_{H_{\boldsymbol\lambda}} \tag{6.42}$$

that sends the class $[P]$ of the Laurent polynomial $P(\acute{\boldsymbol\Gamma};\acute{\boldsymbol z})$ in $K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ to the section of $H_{\boldsymbol\lambda}$ with values $[\acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\, C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\, G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)]_{\boldsymbol z}$. By Proposition 6.23, the map $\text{Б}^{H}_{\boldsymbol\lambda}$ sends the class $[P]$ to the principal term of the solution $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^{\circ}_{P}$ of the joint system of quantum differential equations (6.23) and qKZ difference equations in cohomology (6.25).

Theorem 6.24. *Recall the space $\mathcal O_{H_{\boldsymbol\lambda}}$ of sections of $H_{\boldsymbol\lambda}$ holomorphic in $\boldsymbol z$ for $\boldsymbol z\in L^{\circ}$ and not depending on $\boldsymbol p$, and the space $\mathscr S^{\mathcal O}_{H_{\boldsymbol\lambda}}$ of solutions of quantum differential equations* (6.23) *holomorphic in $\boldsymbol z$ for $\boldsymbol z\in L^{\circ}$. Then the map $\text{Б}^{H}_{\boldsymbol\lambda} : K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)\to\mathcal O_{H_{\boldsymbol\lambda}}$ is well-defined and the following diagram is commutative,*

$$\begin{array}{ccc} K_T(\mathcal F_{\boldsymbol\lambda};\mathbb C) & \xrightarrow{\ \text{Б}^{H}_{\boldsymbol\lambda}\ } & \mathcal O_{H_{\boldsymbol\lambda}} \\ & {\scriptstyle\mu^{K}_{H_{\boldsymbol\lambda}}}\searrow \qquad \swarrow{\scriptstyle\mu^{\mathcal O}_{H_{\boldsymbol\lambda}}} & \\ & \mathscr S^{\mathcal O}_{H_{\boldsymbol\lambda}} & \end{array} \tag{6.43}$$

Proof. The statement follows from Proposition 5.24 by applying the stable envelope map $\mathrm{Stab}_{\boldsymbol\lambda}$. In more detail, by the standard reasoning the functions $C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$, $G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)$, $\acute P(\boldsymbol\Gamma;\boldsymbol z;\kappa)$, and $\mathcal E_{\boldsymbol z}\big\langle \acute P^{H}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\, C^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\, G^{H}_{\boldsymbol\lambda}(\boldsymbol\Gamma;\boldsymbol z;\kappa)\,\mathrm{Stab}^{\mathrm{op}}_{I}(\boldsymbol\Gamma;\boldsymbol z)\big\rangle$ are holomorphic in $\boldsymbol z$ for $\boldsymbol z\in L^{\circ}$. Hence, the map $\text{Б}^{H}_{\boldsymbol\lambda}$ is well-defined. The commutativity of diagram (5.50) follows from Proposition 6.23. □

6.7. **The nonequivariant case $\boldsymbol z = 0$.** In this section, we will discuss solutions of the quantum differential equations for the cohomology algebra $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ of the partial flag variety $\mathcal F_{\boldsymbol\lambda}$ by specializing the results obtained for the equivariant case at $\boldsymbol z=0$.

The cohomology algebra $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ of the partial flag variety $\mathcal F_{\boldsymbol\lambda}$ has the form

$$H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C) \,=\, \mathbb C[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}\Big/\Big\langle \prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\gamma_{i,j}) \,=\, u^n\Big\rangle, \tag{6.44}$$

where u is a formal variable. It is isomorphic to the algebra $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z^0}$ at $\boldsymbol z^0=0$, see (6.3). We denote the class of a polynomial $f(\boldsymbol\Gamma)\in\mathbb C[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}$ in $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ by $[f]_0$.

The integration map on $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$ coincides with the map $\mathcal E_{\boldsymbol z^0}$ at $\boldsymbol z^0=0$, see (6.9),

$$\mathcal E_0 : H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)\to\mathbb C\,, \qquad [f]_0\mapsto\mathcal E_0\langle f\rangle\,, \tag{6.45}$$

where

$$\mathcal E_0\langle f\rangle \,=\, \bigg(\sum_{I\in\mathcal I_{\boldsymbol\lambda}}\frac{f(\boldsymbol z_{\sigma_I})}{R_{\boldsymbol\lambda}(\boldsymbol z_{\sigma_I})}\bigg)\bigg|_{\boldsymbol z=0}\,,$$

see Lemmas 6.3, 6.4.

Consider the polynomials $\mathrm{Stab}_{I,0}(\boldsymbol{\Gamma}) = \mathrm{Stab}_I(\boldsymbol{\Gamma};0)$ and $\mathrm{Stab}^{\mathrm{op}}_{I,0}(\boldsymbol{\Gamma}) = \mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};0)$. By formulae (6.12),

$$\mathrm{Stab}_{\sigma_0(I^{\min}_{\boldsymbol{\lambda}}),0}(\boldsymbol{\Gamma}) \,=\, \mathrm{Stab}^{\mathrm{op}}_{I^{\min}_{\boldsymbol{\lambda}},0}(\boldsymbol{\Gamma}) \,=\, 1\,,$$

$$\mathrm{Stab}_{I^{\min}_{\boldsymbol{\lambda}},0}(\boldsymbol{\Gamma}) \,=\, \mathrm{Stab}^{\mathrm{op}}_{\sigma_0(I^{\min}_{\boldsymbol{\lambda}}),0}(\boldsymbol{\Gamma}) \,=\, \prod_{i=1}^{N-1}\prod_{j=1}^{\lambda_i}\gamma_{i,j}^{n-\lambda^{(i)}}.$$

In general, the polynomials $\mathrm{Stab}_{I,0}(\boldsymbol{\Gamma})$, $\mathrm{Stab}^{\mathrm{op}}_{I,0}(\boldsymbol{\Gamma})$ coincide with the A-type Schubert polynomials, see (6.13),

$$\mathrm{Stab}_{I,0}(\boldsymbol{\Gamma}) \,=\, \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\boldsymbol{\Gamma};0)\,, \qquad \mathrm{Stab}^{\mathrm{op}}_{I,0}(\boldsymbol{\Gamma}) \,=\, \mathfrak{S}_{\sigma_I}(\boldsymbol{\Gamma};0)\,. \tag{6.46}$$

By Lemma 6.6, the polynomials $\mathrm{Stab}_{I,0}(\boldsymbol{\Gamma})$ and $\mathrm{Stab}^{\mathrm{op}}_{I,0}(\boldsymbol{\Gamma})$ are biorthogonal,

$$\mathcal{E}_0\langle \mathrm{Stab}_{I,0}(\boldsymbol{\Gamma})\,\mathrm{Stab}^{\mathrm{op}}_{J,0}(\boldsymbol{\Gamma})\rangle \,=\, \delta_{I,J}\,. \tag{6.47}$$

Denote by $\mathrm{Stab}_{I,0}$, $\mathrm{Stab}^{\mathrm{op}}_{I,0}$ the classes of $\mathrm{Stab}_{I,0}(\boldsymbol{\Gamma})$, $\mathrm{Stab}^{\mathrm{op}}_{I,0}(\boldsymbol{\Gamma})$ in $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. They are the fundamental classes of the Schubert subvarieties in $\mathcal{F}_{\boldsymbol{\lambda}}$ defined for the opposite orderings of the chosen basis of $\mathbb{C}^n$ in the standard way,

The quantum multiplication in $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, is a deformation of the multiplication in $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ depending on quantum parameters $\boldsymbol{p}$. For $Y\in H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, let $Y*_{\boldsymbol{p}}$ be the operator of quantum multiplication by Y depending on $\boldsymbol{p}$.

For $i=1,\dots,N$, denote $D_i=\gamma_{i,1}+\ldots+\gamma_{i,\lambda_i}$, cf. (6.2). The quantum differential equations for $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$-valued functions of $\boldsymbol{p}$ is the following system of compatible differential equations,

$$\kappa\, p_i\frac{\partial f}{\partial p_i} \,=\, [D_i]_0 *_{\boldsymbol{p}} f\,, \qquad i=1,\dots,N\,, \tag{6.48}$$

where κ is a parameter of the equations. Equations (6.48) coincide with the restriction of equivariant quantum differential equations (6.23) to the subbundle $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\times\mathbb{C}^N\to\mathbb{C}^N$ of the bundle $H_{\boldsymbol{\lambda}}\to\mathbb{C}^n\times\mathbb{C}^N$ located over the points $(0,\boldsymbol{p})\in\mathbb{C}^n\times\mathbb{C}^N$ of the base.

Consider the K-theory algebra $K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. Then

$$K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C}) \,=\, \mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]\Big/\Big\langle \prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\acute{\gamma}_{i,j}) \,=\, \prod_{a=1}^{n}(u-1)\Big\rangle\,, \tag{6.49}$$

where u is a formal variable. The evaluation map

$$\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}] \,\to\, \mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\,, \qquad P(\acute{\boldsymbol{\Gamma}};\acute{\boldsymbol{z}})\mapsto P\big(\acute{\boldsymbol{\Gamma}};(1,\dots,1)\big)\,,$$

induces the isomorphism of $K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\to K_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})/\langle \boldsymbol{z}=(1,\dots,1)\rangle$. Denote by $[P]_1$ the class of $P(\acute{\boldsymbol{\Gamma}})\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ in $K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$.

Identify $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ with the subspace of $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$ of Laurent polynomials not depending on $\boldsymbol{z}$. For each $P\in\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$, the solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ of equivariant quantum differential equations (6.23), see Section 6.5, is regular at $\boldsymbol{z}=0$, and the solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(0;\boldsymbol{p};\kappa)$ of equations (6.48) depends only on the class of P in $K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. That is, there is a well-defined map

$$\mu^{K}_{\mathscr{H}_{\boldsymbol{\lambda}}}:\,K(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\to\mathscr{H}_{\boldsymbol{\lambda}}\,,\qquad [P]_1\mapsto \mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(0;\boldsymbol{p};\kappa)\,,\tag{6.50}$$

where $\mathscr{H}_{\boldsymbol{\lambda}}$ is the space of solutions of equations (6.48).

Proposition 6.25. *The map $\mu^{K}_{\mathscr{H}_{\boldsymbol{\lambda}}}$ is an isomorphism.*

Proof. The statement follows from Theorem 5.18 at $\boldsymbol{z}=0$. □

The Levelt fundamental solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa)$ of equivariant quantum differential equations (6.23), see Section 6.5, is regular at $\boldsymbol{z}=0$ and the function $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)$ is an $\mathrm{End}\big(H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\big)$-valued solution of quantum differential equations (6.48). We call $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)$ the Levelt fundamental solution of quantum differential equations (6.48). It induces the map

$$\mu^{H}_{\mathscr{H}_{\boldsymbol{\lambda}}}:\,H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})\to\mathscr{H}_{\boldsymbol{\lambda}}\,,\qquad f\mapsto \mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)\,f\,,\tag{6.51}$$

to the space $\mathscr{H}_{\boldsymbol{\lambda}}$ of solutions of equations (6.48).

Proposition 6.26. *The map $\mu^{H}_{\mathscr{H}_{\boldsymbol{\lambda}}}$ is an isomorphism.*

Proof. The statement follows from Proposition 6.22 at $\boldsymbol{z}=0$. □

For a solution $f(\boldsymbol{p})\in\mathscr{H}_{\boldsymbol{\lambda}}$ of quantum differential equations (6.48), we call the element

$$f^{\varnothing,0}=\big(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)\big)^{-1}f(\boldsymbol{p})\tag{6.52}$$

the *principal term* of $f(\boldsymbol{p})$. The principal term of the solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(0;\boldsymbol{p};\kappa)$ corresponding to a Laurent polynomial $P(\acute{\boldsymbol{\Gamma}})$ is given by Proposition 6.28 below.

