Higher-order Darboux transformations for two-dimensional Dirac systems with diagonal matrix potential

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Abstract. We construct the explicit form of higher-order Darboux transformations for the two-dimensional Dirac equation with diagonal matrix potential. The matrix potential entries can depend arbitrarily on the two variables. Our construction is based on results for coupled Korteweg-de Vries equations [27].

Keywords: Two-dimensional Dirac equation, Darboux transformation, diagonal matrix potential

1. Introduction

Bäcklund and Darboux transformations are methods for generating solutions to a variety of ordinary and partial differential equations [21] by relating their solutions through differential operators. Bäcklund transformations were originally introduced in the context of the sine-Gordon equation [2], and in the meantime they have been constructed for a vast amount of equations and models. Recent applications include the connection between KdV and mKdV equations [10], the Boussinesq equation [20], systems of Burgers equations in higher dimension [28], generalized KdV hierarchies [7], among many others. On the other hand, Darboux transformations interrelate solutions of equations that have the same form, and in addition they interrelate certain parameters that enter in those equations. As an example we consider stationary Schrödinger equations, where the Darboux transformation simultaneously provides interrelations between the respective solutions and potentials. In the Schrödinger context, the Darboux transformation turned out to be the mathematical core of the supersymmetric quantum mechanics (SUSY) formalism [3] [9]. This formalism has generated a vast amount of applications, recent examples of which include spectral design in systems featuring certain hyperbolic potentials [8], the construction of non-hermitian Hamiltonians [29], models with time-dependent boundary conditions [5], just to name a few. While the Darboux transformation in its initial form [6] [16] [17] was applicable to linear equations of second order, it has been generalized to many different scenarios. Besides the Schrödinger case, Darboux transformations were established for many models governed by linear and nonlinear equations, see [11] [15] for a comprehensive overview. Among these, one of the most important equations within relativistic quantum theory is the Dirac equation. Darboux transformations were first constructed for the latter equation in the one-dimensional case as a conjugate mapping to the Schrödinger scenario [18]. An analogous derivation can be found in the two-dimensional situation [19] [25], such that Darboux transformations for the Dirac and Schrödinger equations can be seen as equivalent, up to a decoupling and recoupling procedure. While the aforementioned Darboux transformation applies to the conventional form of the Schrödinger equation, a more general version can be constructed for systems with quadratical energy dependence. This Darboux transformation can be derived from results on coupled Korteweg-de Vries equations [27]. It consists of two different algorithms, the first-order case of which was introduced in [14]. Generalizations to the higher-order context were found and applied in [26]. A representation for the two algorithms and of a mixed version can be given in terms of generalized Wronskians [24], containing the Darboux transformation for the conventional Schrödinger equation as a special case. Similar to the conventional case, the Darboux transformations for systems with quadratical energy dependence in the potential can be adapted to the Dirac scenario by decoupling and recoupling, as shown in the aforementioned reference. A further extension of the latter Darboux transformation can be achieved by generalizing it to the two-dimensional Dirac equation. Based on the results in [27], such an extension was found for first-order transformations [23]. The purpose of this work is to construct the higher-order version of Darboux transformations for the Dirac equation in two dimensions. It should be mentioned that this two-dimensional scenario has been studied in many recent applications, for example [4] [12] [13] [22], among many more. However, in the aforementioned and related research the restriction is imposed on the potential matrix to depend on a single variable only. In the present work we drop this restriction, allowing the diagonal potential to depend arbitrarily on the two variables. As a consequence, our higher-order Darboux transformations can generate solvable Dirac systems that cannot be accessed through known methods. For the sake of completeness we review the one-dimensional situation in section 2. Our decoupling procedure for the two-dimensional Dirac equation is shown in section 3, while section 4 is devoted to the construction of our Darboux transformations. Afterwards we present two simple examples in section 5.

2. Preliminaries

We will now present a brief summary of results on higher-order Darboux transformations in the one-dimensional case [27]. For a real-valued constant λ, we define a differential operator D_λ as

$$D_\lambda = \frac{d}{dx} + \lambda. \tag{1}$$

Next, we implement this operator in a definition of generalized Wronskians. In this generalization, the derivative is replaced by our operator (1). For sufficiently smooth functions $u_1, ..., u_n$ and constants $\lambda_1, ..., \lambda_n$ we define

$$\mathcal{W}_{u_1,...,u_n}(x) = \det \begin{pmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ D_{\lambda_1} u_1(x) & D_{\lambda_2} u_2(x) & \cdots & D_{\lambda_n} u_n(x) \\ D^2_{\lambda_1} u_1(x) & D^2_{\lambda_2} u_2(x) & \cdots & D^2_{\lambda_n} u_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}_{\lambda_1} u_1(x) & D^{n-1}_{\lambda_2} u_2(x) & \cdots & D^{n-1}_{\lambda_n} u_n(x) \end{pmatrix}. \tag{2}$$

Now we introduce functions $\chi_1, ..., \chi_n$ that are obtained as follows

$$\chi_j(x) = u_j'(x) - f(x)\; u_j(x) + \lambda_j\; u_j(x), \quad j = 1, ..., n. \tag{3}$$

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where f is a function that enters in the equation that we will apply our Darboux transformations to. More precisely, the pair of equations to be connected through Darboux transformations read

$$\psi''(x) - f(x)\ \psi'(x) + \left[-\lambda^2 + \lambda\ f(x) + e(x)\right] \psi(x) \quad = \quad 0 \tag{4}$$

$$\hat{\psi}''(x) - \hat{f}(x)\ \hat{\psi}'(x) + \left[-\lambda^2 + \lambda\ \hat{f}(x) + \hat{e}(x)\right] \hat{\psi}(x) \quad = \quad 0, \tag{5}$$

where all involved functions are assumed to be sufficiently smooth. The first n-th order Darboux transformation, applied to a solution ψ of (4), is given by

$$\hat{\psi}(x) \quad = \quad \frac{\mathcal{W}_{u_1,\ldots,u_n,\psi}(x)}{\mathcal{W}_{u_1,\ldots,u_n}(x)}. \tag{6}$$

The function $\hat{\psi}$ is a solution of the transformed equation (5) if the following interrelations hold

$$\hat{e}(x) \quad = \quad e(x) + \frac{d^2}{dx^2}\ \log\left[\mathcal{W