For a function $F(\boldsymbol{\Gamma})$ holomorphic in a neighborhood of $\boldsymbol{\Gamma}=0$ and symmetric in $\gamma_{i,1},\dots,\gamma_{i,\lambda_i}$ for each $i=1,\dots,N$, define its class $[F]_0$ in $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ by expanding $F(\boldsymbol{\Gamma})$ in the power series about $\boldsymbol{\Gamma}=0$ and replacing each term by the corresponding class in $H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$. The resulting sum contains only finitely many nonzero terms and, hence, is well-defined. Alternatively, the class $[F]_0$ can be evaluated as follows. Set

$$F^{H}(\boldsymbol{\Gamma};\boldsymbol{z})\,=\sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{F(\boldsymbol{z}_{\sigma_I})\,V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_I})}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})}\,.\tag{6.53}$$

Lemma 6.27. *The function $F^{H}(\boldsymbol{\Gamma};\boldsymbol{z})$ is regular at $\boldsymbol{z}=0$ and $[F]_0=[F^{H}(\boldsymbol{\Gamma};0)]_0$.*

Proof. By the standard facts on power series, it suffices to prove the statement for a polynomial $F(\boldsymbol{\Gamma})$. In the polynomial case, formulae (6.53), (6.4) yield $F(\boldsymbol{\Gamma})=F^{H}(\boldsymbol{\Gamma};\boldsymbol{\Gamma})$. Then $[F(\boldsymbol{\Gamma})]_0=[F^{H}(\boldsymbol{\Gamma};\boldsymbol{\Gamma})]_0=[F^{H}(\boldsymbol{\Gamma};0)]_0$, because $F^{H}(\boldsymbol{\Gamma};\boldsymbol{z})$ is symmetric in $\boldsymbol{z}$. □

Recall the functions $C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$, $G^+_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$, see (6.41), and the function $\acute P(\boldsymbol\Gamma;\kappa)$ obtained from a Laurent polynomial $P(\acute{\boldsymbol\Gamma})$ by substituting the variables $\acute\gamma_{i,j}$ with the exponentials $e^{2\pi\sqrt{-1}\,\gamma_{i,j}/\kappa}$. The class of $C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$ is a product of exponentials of the first Chern classes $c_1(E_i)=[\gamma_{i,1}+\ldots+\gamma_{i,\lambda_i}]$ of the vector bundles E_i over $\mathcal F_{\boldsymbol\lambda}$ with fibers F_i/F_{i-1}, the class of $G^+_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)$ is the Gamma-class of $\mathcal F_{\boldsymbol\lambda}$, and the class of $\acute P(\boldsymbol\Gamma;\kappa)$ is the Chern character of the class of the Laurent polynomial $P(\acute{\boldsymbol\Gamma})$ in $K(\mathcal F_{\boldsymbol\lambda};\mathbb C)$.

Proposition 6.28. *For a Laurent polynomial $P(\acute{\boldsymbol\Gamma})$, the principal term $\Psi^{\varnothing,0}_P$ of the solution $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P(0;\boldsymbol p;\kappa)$ equals*

$$\Psi^{\varnothing,0}_P(\kappa)\,=\,\big[\acute P(\boldsymbol\Gamma;\kappa)\,C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\,G^+_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\big]_0\,. \tag{6.54}$$

Proof. The statement follows from Proposition 6.28 at $\boldsymbol z=0$ and Lemma 6.27. □

Define a map

$$\text{Б}^{H,0}_{\boldsymbol\lambda}:K(\mathcal F_{\boldsymbol\lambda};\mathbb C)\to H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)\,,\qquad [P]_1\to\big[\acute P(\boldsymbol\Gamma;\kappa)\,C^\circ_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\,G^+_{\boldsymbol\lambda}(\boldsymbol\Gamma;\kappa)\big]_0\,, \tag{6.55}$$

By Proposition 6.28, the map $\text{Б}^{H,0}_{\boldsymbol\lambda}$ sends the class $[P]_1$ to the principal term of the solution $\mathrm{Stab}_{\boldsymbol\lambda}\Psi^\circ_P(0;\boldsymbol p;\kappa)$ of quantum differential equations (6.48).

Theorem 6.29. *Recall the space $\mathscr H_{\boldsymbol\lambda}$ of solutions of quantum differential equations (6.48). The following diagram is commutative,*

$$\begin{array}{ccc} K(\mathcal F_{\boldsymbol\lambda};\mathbb C) & \xrightarrow{\ \text{Б}^{H,0}_{\boldsymbol\lambda}\ } & H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)\\ & {\scriptstyle\mu^K_{\mathscr H_{\boldsymbol\lambda}}}\searrow\quad\swarrow{\scriptstyle\mu^H_{\mathscr H_{\boldsymbol\lambda}}} & \\ & \mathscr H_{\boldsymbol\lambda} & \end{array} \tag{6.56}$$

Proof. The statement is equivalent to Proposition 6.28. □

6.8. **Topological-enumerative morphism and J-function.** Recall the Levelt fundamental solution $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(0;\boldsymbol p;\kappa)$ of quantum differential equations (6.48). By Lemma 6.8 and biorthogonality relation (6.6), for any $I\in\mathcal I_{\boldsymbol\lambda}$, there exists a unique $H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$-valued function $\hat{\mathcal J}^H_I(\boldsymbol p;\kappa)$ such that for any $f\in H^*(\mathcal F_{\boldsymbol\lambda};\mathbb C)$,

$$\mathcal E_0\big\langle\big(\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(0;\boldsymbol p;\kappa)\,f\big)\,\mathrm{Stab}^{\mathrm{op}}_{I,0}\big\rangle\,=\,\mathcal E_0\big\langle\hat{\mathcal J}^H_I(\boldsymbol p;\kappa)\,f\,\big\rangle\,. \tag{6.57}$$

Since $\mathrm{Stab}^{\mathrm{op}}_{I^{\min}_{\boldsymbol\lambda},0}=1$, see (6.12), formula (6.57) for $I=I^{\min}_{\boldsymbol\lambda}$ takes the form

$$\mathcal E_0\big\langle\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ(0;\boldsymbol p;\kappa)\,f\,\big\rangle\,=\,\mathcal E_0\big\langle\hat{\mathcal J}^H_{I^{\min}_{\boldsymbol\lambda}}(\boldsymbol p;\kappa)\,f\,\big\rangle\,. \tag{6.58}$$

Proposition 6.30. *For any $I\in\mathcal I_{\boldsymbol\lambda}$,*

$$\hat{\mathcal J}^H_I(\boldsymbol p;\kappa)\,=\,\Omega^\circ_{\boldsymbol\lambda}(\boldsymbol p;\kappa)\sum_{\boldsymbol m\in\mathbb Z^{N-1}_{\geqslant0}}[\hat{\mathcal J}^H_{I,\boldsymbol m}(\boldsymbol\Gamma;0;\kappa)]_0\prod_{i=1}^{N-1}\big((-\kappa)^{-\lambda_i-\lambda_{i+1}}\,p_{i+1}/p_i\big)^{m_i}\prod_{i=1}^{N}p_i^{[D_i]_0/\kappa}, \tag{6.59}$$

where the functions $\hat{\mathcal{J}}^{H}_{I,\boldsymbol{m}}$ *are given by* (6.30), $\Omega^{\circ}_{\boldsymbol{\lambda}} = e^{\sum^{\circ}_{i<j} p_j/(\kappa p_i)}$ *in which the sum is taken over all pairs* $i<j$ *such that* $\lambda_i = 1$ *and* $\lambda_{i+1} = \ldots = \lambda_j = 0$, *see* (5.18), *and* $D_i = \gamma_{i,1} + \ldots + \gamma_{i,\lambda_i}$.

Proof. By Proposition 6.19, the functions $\hat{\mathcal{J}}^{H}_{I,\boldsymbol{m}}(\boldsymbol{\Gamma};\boldsymbol{z};\kappa)$ are regular at $\boldsymbol{z} = 0$. Thus the classes $[\hat{\mathcal{J}}^{H}_{I,\boldsymbol{m}}(\boldsymbol{\Gamma};0;\kappa)]_0$ in formula (6.59) are well defined. Then the statement follows from Proposition 6.21 and formulae (6.57), (6.33), (6.29). □

By inspection, the function $\hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{p};\kappa)$ in this paper coincides with the nonequivariant J-function of $\mathcal{F}_{\boldsymbol{\lambda}}$ obtained in [BCK] up to a change of notation.

In more detail, formula (6.30) for the function $\hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol{\lambda}},\boldsymbol{m}}(\Gamma;\boldsymbol{z};\kappa)$ takes the form

$$\hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol{\lambda}},\boldsymbol{m}}(\boldsymbol{\Gamma};\boldsymbol{z};\kappa) \,=\, \sum_{i=1}^{N-1}\ \sum_{\substack{\boldsymbol{l}\in\mathbb{Z}^{\lambda\{1\}}\\ |\boldsymbol{l}^{(i)}|=m_i}}\ \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \frac{A^{\circ}(\Sigma_J - \boldsymbol{l};\boldsymbol{z};\kappa)\, V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_J})}{A^{\circ}(\Sigma_J;\boldsymbol{z})\, R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_J})}\,, \tag{6.60}$$

because $W^{\circ}_{I^{\min}_{\boldsymbol{\lambda}}}(\Sigma_J - \boldsymbol{l}\kappa;\boldsymbol{z}) = 1$, see Lemma 5.2. In (6.60), $\boldsymbol{m} = (m_1,\ldots,m_{N-1}) \in \mathbb{Z}^{N-1}_{\geqslant 0}$, $\boldsymbol{l} = \bigl(l^{(1)}_1,\ldots,l^{(1)}_{\lambda^{(1)}},\,\ldots\,,l^{(N-1)}_1,\ldots,l^{(N-1)}_{\lambda^{(N-1)}}\bigr) \in \mathbb{Z}^{\lambda\{1\}}$, $|\boldsymbol{l}^{(i)}| = l^{(i)}_j + \ldots + l^{(i)}_{\lambda^{(i)}}$,

$$A^{\circ}(\boldsymbol{t};\boldsymbol{z};\kappa) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Biggl(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\Gamma\bigl(1 + (t^{(i)}_b - t^{(i)}_a)/\kappa\bigr)\prod_{c=1}^{\lambda^{(i+1)}}\frac{1}{\Gamma\bigl(1 + (t^{(i+1)}_c - t^{(i)}_a)/\kappa\bigr)}\Biggr),$$

where $\lambda^{(N)} = n$, $t^{(N)}_a = z_a$, and $l^{(i)}_a = 0$ for $a > \lambda^{(i)}$ or $i = N$, and Σ_J is given by (4.1). Then the expression in [BCK, Theorem 1.3] for $J^F_d(\hbar)$ with $l = N-1$, $d_i = m_i$, $d_{i,j} = l^{(i)}_j$, $\hbar = \kappa$, and $H_{i,j} = \gamma_{a,j-\lambda^{(a-1)}}$ for $\lambda^{(a-1)} < j \leqslant \lambda^{(a)}$, agrees term by term with the expression for the class $[\hat{\mathcal{J}}^{H}_{I,\boldsymbol{m}}(\boldsymbol{\Gamma};0;\kappa)]_0$ obtained from (6.60). Also, formula (6.59) for $\hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{p};\kappa)$ is the same as the series expansion of the nonequivariant J-function of $\mathcal{F}_{\boldsymbol{\lambda}}$ in [BCK].

Observe that relation (6.58) between the J-function $\hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{p};\kappa)$ and the Levelt fundamental solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)$ matches the general relation between the J-function and the topological-enumerative morphism, see Definition 5.3 and formula (5.8) in [CV], where the J-function is defined in terms of the topological-enumerative morphism. This naturally suggests the following conjecture.

Conjecture 6.31. *The Levelt fundamental solution* $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(0;\boldsymbol{p};\kappa)$ *of quantum differential equations* (6.48) *is the topological-enumerative morphism of* $\mathcal{F}_{\boldsymbol{\lambda}}$.

Observe that formulae (6.57) – (6.59) are specializations of respective formulae for the Levelt fundamental solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa)$ of equivariant quantum differential equations (6.23). By Lemma 6.8 and biorthogonality relation (6.6), for any $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, there exists a unique section $\hat{\mathcal{J}}^{H}_{I}(\boldsymbol{z};\boldsymbol{p};\kappa)$ of the bundle $H_{\boldsymbol{\lambda}}$ such that for any $f \in H^*(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$ considered as a section of $H_{\boldsymbol{\lambda}}$ not depending on $\boldsymbol{p}$,

$$\mathcal{E}_{\boldsymbol{z}}\bigl\langle\bigl(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa)\,[f]_{\boldsymbol{z}}\bigr)\,\mathrm{Stab}^{\mathrm{op}}_{I,\boldsymbol{z}}\bigr\rangle \,=\, \mathcal{E}_{\boldsymbol{z}}\bigl\langle\hat{\mathcal{J}}^{H}_{I}(\boldsymbol{z};\boldsymbol{p};\kappa)\,[f]_{\boldsymbol{z}}\bigr\rangle\,. \tag{6.61}$$

Since $\mathrm{Stab}^{\mathrm{op}}_{I^{\min}_{\boldsymbol\lambda},\boldsymbol z} = 1$, see (6.12), formula (6.57) for $I = I^{\min}_{\boldsymbol\lambda}$ takes the form

$$\mathcal{E}_{\boldsymbol z}\big\langle \mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^{\circ}(\boldsymbol z;\boldsymbol p;\kappa)\,[f]_{\boldsymbol z}\big\rangle \,=\, \mathcal{E}_{\boldsymbol z}\big\langle \hat{\mathcal{J}}^{H}_{I^{\min}_{\boldsymbol\lambda}}(\boldsymbol z;\boldsymbol p;\kappa)\,[f]_{\boldsymbol z}\big\rangle\,. \tag{6.62}$$

Furthermore, by Proposition 6.21 and formulae (6.61), (6.33), (6.29),

$$\hat{\mathcal{J}}^{H}_{I}(\boldsymbol z;\boldsymbol p;\kappa) \,= \tag{6.63}$$
$$=\, \Omega^{\circ}_{\boldsymbol\lambda}(\boldsymbol p;\kappa)\sum_{\boldsymbol m\in\mathbb{Z}^{N-1}_{\geqslant 0}}[\hat{\mathcal{J}}^{H}_{I,\boldsymbol m}(\boldsymbol\Gamma;\boldsymbol z;\kappa)]_z\prod_{i=1}^{N-1}\big((-\kappa)^{-\lambda_i-\lambda_{i+1}}\,p_{i+1}/p_i\big)^{m_i}\prod_{i=1}^{N}p_i^{[D_i]_{\boldsymbol z}/\kappa},$$

where the notation is the same as in formula (6.59). This suggests the equivariant version of Conjecture 6.31.

Conjecture 6.32. *The Levelt fundamental solution* $\mathrm{Stab}_{\boldsymbol\lambda}\widehat\Psi^{\circ}(\boldsymbol z;\boldsymbol p;\kappa)$ *of equivariant quantum differential equations* (6.23) *is the equivariant topological-enumerative morphism of* $\mathcal{F}_{\boldsymbol\lambda}$.

6.9. **Example $\boldsymbol\lambda = (1, n-1)$, $\mathcal{F}_{\boldsymbol\lambda} = \mathbb{CP}^{n-1}$.** Throughout this section, let $N = 2$, $n \geqslant 2$, $\boldsymbol\lambda = (1, n-1)$, $\mathcal{F}_{\boldsymbol\lambda} = \mathbb{CP}^{n-1}$.

Denote $\gamma = \gamma_{1,1}$. Identify $\mathbb{C}[\gamma]$ with the subspace of $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}$ consisting of polynomials not depending on $\gamma_{2,1},\dots,\gamma_{2,n-1}$, and $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}$ with the subspace of $\mathbb{C}[\boldsymbol\Gamma]^{S_{\boldsymbol\lambda}}\otimes\mathbb{C}[\boldsymbol z]$ consisting of polynomials not depending on $\boldsymbol z$. Then the equivariant cohomology algebra $H^*_T(\mathbb{CP}^{n-1};\mathbb{C})$, see (6.1), can be presented as follows

$$H^*_T(\mathbb{CP}^{n-1};\mathbb{C}) \,=\, \mathbb{C}[\gamma]\otimes\mathbb{C}[\boldsymbol z]\Big/\Big\langle\prod_{a=1}^{n}(\gamma - z_a) \,=\, 0\Big\rangle\,. \tag{6.64}$$

Similarly, for any $\boldsymbol z^0\in\mathbb{C}^n$, the algebra $H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol z^0}$ can be presented as follows

$$H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol z^0} \,=\, \mathbb{C}[\gamma]\Big/\Big\langle\prod_{a=1}^{n}(\gamma - z^0_a) \,=\, 0\Big\rangle\,. \tag{6.65}$$

For any polynomial $f(\gamma;\boldsymbol z)$, the polynomial $\mathcal{E}\langle f\rangle(\boldsymbol z)$, see (6.7), can be evaluated as an integral

$$\mathcal{E}\langle f\rangle(\boldsymbol z) \,=\, \frac{1}{2\pi\sqrt{-1}}\int_C\frac{f(\gamma;\boldsymbol z)}{(\gamma-z_1)\dots(\gamma-z_n)}\,d\gamma \tag{6.66}$$

over a contour C encircling $z_1,\dots,z_n$ counterclockwise. Similarly, for any polynomial $f(\gamma)$ and $\boldsymbol z^0\in\mathbb{C}^n$,

$$\mathcal{E}_{\boldsymbol z^0}\langle f\rangle \,=\, \frac{1}{2\pi\sqrt{-1}}\int_C\frac{f(\gamma)}{(\gamma-z^0_1)\dots(\gamma-z^0_n)}\,d\gamma \tag{6.67}$$

for a contour C encircling $z^0_1,\dots,z^0_n$ counterclockwise.

Following Section 5.7, denote by $[a]$ the element $\big(\{a\},\{1,\dots,a-1,a+1,\dots,n\}\big)\in\mathcal{I}_{\boldsymbol\lambda}$.

The polynomials $\mathrm{Stab}_{[a]}(\gamma;\boldsymbol z)$, $\mathrm{Stab}^{\mathrm{op}}_{[a]}(\gamma;\boldsymbol z)$ are respectively obtained from the functions $\check W^\circ_{[a]}(t;\boldsymbol z)$, $W^\circ_{[a]}(t;\boldsymbol z)$, see (5.57), (5.55), by the substitution $t=\gamma$,

$$\mathrm{Stab}_{[a]}(\gamma;\boldsymbol z) \,=\, \prod_{c=a+1}^{n}(\gamma-z_b)\,,\qquad \mathrm{Stab}^{\mathrm{op}}_{[a]}(\gamma;\boldsymbol z) \,=\, \prod_{c=1}^{a-1}(\gamma-z_b)\,,\qquad a=1,\dots,n\,. \tag{6.68}$$

The biorthogonality relation, see Lemma 6.6,

$$\mathcal{E}\langle \mathrm{Stab}_{[a]}(\gamma;\boldsymbol z)\,\mathrm{Stab}^{\mathrm{op}}_{[b]}(\gamma;\boldsymbol z)\rangle \,=\, \delta_{a,b} \tag{6.69}$$

is clear from formula (6.66).

Consider a polynomial

$$\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z) \,=\, \frac{1}{t-\gamma}\left(\prod_{a=1}^{n}(t-z_a)-\prod_{a=1}^{n}(\gamma-z_a)\right), \tag{6.70}$$

and its image $\bigl[\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z)\bigr]$ in $\mathbb C[t]\otimes H^*_T(\mathbb{CP}^{n-1};\mathbb C)$. Recall the polynomial $\widehat W^\circ_{\boldsymbol\lambda}(t;\boldsymbol\Gamma)$, see (6.18), and its image $\bigl[\widehat W^\circ_{\boldsymbol\lambda}(t;\boldsymbol\Gamma)\bigr]$ in $\mathbb C[t]\otimes H^*_T(\mathbb{CP}^{n-1};\mathbb C)$. By relations (6.1), (6.64),

$$\bigl[\widehat W^\circ_{\boldsymbol\lambda}(t;\boldsymbol\Gamma)\bigr] \,=\, \bigl[\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z)\bigr]\,.$$

Thus Theorem 6.9 for $\mathbb{CP}^{n-1}$ reads as follows.

$$\bigl[\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z)\bigr] \,=\, \sum_{a=1}^{n} W^\circ_{[a]}(t;\boldsymbol z)\,\bigl[\mathrm{Stab}_{[a]}(\gamma;\boldsymbol z)\bigr]\,. \tag{6.71}$$

In fact, formula (6.71) is the image $\mathbb C[t]\otimes H^*_T(\mathbb{CP}^{n-1};\mathbb C)$ of the equality of polynomials

$$\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z) \,=\, \sum_{a=1}^{n} W^\circ_{[a]}(t;\boldsymbol z)\,\mathrm{Stab}_{[a]}(\gamma;\boldsymbol z)\,. \tag{6.72}$$

Recall that for $\boldsymbol z\in\mathbb C^n$ and $a=1,\dots,n$, the classes of the polynomials $\mathrm{Stab}_{[a]}(\gamma;\boldsymbol z)$, $\mathrm{Stab}^{\mathrm{op}}_{[a]}(\gamma;\boldsymbol z)$ in $H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$ are denoted by $\mathrm{Stab}_{[a],\boldsymbol z}$, $\mathrm{Stab}^{\mathrm{op}}_{[a],\boldsymbol z}$. Consider a basis $v_{[1]}$, $\dots,v_{[n]}$ of $(\mathbb C^2)^{\otimes n}_{\boldsymbol\lambda}$, where $v_{[a]}=v_2^{\otimes(a-1)}\otimes v_1\otimes v_2^{\otimes(n-a)}$. For every $\boldsymbol z\in\mathbb C^n$, the map

$$\mathrm{Stab}_{\boldsymbol\lambda,\boldsymbol z}\,:\,(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}\,\to\,H^*_T(\mathcal F_{\boldsymbol\lambda};\mathbb C)_{\boldsymbol z}\,,\qquad v_{[a]}\,\mapsto\,\mathrm{Stab}_{[a],\boldsymbol z}\,,\quad a=1,\dots,n\,, \tag{6.73}$$

gives an isomorphism of $(\mathbb C^N)^{\otimes n}_{\boldsymbol\lambda}$ and $H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$.

Set $D_1=\gamma$, $D_2=z_1+\ldots+z_n-\gamma$. The operators of quantum multiplication $[D_1]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}$, $[D_2]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}$ act on $H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$ as follows

$$\begin{aligned}
&[D_1]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}\mathrm{Stab}_{[1],\boldsymbol z} \,=\, z_1\,\mathrm{Stab}_{[1],\boldsymbol z}+\frac{p_2}{p_1}\,\mathrm{Stab}_{[n],\boldsymbol z}\,, \\
&[D_1]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}\mathrm{Stab}_{[a],\boldsymbol z} \,=\, z_a\,\mathrm{Stab}_{[a],\boldsymbol z}+\mathrm{Stab}_{[a-1],\boldsymbol z}\,, && a=2,\dots,n\,, \\
&[D_2]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}\mathrm{Stab}_{[b],\boldsymbol z} \,=\, \Bigl(-[D_1]_{\boldsymbol z}*_{\boldsymbol p,\boldsymbol z}+\sum_{c=1}^{n}z_c\Bigr)\mathrm{Stab}_{[b],\boldsymbol z}\,, && b=1,\dots,n\,,
\end{aligned} \tag{6.74}$$

see for instance [TV7]. As Corollary 6.14 states, the map $\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}}$ intertwines the dynamical operators $X_1^\circ(\boldsymbol{z};\boldsymbol{p})$, $X_2^\circ(\boldsymbol{z};\boldsymbol{p})$, see (5.53), acting on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ and the operators of quantum multiplication, $[D_1]_{\boldsymbol{z}} *_{\boldsymbol{p},\boldsymbol{z}}$, $[D_2]_{\boldsymbol{z}} *_{\boldsymbol{p},\boldsymbol{z}}$,

$$\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}} \circ X_i^\circ(\boldsymbol{z};\boldsymbol{p}) \,=\, [D_i]_{\boldsymbol{z}} *_{\boldsymbol{p},\boldsymbol{z}} \circ \mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}}\,, \qquad i=1,2\,. \tag{6.75}$$

Consider the trivial vector bundle $H_{\boldsymbol{\lambda}} \to \mathbb{C}^n \times \mathbb{C}^2$ with fiber over a point $(\boldsymbol{z}^0,\boldsymbol{p}^0)$ given by $H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol{z}^0}$. The equivariant quantum differential equations for sections of $H_{\boldsymbol{\lambda}}$ is the system of differential equations

$$\kappa\, p_i \frac{\partial f}{\partial p_i} \,=\, [D_i]_{\boldsymbol{z}} *_{\boldsymbol{p},\boldsymbol{z}} f\,, \qquad i=1,2, \tag{6.76}$$

where κ is the parameter of the equations. The isomorphisms $\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}}$ identify equations (6.76) with the limiting dynamical differential equation (5.54). Furthermore, the isomorphisms $\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}}$ and the limiting *qKZ* equations (5.52) define the *qKZ* difference equations in cohomology

$$f(z_1,\dots,z_a+\kappa,\dots,z_n;\boldsymbol{p};\kappa) \,=\, K^H_a(\boldsymbol{z};\boldsymbol{p};\kappa)\, f(\boldsymbol{z};\boldsymbol{p};\kappa)\,, \qquad a=1,\dots,n\,, \tag{6.77}$$

where for each $a=1,\dots,n$, and fixed $\boldsymbol{z},\boldsymbol{p}$, the operator $K^H_a(\boldsymbol{z};\boldsymbol{p};\kappa)$ is a map of fibers,

$$K^H_a(\boldsymbol{z};\boldsymbol{p};\kappa)\,:\, H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{\boldsymbol{z}} \,\to\, H^*_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})_{(z_1,\dots,z_a+\kappa,\dots,z_n)}\,,$$

$$K^H_a(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, \mathrm{Stab}_{\boldsymbol{\lambda},(z_1,\dots,z_a+\kappa,\dots,z_n)} \circ K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa) \circ (\mathrm{Stab}_{\boldsymbol{\lambda},\boldsymbol{z}})^{-1},$$

and the operator $K^\circ_a(\boldsymbol{z};\boldsymbol{p};\kappa)$ is given by (5.51). By Theorem 6.16, quantum differential equations (6.76) and *qKZ* difference equations (6.77) is a compatible system of differential and difference equations for sections of $H_{\boldsymbol{\lambda}}$.

Denote $\acute{\gamma} = \acute{\gamma}_{1,1}$. Identify $\mathbb{C}[\acute{\gamma}^{\pm 1}]$ with the subspace of $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ consisting of Laurent polynomials not depending on $\acute{\gamma}_{2,1},\dots,\acute{\gamma}_{2,n-1}$, and $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}}$ with the subspace of $\mathbb{C}[\acute{\boldsymbol{\Gamma}}^{\pm1}]^{S_{\boldsymbol{\lambda}}} \otimes \mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$ consisting of polynomials not depending on $\acute{\boldsymbol{z}}$. Then the equivariant K-theory algebra $K_T(\mathbb{CP}^{n-1};\mathbb{C})$, see (6.26), can be presented as in (5.63),

$$K_T(\mathbb{CP}^{n-1};\mathbb{C}) \,=\, \mathbb{C}[\acute{\gamma}^{\pm1},\acute{\boldsymbol{z}}^{\pm1}] \Big/ \Big\langle \prod_{a=1}^n (\acute{\gamma}-\acute{z}_a) \,=\, 0 \Big\rangle. \tag{6.78}$$

Recall that for a Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}})$, we denote

$$\acute{P}(\gamma;\boldsymbol{z};\kappa) \,=\, P(e^{2\pi\sqrt{-1}\,\gamma/\kappa}; e^{2\pi\sqrt{-1}\,z_1/\kappa},\dots,e^{2\pi\sqrt{-1}\,z_n/\kappa})\,.$$

In Section 5.7, we described solutions $\Psi^\circ_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ of equations (5.54), (5.52) labeled by Laurent polynomials $P(\acute{\gamma};\acute{\boldsymbol{z}})$, see (5.61). Denote by $\mathscr{S}^K_{H_{\boldsymbol{\lambda}}}$ the space spanned over $\mathbb{C}$ by the solutions $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^\circ_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ of equations (5.54), (5.52). Each element of $\mathscr{S}^K_{H_{\boldsymbol{\lambda}}}$ is a section of $H_{\boldsymbol{\lambda}}$ holomorphic in p_1, p_2 provided branches of $\log p_1$ and $\log p_2$ are fixed, and entire in $\boldsymbol{z}$, see Proposition 5.17. The space $\mathscr{S}^K_{H_{\boldsymbol{\lambda}}}$ is a $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm1}]$-module, with a Laurent polynomial $P(\acute{\boldsymbol{z}})$ acting as multiplication by $\acute{P}(\boldsymbol{z};\kappa)$.

Since the function $\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ depends only on the class of P in $K_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C})$, then so does the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$ of $H_{\boldsymbol{\lambda}}$. Thus the map

$$\mu^K_{H_{\boldsymbol{\lambda}}} : K_T(\mathcal{F}_{\boldsymbol{\lambda}};\mathbb{C}) \to \mathscr{S}^K_{H_{\boldsymbol{\lambda}}}, \qquad [P] \mapsto \mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P, \tag{6.79}$$

is well-defined, cf. (5.29). By Corollary 5.19, the map $\mu^K_{H_{\boldsymbol{\lambda}}}$ is an isomorphism of $\mathbb{C}[\acute{\boldsymbol{z}}^{\pm 1}]$-modules.

There is an integral formula for the function $\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$, see (5.62). Taking into account formula (6.71), we get an integral presentation for the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa)$,

$$\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_P(\boldsymbol{z};\boldsymbol{p};\kappa) = \frac{1}{2\pi\sqrt{-1}\,\kappa}\int_C \acute{P}(t;\boldsymbol{z};\kappa)\,\Phi^{\circ}_{\boldsymbol{\lambda}}(t;\boldsymbol{z};\boldsymbol{p};\kappa)\,\bigl[\bar{W}_{\boldsymbol{\lambda}}(t;\gamma;\boldsymbol{z})\bigr]_{\boldsymbol{z}}\,dt, \tag{6.80}$$

where

$$\Phi^{\circ}_{\boldsymbol{\lambda}}(t;\boldsymbol{z};\boldsymbol{p};\kappa) = (p_2/\kappa)^{\sum_{a=1}^n z_a/\kappa}\,(\kappa^n p_1/p_2)^{t/\kappa}\prod_{a=1}^n \Gamma\bigl((t-z_a)/\kappa\bigr) \tag{6.81}$$

is the master function, see (5.58), the polynomial $\bar{W}_{\boldsymbol{\lambda}}(t;\gamma;\boldsymbol{z})$ is given by (6.70), a contour C encircles the poles of the product $\prod_{a=1}^n \Gamma\bigl((t-z_a)/\kappa\bigr)$ counterclockwise. For instance, C can be the parabola

$$C = \{\,\kappa\bigl(A - s^2 + s\sqrt{-1}\bigr) \mid s\in\mathbb{R}\,\},$$

where A is a sufficiently large positive real number.

In Section 6.5, we introduced the Levelt fundamental solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}$ of quantum differential equations. For the example of equations (6.76), $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}$ is the section of the vector bundle $EH_{\boldsymbol{\lambda}} \to \mathbb{C}^n\times\mathbb{C}^2$ with fiber over a point $(\boldsymbol{z}^0,\boldsymbol{p}^0)$ given by $\mathrm{End}\bigl(H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol{z}^0}\bigr)$.

By (6.29), the values of $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}$ have the form

$$\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa) = \bigl(\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\bullet}(\boldsymbol{z};\boldsymbol{p};\kappa)\bigr)\,(p_1/p_2)^{[\gamma]_{\boldsymbol{z}}/\kappa}\,p_2^{\sum_{c=1}^n z_c/\kappa}, \tag{6.82}$$

where $(p_1/p_2)^{[\gamma]_{\boldsymbol{z}}/\kappa}\,p_2^{\sum_{c=1}^n z_c/\kappa}$ acts on the fiber $H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol{z}}$ as multiplication by itself and the section $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\bullet}(\boldsymbol{z};\boldsymbol{p};\kappa)$ of $EH_{\boldsymbol{\lambda}}$ is an entire function of p_2/p_1 equal to the identity map at $p_2 = 0$. Formulae (6.30)–(6.33), (6.67), (6.71) yield an integral expression for $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\bullet}(\boldsymbol{z};\boldsymbol{p};\kappa)$, see formulae (6.83), (6.84) below.

Recall the complement $L^{\circ}\subset\mathbb{C}^n$ of the union of hyperplanes (6.36),

$$z_a - z_b \in \kappa\mathbb{Z}_{\neq 0}, \qquad a,b = 1,\dots,n, \quad a\neq b.$$

For $l\in\mathbb{Z}_{\geqslant 0}$, let $\tilde{\mathcal{J}}^H_l$ be the section of the bundle $EH_{\boldsymbol{\lambda}}$ with values

$$\tilde{\mathcal{J}}^H_l(\boldsymbol{z};\kappa) : H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol{z}} \to H^*_T(\mathbb{CP}^{n-1};\mathbb{C})_{\boldsymbol{z}}, \tag{6.83}$$

$$\tilde{\mathcal{J}}^H_l(\boldsymbol{z};\kappa) : [f]_{\boldsymbol{z}} \mapsto \frac{1}{2\pi\sqrt{-1}}\int_{C_{\boldsymbol{z}}} f(t)\,\bigl[\bar{W}_{\boldsymbol{\lambda}}(t-l\kappa;\gamma;\boldsymbol{z})\bigr]_{\boldsymbol{z}}\prod_{a=1}^n\prod_{m=0}^l \frac{1}{t-z_a-m\kappa}\,dt,$$

where the integral is over a contour $C_{\boldsymbol z}$ encircling the points $z_1,\dots,z_n$ counterclockwise and separating them from the sets $z_a+\kappa\mathbb{Z}_{>0}$, $a=1,\dots,n$. It is assumed in (6.83) that $\boldsymbol z\in L^\circ$. By Proposition 6.20,

$$\operatorname{Stab}_{\boldsymbol\lambda}\Psi^\bullet(\boldsymbol z;\boldsymbol p;\kappa)\,=\,\sum_{l=0}^\infty \tilde{\mathcal J}^{\,H}_l(\boldsymbol z;\kappa)\,(p_2/p_1)^l\,. \tag{6.84}$$

Notice that $\tilde{\mathcal J}^{\,H}_0(\boldsymbol z;\kappa)$ is the identity map because the classes in $H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$ of a polynomial f and the polynomial

$$f_*(\gamma)\,=\,\frac{1}{2\pi\sqrt{-1}}\int\limits_{C_{\boldsymbol z}}\frac{f(t)\,\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z)}{(t-z_1)\dots(t-z_n)}\,dt \tag{6.85}$$

coincide, $[f_*]_{\boldsymbol z}=[f]_{\boldsymbol z}\in H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$.

Formulae (6.83), (6.84) are counterparts of formulae (5.64) for dynamical differential equations (5.54).

Recall the space $\mathcal O_{H_{\boldsymbol\lambda}}$ of sections of $H_{\boldsymbol\lambda}$ holomorphic in $\boldsymbol z$ for $\boldsymbol z\in L^\circ$ and not depending on $\boldsymbol p$, and the space $\mathscr S^{\mathcal O}_{H_{\boldsymbol\lambda}}$ of sections of $H_{\boldsymbol\lambda}$ that are solutions of quantum differential equations (6.76) holomorphic in $\boldsymbol z$ for $\boldsymbol z\in L^\circ$. By Proposition 6.22, the Levelt solution $\operatorname{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ$ defines an isomorphism

$$\mu^{\mathcal O}_{H_{\boldsymbol\lambda}}:\mathcal O_{H_{\boldsymbol\lambda}}\to\mathscr S^{\mathcal O}_{H_{\boldsymbol\lambda}},\qquad f\,\mapsto\,\operatorname{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ f\,. \tag{6.86}$$

of $\mathcal O_{H_{\boldsymbol\lambda}}$ and $\mathscr S^{\mathcal O}_{H_{\boldsymbol\lambda}}$. The section f is called the principal term of $\operatorname{Stab}_{\boldsymbol\lambda}\widehat\Psi^\circ f$. The principal term of the solution $\operatorname{Stab}_{\boldsymbol\lambda}\Psi^\circ_P(\boldsymbol z;\boldsymbol p;\kappa)$ corresponding to a Laurent polynomial $P(\acute\gamma;\acute{\boldsymbol z})$ is described below, see (6.89).

For a function $f(t)$ holomorphic in a neighbourhood of the points $z_1,\dots,z_n$, define the polynomial $f_*(\gamma)$ by the rule

$$f_*(\gamma)\,=\,\frac{1}{2\pi\sqrt{-1}}\int\limits_{C}\frac{f(t)\,\bar W_{\boldsymbol\lambda}(t;\gamma;\boldsymbol z)}{(t-z_1)\dots(t-z_n)}\,dt\,, \tag{6.87}$$

where a contour C encircles the points $z_1,\dots,z_n$ counterclockwise and $f(t)$ is holomorphic inside C, cf. (6.85). Define the class $[f]_{\boldsymbol z}\in H^*_T(\mathbb{CP}^{n-1};\mathbb C)_{\boldsymbol z}$ by the rule $[f]_{\boldsymbol z}=[f_*]_{\boldsymbol z}$.

Set

$$\widetilde C^\circ_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\,=\,\kappa^{\left(n\gamma-\sum_{c=1}^n z_c\right)/\kappa},\qquad \widetilde G^+_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\,=\,\kappa^{n-1}\prod_{a=1}^n\Gamma\bigl(1+(\gamma-z_a)/\kappa\bigr)\,, \tag{6.88}$$

cf. (6.41). By Proposition 6.23, the principal term $\Psi^\varnothing_P$ of the solution $\operatorname{Stab}_{\boldsymbol\lambda}\Psi^\circ_P$ corresponding to a Laurent polynomial $P(\acute\gamma;\acute{\boldsymbol z})$ is

$$\Psi^\varnothing_P(\boldsymbol z;\kappa)\,=\,\bigl[\acute P(\gamma;\boldsymbol z;\kappa)\,\widetilde C^\circ_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\,\widetilde G^+_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\bigr]_{\boldsymbol z}\,. \tag{6.89}$$

The right-hand side of the last formula equals the product of the equivariant Chern character $\bigl[\acute P(\gamma;\boldsymbol z;\kappa)\bigr]_{\boldsymbol z}$ of the class of the Laurent polynomial $P(\acute\gamma;\acute{\boldsymbol z})$ in $K_T(\mathbb{CP}^{n-1};\mathbb C)$, the exponential $\bigl[\widetilde C^\circ_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\bigr]_{\boldsymbol z}$ of the equivariant first Chern class $[\gamma]_{\boldsymbol z}$ of the tangent bundle of $\mathbb{CP}^{n-1}$, and is the equivariant Gamma-class $\bigl[\widetilde G^+_{\boldsymbol\lambda}(\gamma;\boldsymbol z;\kappa)\bigr]_{\boldsymbol z}$ of $\mathbb{CP}^{n-1}$.

Recall the map (6.42),

$$\text{Б}^{H}_{\boldsymbol{\lambda}} : K_T(\mathbb{CP}^{n-1};\mathbb{C}) \to \mathcal{O}_{H_{\boldsymbol{\lambda}}}\,,$$

that sends the class $[P]$ of the Laurent polynomial $P(\acute{\gamma};\acute{\boldsymbol{z}})$ in $K_T(\mathbb{CP}^{n-1};\mathbb{C})$ to the section of $H_{\boldsymbol{\lambda}}$ with values $\bigl[\acute{P}(\gamma;\boldsymbol{z};\kappa)\,\widetilde{C}^{\circ}_{\boldsymbol{\lambda}}(\gamma;\boldsymbol{z};\kappa)\,\widetilde{G}^{+}_{\boldsymbol{\lambda}}(\gamma;\boldsymbol{z};\kappa)\bigr]_{\boldsymbol{z}}$. By (6.89), the map $\text{Б}^{H}_{\boldsymbol{\lambda}}$ sends the class $[P]$ to the principal term of the solution $\mathrm{Stab}_{\boldsymbol{\lambda}}\Psi^{\circ}_{P}$ of the joint system of differential equations (6.76) and difference equations (6.77). Furthermore, by Theorem 6.24, the following diagram is commutative,

$$\begin{array}{ccc} K_T(\mathbb{CP}^{n-1};\mathbb{C}) & \xrightarrow{\ \text{Б}^{H}_{\boldsymbol{\lambda}}\ } & \mathcal{O}_{H_{\boldsymbol{\lambda}}} \\ & {\scriptstyle \mu^{K}_{H_{\boldsymbol{\lambda}}}}\searrow \quad \swarrow{\scriptstyle \mu^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}} & \\ & \mathscr{S}^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}} & \end{array} \tag{6.90}$$

where $\mu^{K}_{H_{\boldsymbol{\lambda}}}$ and $\mu^{\mathcal{O}}_{H_{\boldsymbol{\lambda}}}$ are the maps (6.79) and (6.86), respectively.

The topological-enumerative morphism for $\mathbb{CP}^{n-1}$ was studied in [CV]. To refer to formulae and statements in [CV], we will use the superscript ${}^{\mathrm{CV}}$. To compare formulae in this paper with their counterparts in [CV], one should make the following substitution,

$$[\gamma] = x\,, \qquad p_1 = q^{-1}, \qquad p_2 = 1\,, \qquad \kappa = -1\,.$$

Then the classes of polynomials $\mathrm{Stab}_{[1]}(\gamma;\boldsymbol{z}),\dots,\mathrm{Stab}_{[n]}(\gamma;\boldsymbol{z})$ in $H^{*}_{T}(\mathbb{CP}^{n-1};\mathbb{C})$ coincide with the classes $g_1,\dots,g_n$ given by $(4.4)^{\mathrm{CV}}$, formulae (6.75) for quantum multiplication agree with formulae $(5.13)^{\mathrm{CV}}$, $(5.14)^{\mathrm{CV}}$, and quantum differential equations (6.76) and $(5.19)^{\mathrm{CV}}$ match.

By the results of [CV], Conjecture 6.32 holds true for $\mathbb{CP}^{n-1}$.

Theorem 6.33. *The Levelt fundamental solution* $\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa)$ *of equivariant quantum differential equations* (6.76) *is the equivariant topological-enumerative morphism of* $\mathbb{CP}^{n-1}$ *restricted to the small equivariant quantum locus.*

Proof. The statement follows from formulae (6.82)–(6.84) and Theorem 6.4^{CV}. □

According to Definition 5.3^{CV}, the equivariant J-function for $\mathbb{CP}^{n-1}$ is a unique section $J(\boldsymbol{z};\boldsymbol{p};\kappa)$ of $H_{\boldsymbol{\lambda}}$ such that for any $f\in H^{*}(\mathbb{CP}^{n-1};\mathbb{C})$ considered as a section of $H_{\boldsymbol{\lambda}}$ not depending on $\boldsymbol{p}$,

$$\mathcal{E}_{\boldsymbol{z}}\bigl\langle\bigl(\mathrm{Stab}_{\boldsymbol{\lambda}}\widehat{\Psi}^{\circ}(\boldsymbol{z};\boldsymbol{p};\kappa)\bigr)\,[f]_{\boldsymbol{z}}\bigr\rangle \,=\, \mathcal{E}_{\boldsymbol{z}}\bigl\langle J(\boldsymbol{z};\boldsymbol{p};\kappa)\,[f]_{\boldsymbol{z}}\bigr\rangle\,. \tag{6.91}$$

Formulae (6.67), (6.82)–(6.84), and

$$\frac{1}{2\pi\sqrt{-1}}\int\limits_{C}\frac{\bar{W}_{\boldsymbol{\lambda}}(t;\gamma;\boldsymbol{z})}{(\gamma-z_1)\dots(\gamma-z_n)}\,d\gamma \,=\, 1$$

for a contour C encircling $z_1,\dots,z_n$ counterclockwise, yield the following expression for the equivariant J-function for $\mathbb{CP}^{n-1}$,

$$J(\boldsymbol{z};\boldsymbol{p};\kappa) \,=\, p_2^{\sum_{c=1}^n z_c/\kappa}(p_1/p_2)^{[\gamma]_{\boldsymbol{z}}/\kappa}\Bigg[\,\sum_{l=0}^{\infty}(p_2/p_1)^l\prod_{a=1}^n\prod_{m=1}^l\frac{1}{\gamma-z_a-m\kappa}\Bigg]_{\boldsymbol{z}}. \tag{6.92}$$

This expression matches the formula for the equivariant J-function for $\mathbb{CP}^{n-1}$ obtained in [Gi1], [LLY].

All the results of this section admit straightforward specialization to the nonequivariant case by setting $\boldsymbol{z}=0$.

Appendix A. Polynomial identities

Recall $\lambda=(\lambda_1,\dots,\lambda_N)$, $|\boldsymbol{\lambda}|=n$, the set $\mathcal{I}_{\boldsymbol{\lambda}}$, $I^{\min}_{\boldsymbol{\lambda}}\in\mathcal{I}_{\boldsymbol{\lambda}}$, and the permutation $\sigma_I\in S_n$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, such that $\sigma_I(I^{\min}_{\boldsymbol{\lambda}})=I$. Let $S_{\boldsymbol{\lambda}}=S_{\lambda_1}\times\ldots\times S_{\lambda_N}\subset S_n$ be the isotropy subgroup of $I^{\min}_{\boldsymbol{\lambda}}$. Each coset in $S_n/S_{\boldsymbol{\lambda}}$ contains exactly one permutation of the form σ_I, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$.

Consider the variables $\boldsymbol{x}=(x_1,\dots,x_n)$, $\boldsymbol{y}=(y_1,\dots,y_n)$. Define

$$V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{y}) \,=\, \prod_{a=1}^{\lambda^{(i)}}\prod_{b=\lambda^{(i)}+1}^{n}(x_a-y_b)\,. \tag{A.1}$$

Clearly, $V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{y})\in\mathbb{C}[\boldsymbol{x}]^{S_{\boldsymbol{\lambda}}}\otimes\mathbb{C}[\boldsymbol{y}]^{S_{\boldsymbol{\lambda}}}$. For $\sigma\in S_n$, denote $\boldsymbol{x}_\sigma=(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Lemma A.1. *We have* $V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{x}_\sigma)=0$ *unless* $\sigma\in S_{\boldsymbol{\lambda}}$, *and* $V_{\boldsymbol{\lambda}}(\boldsymbol{x}_{\sigma_I};\boldsymbol{x}_{\sigma_I})=(-1)^{|\sigma_I|}V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{x})$.

Proof. Straightforward. □

For any function $f(\boldsymbol{x})$, define

$$\mathrm{Sym}^{\boldsymbol{\lambda}}_{\boldsymbol{x}}\,f(\boldsymbol{x}) \,=\, \sum_{I\in\mathcal{I}_{\boldsymbol{\lambda}}} f(\boldsymbol{x}_{\sigma_I})\,.$$

Proposition A.2. *We have*

$$V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{z}) \,=\, \mathrm{Sym}^{\boldsymbol{\lambda}}_{\boldsymbol{y}}\bigg(\frac{V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{y})\,V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{z})}{V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{y})}\bigg)\,. \tag{A.2}$$

Proof. By formula (A.1), the expression $V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{y})\,V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{z})/V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{y})$ is invariant under permutations of $y_1,\dots,y_n$ by elements of the subgroup $S_{\boldsymbol{\lambda}}$. Thus the right-hand side of formula (A.2) is a symmetric polynomial in $y_1,\dots,y_n$ of degree zero in each of the variables $y_1,\dots,$ y_n, hence a constant. The constant can be found by evaluating this polynomial at $\boldsymbol{y}=\boldsymbol{x}$ or $\boldsymbol{y}=\boldsymbol{z}$. □

Proposition A.3. *For any polynomial* $f\in\mathbb{C}[\boldsymbol{x}]^{S_{\boldsymbol{\lambda}}}$, *we have*

$$f(\boldsymbol{x}) \,=\, \mathrm{Sym}^{\boldsymbol{\lambda}}_{\boldsymbol{y}}\bigg(\frac{V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{y})}{V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{y})}\;\mathrm{Sym}^{\boldsymbol{\lambda}}_{\boldsymbol{x}}\bigg(\frac{f(\boldsymbol{x})\,V_{\boldsymbol{\lambda}}(\boldsymbol{y};\boldsymbol{x})}{V_{\boldsymbol{\lambda}}(\boldsymbol{x};\boldsymbol{x})}\bigg)\bigg)\,. \tag{A.3}$$

Proof. Changing the order of symmetrizations in the right-hand side of formula (A.3) and applying formula (A.2) yields

$$\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol y}\biggl(\frac{V_{\boldsymbol\lambda}(\boldsymbol x;\boldsymbol y)}{V_{\boldsymbol\lambda}(\boldsymbol y;\boldsymbol y)}\,\biggl(\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol z}\biggl(\frac{f(\boldsymbol z)\,V_{\boldsymbol\lambda}(\boldsymbol y;\boldsymbol z)}{V_{\boldsymbol\lambda}(\boldsymbol z;\boldsymbol z)}\biggr)\biggr)\biggr|_{\boldsymbol z=\boldsymbol x}\biggr)\,=$$

$$=\,\biggl(\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol z}\biggl(\frac{f(\boldsymbol z)}{V_{\boldsymbol\lambda}(\boldsymbol z;\boldsymbol z)}\;\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol y}\biggl(\frac{V_{\boldsymbol\lambda}(\boldsymbol x;\boldsymbol y)\,V_{\boldsymbol\lambda}(\boldsymbol y;\boldsymbol z)}{V_{\boldsymbol\lambda}(\boldsymbol y;\boldsymbol y)}\biggr)\biggr)\biggr)\biggr|_{\boldsymbol z=\boldsymbol x}=$$

$$=\,\biggl(\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol z}\biggl(\frac{f(\boldsymbol z)\,V_{\boldsymbol\lambda}(\boldsymbol x;\boldsymbol z)}{V_{\boldsymbol\lambda}(\boldsymbol z;\boldsymbol z)}\biggr)\biggr)\biggr|_{\boldsymbol z=\boldsymbol x}=\,f(\boldsymbol x)\,.$$

□

The expression $\mathrm{Sym}^{\boldsymbol\lambda}_{\boldsymbol x}\bigl(f(\boldsymbol x)\,V_{\boldsymbol\lambda}(\boldsymbol y;\boldsymbol x)/V_{\boldsymbol\lambda}(\boldsymbol x;\boldsymbol x)\bigr)$ is a symmetric polynomial in $x_1,\dots,x_n$. Thus Lemma A.1 and Proposition A.3 show that the polynomials $V_{\boldsymbol\lambda}(\boldsymbol x;\boldsymbol y_{\sigma_I})$, $I\in\mathcal I_{\boldsymbol\lambda}$, give a basis of the space $\mathbb C[\boldsymbol x]^{S_{\boldsymbol\lambda}}$ as a free module over the ring of symmetric polynomials $\mathbb C[\boldsymbol x]^{S_n}$.

Proposition A.4. *We have*

$$\det\bigl(V_{\boldsymbol\lambda}(\boldsymbol x_{\sigma_I};\boldsymbol y_{\sigma_J})\bigr)_{I,J\in\mathcal I_{\boldsymbol\lambda}}\,=\,\prod_{a=1}^{n-1}\,\prod_{b=a+1}^{n}\bigl((x_a-x_b)\,(y_b-y_a)\bigr)^{d^{(2)}_{\boldsymbol\lambda}},\tag{A.4}$$

where

$$d^{(2)}_{\boldsymbol\lambda}\,=\,\frac{(n-2)!}{\lambda_1!\dots\lambda_N!}\,\sum_{i=1}^{N-1}\,\sum_{j=i+1}^{N}\lambda_i\lambda_j\,.$$

Proof. Consider the matrix $M(\boldsymbol x;\boldsymbol y)=\bigl(V_{\boldsymbol\lambda}(\boldsymbol x_{\sigma_I};\boldsymbol y_{\sigma_J})\bigr)_{I,J\in\mathcal I_{\boldsymbol\lambda}}$. Let $F(\boldsymbol x;\boldsymbol y)=\det M(\boldsymbol x;\boldsymbol y)$. Formula (A.2) reads $M(\boldsymbol x;\boldsymbol z)=M(\boldsymbol x;\boldsymbol y)\bigl(M(\boldsymbol y;\boldsymbol y)^{-1}M(\boldsymbol y;\boldsymbol z)$, thus

$$F(\boldsymbol x;\boldsymbol z)\,=\,\frac{F(\boldsymbol x;\boldsymbol y)\,F(\boldsymbol y;\boldsymbol z)}{F(\boldsymbol y;\boldsymbol y)}\;,\tag{A.5}$$

so that $F(\boldsymbol x;\boldsymbol y)=G(\boldsymbol x)\,\widetilde G(\boldsymbol y)$ for some functions $G,\widetilde G$. Also, $F(\boldsymbol x;\boldsymbol y)=F(\boldsymbol y;\boldsymbol x)$, since the matrix $M(\boldsymbol y;\boldsymbol x)$ is the transpose of $M(\boldsymbol x;\boldsymbol y)$. Hence the functions G and $\widetilde G$ are proportional and one can take $\widetilde G=G$. Finally, the matrix $M(\boldsymbol x;\boldsymbol x)$ is diagonal and

$$\bigl(G(\boldsymbol x)\bigr)^2\,=\,F(\boldsymbol x;\boldsymbol x)\,=\,\prod_{I\in\mathcal I_{\boldsymbol\lambda}}V_{\boldsymbol\lambda}(\boldsymbol x_{\sigma_I};\boldsymbol x_{\sigma_I})\,=\,\prod_{a=1}^{n}\,\prod_{\substack{b=1\\ b\neq a}}^{n}(x_a-x_b)^{d^{(2)}_{\boldsymbol\lambda}},$$

which implies formula (A.4). □

For $I\in\mathcal I_{\boldsymbol\lambda}$, set

$$V_I(\boldsymbol x)\,=\,V_{\boldsymbol\lambda}\bigl(x_1,\dots,x_n;\sigma_I(n)-1,\dots,\sigma_I(1)-1\bigr)\,.\tag{A.6}$$

We use the polynomials $V_I(\boldsymbol{x})$ in Section 4.6 to give a basis of the algebra $\mathcal{K}_{\boldsymbol{\lambda}}$. By formula (A.4),

$$\det\big(V_I(\boldsymbol{x}_{\sigma_J})\big)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} = \prod_{k=2}^{n-1} j^{(n-j)\,d^{(2)}_{\boldsymbol{\lambda}}} \prod_{a=1}^{n-1}\prod_{b=a+1}^{n}(x_b-x_a)^{d^{(2)}_{\boldsymbol{\lambda}}}. \tag{A.7}$$

Appendix B. Schubert polynomials

Consider the operators $\Delta^{\boldsymbol{x}}_1,\dots,\Delta^{\boldsymbol{x}}_{n-1}$ acting on functions of $x_1,\dots,x_n$:

$$\Delta^{\boldsymbol{x}}_a f(\boldsymbol{x}) = \frac{f(\boldsymbol{x})-f(x_1,\dots,x_{a+1},x_a,\dots,x_n)}{x_a-x_{a+1}}\,.$$

They satisfy the nil-Coxeter relations,

$$(\Delta^{\boldsymbol{x}}_a)^2=0\,,\qquad \Delta^{\boldsymbol{x}}_a\Delta^{\boldsymbol{x}}_b=\Delta^{\boldsymbol{x}}_b\Delta^{\boldsymbol{x}}_a\,,\quad |a-b|>1\,,\qquad \Delta^{\boldsymbol{x}}_a\Delta^{\boldsymbol{x}}_{a+1}\Delta^{\boldsymbol{x}}_a=\Delta^{\boldsymbol{x}}_{a+1}\Delta^{\boldsymbol{x}}_a\Delta^{\boldsymbol{x}}_{a+1}\,, \tag{B.1}$$

for any a,b.

Given a permutation $\sigma\in S_n$, define the operator $\Delta^{\boldsymbol{x}}_\sigma$ as follows. For $a=1,\dots,n-1$, let s_a be the transposition of a and $a+1$. If $\sigma=s_{a_1}\dots s_{a_k}$ is the reduced presentation, then $\Delta^{\boldsymbol{x}}_\sigma=\Delta^{\boldsymbol{x}}_{a_1}\dots\Delta^{\boldsymbol{x}}_{a_k}$. In particular, $\Delta^{\boldsymbol{x}}_{s_a}=\Delta_a$. The operators $\Delta^{\boldsymbol{x}}_\sigma$ are well-defined due to relations (B.1).

Lemma B.1. *For $\sigma,\tau\in S_n$, we have $\Delta^{\boldsymbol{x}}_\sigma\Delta^{\boldsymbol{x}}_\tau=\Delta^{\boldsymbol{x}}_{\sigma\tau}$ if $|\sigma\tau|=|\sigma|+|\tau|$, and $\Delta^{\boldsymbol{x}}_\sigma\Delta^{\boldsymbol{x}}_\tau=0$, otherwise.*

Proof. The statement follows from relations (B.1). □

Define the operators $\Delta^{\boldsymbol{y}}_\sigma$, $\sigma\in S_n$, acting on functions of $y_1,\dots,y_n$ similarly.

The A-type double Schubert polynomials $\mathfrak{S}_\sigma$, $\sigma\in S_n$, see [L, Chapter 2], are defined as follows. For the longest permutation σ_0, $\sigma_0(i)=n+1-i$, $i-1,\dots,n$, set

$$\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i}(x_i-y_j)\,,$$

For any $\sigma\in S_n$, set $\mathfrak{S}_\sigma(\boldsymbol{x};\boldsymbol{y})=\Delta^{\boldsymbol{x}}_{\sigma^{-1}\sigma_0}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})$.

Lemma B.2. *For any $\sigma\in S_n$, we have $\Delta^{\boldsymbol{x}}_\sigma\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})=(-1)^{|\sigma|}\,\Delta^{\boldsymbol{y}}_{\sigma_0\sigma^{-1}\sigma_0}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})$.*

Proof. The statement follow by induction on the length of σ from the equalities

$$\Delta^{\boldsymbol{x}}_{s_a}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y}) = \frac{\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})}{x_a-y_{n-a}} = -\,\Delta^{\boldsymbol{y}}_{s_{n-a-1}}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})$$

and $s_{n-a-1}=\sigma_0\,s_a\sigma_0$ for every $a=1,\dots,n-1$. □

Proposition B.3. *For any $\sigma\in S_n$, we have*

$$\mathfrak{S}_\sigma(\boldsymbol{x};\boldsymbol{y}) = (-1)^{|\sigma_0|-|\sigma|}\,\Delta^{\boldsymbol{y}}_{\sigma\sigma_0}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y}) = (-1)^{|\sigma|}\,\mathfrak{S}_{\sigma^{-1}}(\boldsymbol{y};\boldsymbol{x})\,.$$

Proof. The statement follows from the equality $\mathfrak{S}_{\sigma_0}(\boldsymbol{y};\boldsymbol{x}) = (-1)^{|\sigma_0|}\,\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})$ and Lemma B.2. □

Denote by $\sigma_{\boldsymbol{\lambda}}$ the longest element of the subgroup $S_{\boldsymbol{\lambda}} = S_{\lambda_1}\times\ldots\times S_{\lambda_N}\subset S_n$,

$$\sigma_{\boldsymbol{\lambda}}(a) = \lambda^{(i-1)} + \lambda^{(i)} + 1 - a\,, \qquad \lambda^{(i-1)} + 1 < a \leqslant \lambda^{(i)}, \quad i = 1,\ldots,N\,.$$

Lemma B.4. *We have*

$$\Delta^{\boldsymbol{x}}_{\sigma_{\boldsymbol{\lambda}}}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y}) \,=\, \prod_{a=1}^{\lambda^{(i)}}\,\prod_{b=\lambda^{(i)}+1}^{n}(x_a - y_{n-b+1})\,.$$

Proof. Straightforward. □

Recall the polynomials $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, $\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z})$, see Section 6.2.

Proposition B.5. *For any* $I\in\mathcal{I}_{\boldsymbol{\lambda}}$,

$$\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \mathfrak{S}_{\sigma_{\sigma_0(I)}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0})\,, \qquad \mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \mathfrak{S}_{\sigma_I}(\boldsymbol{\Gamma};\boldsymbol{z})\,. \tag{B.2}$$

Proof. By formula (6.12) and Lemmas 5.4,

$$\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, (-1)^{|\sigma_{\sigma_0(I)}|-|\sigma_I|}\,\Delta^{\boldsymbol{z}}_{\sigma_I\sigma^{-1}_{\sigma_0(I)}}V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0})\,,$$

where the operator $\Delta^{\boldsymbol{z}}_{\sigma}$ acts on functions of $\boldsymbol{z}$. By formula (6.4) and Lemmas B.4, B.2,

$$V_{\boldsymbol{\lambda}}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0}) \,=\, (-1)^{|\sigma_{\boldsymbol{\lambda}}|}\,\Delta^{\boldsymbol{z}}_{\sigma_0\,\sigma_{\boldsymbol{\lambda}}\sigma_0}\mathfrak{S}_{\sigma_0}(\boldsymbol{x};\boldsymbol{y})$$

because $\sigma^{-1}_{\boldsymbol{\lambda}} = \sigma_{\boldsymbol{\lambda}}$. Since $\sigma_{\sigma_0(I)} = \sigma_0\,\sigma_{\boldsymbol{\lambda}}$ and $|\sigma_I\,\sigma^{-1}_{\sigma_0(I)}| + |\sigma_{\sigma_0(I)}\,\sigma_0| = |\sigma_I\,\sigma_0|$,

$$\mathrm{Stab}^{\mathrm{op}}_I(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, (-1)^{|\sigma_0|-|\sigma|}\,\Delta^{\boldsymbol{z}}_{\sigma\sigma_0}\mathfrak{S}_{\sigma_0}(\boldsymbol{\Gamma};\boldsymbol{z}) \,=\, \mathfrak{S}_{\sigma_I}(\boldsymbol{\Gamma};\boldsymbol{z})\,.$$

by Lemma B.1 and Proposition B.3, that proves the second equality in (B.2). The first equality in (B.2) follows from the relation $\mathrm{Stab}_I(\boldsymbol{\Gamma};\boldsymbol{z}) = \mathrm{Stab}^{\mathrm{op}}_{\sigma_0(I)}(\boldsymbol{\Gamma};\boldsymbol{z}_{\sigma_0})$. □

Appendix C. Proofs of Lemmas 5.2 and 5.3

Proof of Lemma 5.2. The proof is by induction on N. To show that $W^{\circ}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{t};\boldsymbol{z}) = 1$, set $f_1 = 1$, and for $N\geqslant 2$, write

$$f_N(\boldsymbol{t}^{(1)},\ldots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \,=\, W^{\circ}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{t};\boldsymbol{z})\,,$$

indicating the dependence on N explicitly. By formula (5.1), the function f_N is a polynomial in $t^{(N-1)}_1,\ldots,t^{(N-1)}_{\lambda^{(N-1)}}$ of degree zero in each of these variables, hence a constant. Evaluating

$f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z})$ at $\boldsymbol{t}^{(N-1)}=(z_1,\dots,z_{\lambda^{(N-1)}})$ yields

$$f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; f_{N-1}(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};z_1,\dots,z_{\lambda^{(N-1)}})\,.$$

Thus $f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z})=1$ by the induction assumption.

The proof of formula (5.2) is similar. Set $f_1(\boldsymbol{z})=1$, and for $N\geqslant 2$, write

$$f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; W^{\circ}_{\sigma_0(I^{\min}_{\boldsymbol{\lambda}})}(\boldsymbol{t};\boldsymbol{z})\,,$$

Define polynomials $g_1,\dots,g_{N-1}$,

$$g_i(\boldsymbol{t}^{(i)};\boldsymbol{z}) \;=\; \prod_{a=1}^{\lambda^{(i)}}\;\prod_{b=\lambda^{(i)}+1}^{\lambda^{(i+1)}}(t^{(i)}_a-z_{n-b+1})\,.$$

By formula (5.1), the function f_N is a polynomial in $t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}$ of degree λ_N in each of these variables, and f_N is divisible by $g_{N-1}(\boldsymbol{t}^{(N-1)};\boldsymbol{z})$. Hence,

$$f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; r_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};\boldsymbol{z})\,g_{N-1}(\boldsymbol{t}^{(N-1)};\boldsymbol{z})$$

for some polynomial r_N. Evaluating both sides at $\boldsymbol{t}^{(N-1)}=(z_{\lambda_N+1},\dots,z_n)$

$$r_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};\boldsymbol{z}) \;=\; f_{N-1}(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};z_{\lambda_N+1},\dots,z_n)\,.$$

Thus $f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z})=g_1(\boldsymbol{t}^{(1)};\boldsymbol{z})\dots g_{N-1}(\boldsymbol{t}^{(N-1)};\boldsymbol{z})$, which proves formula (5.2). □

Proof of Lemma 5.3. The proof is by induction on N, similarly to that of Lemma 5.2. Set $f_1=1$, and for $N\geqslant 2$, write

$$f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; W^{\circ}_{I^{\min}_{\boldsymbol{\lambda}}}(\boldsymbol{t};\boldsymbol{z})\,,\qquad f^{(i)}_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; W^{\circ}_{s_{\lambda^{(i)},\,\lambda^{(i)}+1}(I^{\min}_{\boldsymbol{\lambda}})}(\boldsymbol{t};\boldsymbol{z})\,,$$

indicating the dependence on N explicitly. By Lemma 5.2, $f_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z})=1$.

By formula (5.1), the function $f^{(N-1)}_N$ is a symmetric linear function in $t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}$ vanishing at $\boldsymbol{t}^{(N-1)}=(z_1,\dots,z_{\lambda^{(N-1)}})$ Hence,

$$f^{(N-1)}_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \;=\; r_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};\boldsymbol{z})\left(\sum_{j=1}^{\lambda^{(N-1)}}(t^{(N-1)}_j-z_j)\right)$$

for some polynomial r_N. Evaluating both sides at $\boldsymbol{t}^{(N-1)}=(z_1,\dots,z_{\lambda^{(N-1)}-1},0)$ yields

$$r_N(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};\boldsymbol{z}) \;=\; f_{N-1}(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};z_1,\dots,z_{\lambda^{(N-1)}-1},0) \;=\; 1$$

by Lemma 5.2, which proves Lemma 5.3 for $i=N-1$.

For $i\neq N-1$, formula (5.1) implies that the function $f^{(i)}_N$ is a polynomial in $t^{(N-1)}_1,\dots,$ $t^{(N-1)}_{\lambda^{(N-1)}}$ of degree zero in each of these variables, hence a constant. Evaluating $f^{(i)}_N(\boldsymbol{t}^{(1)},\dots,$

$\boldsymbol{t}^{(N-1)};\boldsymbol{z})$ at $\boldsymbol{t}^{(N-1)}=(z_1,\dots,z_{\lambda^{(N-1)}})$ yields

$$f_N^{(i)}(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-1)};\boldsymbol{z}) \,=\, f_{N-1}^{(i)}(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(N-2)};z_1,\dots,z_{\lambda^{(N-1)}})\,,=\sum_{j=1}^{\lambda^{(i)}}(t_j^{(i)}-z_j)\,,$$

where the second equality holds by the induction assumption. Lemma 5.3 is proved. □

Appendix D. Stirling's formula

The next two lemmas are used in the proof of Lemma 5.10. The first lemma is well known.

Lemma D.1. *For* $|\arg(h/\kappa)|<\pi$ *and any* $\alpha\in\mathbb{C}$, *we have*

$$\lim_{h\to\infty}\frac{\Gamma(\alpha+h/\kappa)}{(h/\kappa)^\alpha\,\Gamma(h/\kappa)}\,=\,1$$

locally uniformly in α.

Lemma D.2. *For any* $\alpha,r\in\mathbb{C}$, $l\in\mathbb{Z}$, *we have*

$$\lim_{h\to\infty}\frac{r^l\,\Gamma(\alpha+l+h/\kappa)}{(h/\kappa)^l\,\Gamma(\alpha+h/\kappa)\,\Gamma(l+1)}\,=\,\frac{r^l}{\Gamma(l+1)}$$

uniformly in l *and locally uniformly in* α,r.

Proof. Let $y=h/\kappa$ and $f(x,y,\alpha)=(1-xy^{-1})^{-\alpha-y}$. For $r<|y|$ by Cauchy's formula,

$$\frac{r^l\,\Gamma(\alpha+l+y)}{y^l\,\Gamma(\alpha+y)\,\Gamma(l+1)}\,=\,\frac{r^l}{2\pi\sqrt{-1}}\int\limits_{|x|=r}\frac{f(x,y,\alpha)}{x^{l+1}}\,dx\,.$$

Since $\lim\limits_{y\to\infty}f(x,y,\alpha)=e^x$ locally uniformly in x,α, and

$$\frac{r^l}{\Gamma(l+1)}\,=\,\frac{r^l}{2\pi\sqrt{-1}}\int\limits_{|x|=r}\frac{e^x}{x^{l+1}}\,dx\,,$$

Lemma D.2 follows. □

Appendix E. Polynomiality of solutions

Consider dynamical differential equations (2.7) for nonpositive integer values of h/κ. Recall the $\mathrm{End}\big((\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}\big)$-valued Levelt fundamental solution $\widehat\Psi(\boldsymbol{z};h;\boldsymbol{q};\kappa)$, see Theorem 4.21, Proposition 4.24, and formula (4.47) for the dynamical Hamiltonians $X_1,\dots,X_n$ at $\boldsymbol{q}=\emptyset$.

Say that a rational function $f(\boldsymbol{z})$ is over integers if $f(\boldsymbol{z})$ is a ratio of polynomials in $\boldsymbol{z}$ with integer coefficients.

Theorem E.1. *Let* $m\in\mathbb{Z}_{\geqslant0}$ *and* $h=-m\kappa$. *Then the Levelt fundamental solution has the form*

$$\widehat\Psi(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)\,=\,P(\boldsymbol{z}/\kappa;m;\boldsymbol{q})\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-q_j/q_i)^{-m\lambda_i}\prod_{i=1}^{N}q_i^{X_i(\boldsymbol{z}/\kappa;-m;\emptyset)}\,,\tag{E.1}$$

where $P(\boldsymbol{z};m;\boldsymbol{q})$ *is a polynomial in the variables* $q_2/q_1,\dots,q_n/q_{n-1}$ *with coefficients being rational functions of* $\boldsymbol{z}$ *over integers. The degree of* $P(\boldsymbol{z};m;\boldsymbol{q})$ *in the variable* q_{i+1}/q_i *is at most* $m(n-i)\lambda^{(i)}$. *The funciton* $P(\boldsymbol{z};m;\boldsymbol{q})$ *is holomorphic in* $\boldsymbol{z}$ *if* $z_a-z_b\notin\mathbb{Z}_{\neq 0}$ *for all* $1\leqslant a<b\leqslant n$. *The singularities of* $P(\boldsymbol{z};m;\boldsymbol{q})$ *at the hyperplanes* $z_a-z_b\in\mathbb{Z}_{\neq 0}$ *are simple poles. For any* $\boldsymbol{z}$ *such that* $z_a-z_b\notin\kappa\mathbb{Z}_{\neq 0}$ *for all* $1\leqslant a<b\leqslant n$, *the columns of* $\widehat{\Psi}(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)$ *form a basis of the space of* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$*-valued solutions of dynamical differential equations* (2.7).

Proof. By formulae (4.48), (4.60), the function $\widehat{\Psi}(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)$ has the form (E.1) with $P(\boldsymbol{z};m;\boldsymbol{q})$ being a power series in $q_2/q_1,\dots,q_n/q_{n-1}$. The first step of the proof is to show that the power series terminate so that $P(\boldsymbol{z};m;\boldsymbol{q})$ is a polynomial in $q_2/q_1,\dots,q_n/q_{n-1}$.

Denote by $P_{I,J}(\boldsymbol{z};m;\boldsymbol{q})$ the entries of $P(\boldsymbol{z};m;\boldsymbol{q})$ in the standard basis $\{v_I\,,\ I\in\mathcal{I}_{\boldsymbol{\lambda}}\}$ of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$. By formula (4.60),

$$P_{I,J}(\boldsymbol{z};m;\boldsymbol{q}) \,=\, \sum_{\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}} \mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m)\prod_{i=1}^{N-1}(q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}, \tag{E.2}$$

where the coefficients $\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m)$ are given by formula (E.4) below.

Denote $[x,j]=x(x-1)\dots(x-j+1)$. Recall

$$\mathcal{Z}_{\boldsymbol{\lambda}} \,=\, \{\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}\ |\ l_a^{(i)}\geqslant l_a^{(i+1)},\ \ i=1,\dots,N-1\,,\ \ a=1,\dots,\lambda^{(i)}\}\,,$$

see (4.31). For $\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}$, set $F_{\boldsymbol{l}}(\boldsymbol{z};m)=0$ if $\boldsymbol{l}\notin\mathcal{Z}_{\boldsymbol{\lambda}}$, and

$$\begin{aligned} F_{\boldsymbol{l}}(\boldsymbol{z};m) \,=\, \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\biggl(&\frac{(-1)^{l_a^{(i)}-l_a^{(i+1)}}\,[m\,,l_a^{(i)}-l_a^{(i+1)}]}{(l_a^{(i)}-l_a^{(i+1)})!}\,\times\\ &\times\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\frac{[z_b-z_a+l_a^{(i)}-l_b^{(i)},m+1]}{[z_b-z_a\,,m+1]}\prod_{\substack{c=1\\ c\neq a}}^{\lambda^{(i+1)}}\frac{[z_c-z_a\,,m+1]}{[z_c-z_a+l_a^{(i)}-l_c^{(i+1)},m+1]}\biggr) \end{aligned} \tag{E.3}$$

if $\boldsymbol{l}\in\mathcal{Z}_{\boldsymbol{\lambda}}$. Here $\lambda^{(N)}=n$ and $l_a^{(N)}=0$ for all $a=1,\dots,n$.

Notice that the factor $[m\,,l_a^{(i)}-l_a^{(i+1)}]$ in formula (E.3) equals zero if $l_a^{(i)}-l_a^{(i+1)}>m$. Therefore, $F_{\boldsymbol{l}}(\boldsymbol{z};m)=0$ unless $l_a^{(i)}-l_a^{(i+1)}\leqslant m$ for all a,i, and in particular, $F_{\boldsymbol{l}}(\boldsymbol{z};m)=0$ unless $l_a^{(i)}\leqslant m(n-i)$ for all a,i.

Denote

$$\mathcal{Z}_{\boldsymbol{\lambda},m} \,=\, \{\boldsymbol{l}\in\mathbb{Z}_{\geqslant 0}^{\lambda^{\{1\}}}\ |\ 0\leqslant l_a^{(i)}-l_a^{(i+1)}\leqslant m\,,\ \ i=1,\dots,N-1\,,\ \ a=1,\dots,\lambda^{(i)}\}\,,$$

By formulae (4.59), (4.57), (E.3),

$$\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m) \,=\, \sum_{K\in\mathcal{I}_{\boldsymbol{\lambda}}}\frac{F_{\boldsymbol{l}}(\boldsymbol{z}_{\sigma_K};m)\,W_I(\Sigma_K-\boldsymbol{l};\boldsymbol{z};-m)\,\check{W}_J(\Sigma_K;\boldsymbol{z};-m)}{R_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_K})\,Q_{\boldsymbol{\lambda}}(\boldsymbol{z}_{\sigma_K};-m)\,c_{\boldsymbol{\lambda}}(\Sigma_K-\boldsymbol{l};-m)\,c_{\boldsymbol{\lambda}}(\Sigma_K;-m)} \tag{E.4}$$

if $\boldsymbol{l}\in\mathcal{Z}_{\boldsymbol{\lambda},m}$, and $\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m)=0$ if $\boldsymbol{l}\notin\mathcal{Z}_{\boldsymbol{\lambda},m}$. Therefore,

$$P_{I,J}(\boldsymbol{z};m;\boldsymbol{q}) \,=\, \sum_{\boldsymbol{l}\in\mathcal{Z}_{\boldsymbol{\lambda},m}} \mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m)\,\prod_{i=1}^{N-1}\,(q_{i+1}/q_i)^{\sum_{a=1}^{\lambda^{(i)}} l_a^{(i)}}, \tag{E.5}$$

so that $P(\boldsymbol{z};m;\boldsymbol{q})$ is a polynomial in $q_2/q_1,\dots,q_n/q_{n-1}$ of degree at most $m\,(n-i)\,\lambda^{(i)}$ in the variable q_{i+1}/q_i, $i=1,\dots,N$. Moreover, by formulae (E.3), (4.2), (4.5), (4.8), all the coefficients $\mathcal{J}_{I,J,\boldsymbol{l}}(\boldsymbol{z};-m)$ are rational functions of $\boldsymbol{z}$ over integers. The regularity properties of $P(\boldsymbol{z};m;\boldsymbol{q})$ follow from Theorem 4.21, item (ii).

The columns of $\widehat{\Psi}(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)$ form a basis of the space of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$-valued solutions of dynamical differential equations (2.7) by Theorem 4.21, see formula (4.49). □

Theorem E.1 allows us to describe the monodromy of the system of dynamical differential equations (2.7) if h/κ is a nonpositive integer.

Corollary E.2. *Let $h\in\kappa\mathbb{Z}_{\leqslant 0}$. Then the monodromy of the system of dynamical differential equations (2.7) is abelian. The monodromy representation is generated by the matrices $e^{2\pi\sqrt{-1}\,X_1(\boldsymbol{z}/\kappa;h/\kappa;\mathbb{0})},\dots,e^{2\pi\sqrt{-1}\,X_N(\boldsymbol{z}/\kappa;h/\kappa;\mathbb{0})}$. For any $\boldsymbol{z}$ such that $z_a-z_b\notin\kappa\mathbb{Z}$ for all $1\leqslant a<b\leqslant n$, the monodromy representation is the direct sum of one-dimensional representations generated by the solutions $\Psi_I(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)$, $I\in\mathcal{I}_{\boldsymbol{\lambda}}$, defined by (4.20).*

Notice that for integer values of h/κ, the factors $(1-q_j/q_i)^{\lambda_i h/\kappa}$ in formula (4.21) are rational functions and the only nontrivial monodromy of the solutions $\Psi_J(\boldsymbol{z};-m\kappa;\boldsymbol{q};\kappa)$ comes from the factors $q_i^{\sum_{a\in J_i} z_a/\kappa}$ in formula (4.22).

The results of this appendix were motivated by the corresponding statements in [OP] on the quantum differential equation of the Hilbert scheme of points in the plane.

References

[AB] D.V. Anosov, A. A. Bolibruch, *The Riemann-Hilbert problem*, Aspects of Mathematics, E22, Friedr. Vieweg & Sohn, Braunschweig, 1994

[BCK] A. Bertram, I. Ciocan-Fontanine, B. Kim, *Gromov–Witten invariants for abelian and nonabelian quotients*, J. Alg. Geom. **17** (2008) 275–294

[BMO] A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. **227** (2011), no. 1, 421–458

[BM] A. Buch. L. Mihalcea, *Curve neighborhoods of Schubert varieties*, J. Diff. Geom. **99** (2015), no. 2, 255–283

[C] G. Cotti, *Cyclic stratum of Frobenius manifolds, Borel-Laplace (α,β)-multitransforms, and integral representations of solutions of Quantum Differential Equations*, Preprint (2020)arXiv:2005.08262, 1–97

[CDG] G. Cotti, B. Dubrovin, D. Guzzetti, *Helix structures in quantum cohomology of Fano varieties*, arXiv:1811.09235, 1–149

[CV] G. Cotti, A.Varchenko, *Equivariant quantum differential equation and qKZ equations for a projective space: Stokes bases as exceptional collections, Stokes matrices as Gram matrices, and B-Theorem*, "Integrability, Quantization and Geometry — Dubrovin's memorial volume", edited by I. Krichever, S. Novikov, O. Ogievetsky, and

S. Shlosman, In Proceedings of Symposia in Pure Mathematics, vol. 103.1, AMS, 2021

[D1] B. A. Dubrovin, *Geometry and analytic theory of Frobenius manifolds*, In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), 315–326

[D2] B. A. Dubrovin, *Quantum Cohomology and Isomonodromic Deformation*. Lecture at Recent Progress in the Theory of Painlevé Equations: Algebraic, asymptotic and topological aspects, Strasbourg, November 2013

[FRW] L. Feher, R. Rimányi, A. Weber, *Characteristic classes of orbit stratifications, the axiomatic approach*, "Schubert calculus and its applications in combinatorics and representation theory", Springer Proc. Math. Stat., vol. 332, Springer, Singapore, 2020 pp. 223–249

[FR] I. Frenkel, N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60

[Gi1] A. Givental, *Equivariant Gromov–Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663

[Gi2] A. Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, AMS Transl. Ser. 2 **180** (1997), 103–115

[GKi] A. Givental, B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. **168** (1997), no. 3, 609–641

[GK] V. Gorbounov, C. Korff, *Quantum Integrability and Generalised Quantum Schubert Calculus*, Adv. in Math. **313** (2017), 282–356

[GKS] V. Gorbounov, C. Korff, C. Stroppel, *Yang-Baxter algebras, convolution algebras, and Grassmannians*, Russian Math. Surveys **75** (2020), no. 5, 791–842

[GRTV] V. Gorbounov, R. Rimányi, V. Tarasov, A. Varchenko, *Quantum cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra*, J. Geom. Phys. **74** (2013), 56–86, DOI: 10.1016/j.geomphys.2013.07.006

[Gu] D. Guzzetti, *Stokes matrices and monodromy of the quantum cohomology of projective spaces* Comm. Math. Phys., **207** (2) : 341–383, 1999

[L] A. Lascoux, *Polynomials*, 2013,
http://www-igm.univ-mlv.fr/~al/ARTICLES/CoursYGKM.pdf

[LLY] B. H. Lian, K. Liu, S.T. Yau, *Mirror principle I*, Surveys Differ. Geom., **5** (1999) 405–454

[M] L. Mihalcea, *On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms*, Duke Math. J. **140** (2007), no. 2, 321–350

[MO] D. Maulik, A. Okounkov, *Quantum Groups and Quantum Cohomology*, Astérisque **408** (2019), 1–209

[MR] R.J. Marsh, K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, Adv. Math. **366**, 107027, https://doi.org/10.1016/j.aim.2020.107027

[MTV] E. Mukhin, V. Tarasov, A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. (2006), no. 8, P08002, 1–44

[O] T. Ohmoto, *Equivariant Chern classes of singular algebraic varieties with group actions*, Math. Proc. Cambridge Phil. Soc. **140** (2006), 115–134

[OP] A. Okounkov, R. Pandharipande, *The quantum differential equation of the Hilbert scheme of points in the plane*, Transform. Groups **15** (2010), no. 4, 965–982

[R] R. Rimányi, *Thom polynomials, symmetries and incidences of singularities*, Inv. Math. **143** (2001), 499–521

[RTV1] R. Rimányi, V. Tarasov, A.Varchenko, *Partial flag varieties, stable envelopes and weight functions*, Quantum Topology **6** (2015), no. 2, 333–364, DOI: 10.4171/QT/64

[RTV2] R. Rimányi, V. Tarasov, A.Varchenko, *Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety*, Journal of Geometry and Physics, **94** (2015), 81–119, DOI: 10.1016/j.geomphys.2015.04.002

[RV] R. Rimányi, A.Varchenko, *Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae*, Schubert varieties, equivariant cohomology and characteristic classes — IMPANGA 15, EMS Ser. Congr. Rep., vol. 13, EMS, Zurich, 2018, pp. 225–235

[TV1] V. Tarasov, A.Varchenko, *Jackson Integral Representations for Solutions to the Quantized Knizhnik-Zamolodchikov Equation*, St. Petersb. Math. J. **6** (1994), no. 2, 275–313

[TV2] V. Tarasov, A.Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, Invent. Math. **128** (1997), 501–588

[TV3] V. Tarasov and A.Varchenko, *Dynamical differential equations compatible with rational qKZ equations*, Lett. Math. Phys. **71** (2005), no. 2, 101–108

[TV4] V. Tarasov and A.Varchenko, *Combinatorial formulae for nested Bethe vectors*, SIGMA Symmetry Integrability Geom. Methods Appl. **9** (2013), Paper 048, 1–28

[TV5] V. Tarasov and A.Varchenko, *Hypergeometric solutions of the quantum differential equation of the cotangent bundle of a partial flag variety*, Cent. Eur. J. Math. **12** (2014), no. 5, 694–710

[TV6] V. Tarasov and A.Varchenko, *q-Hypergeometric solutions of quantum differential equations, quantum Pieri rules, and Gamma theorem*, J. Geom. Phys. **142** (2019), 179–212

[TV7] V. Tarasov, A.Varchenko, *Equivariant quantum differential equation, Stokes bases, and K-theory for a projective space*, European J. Math. **7** (2021), no. 2, 706–728

[W] W.Wasow, *Asymptotic expansions for ordinary differential equations*, Pure and Applied Mathematics, Vol. 14, Interscience Publishers, 1965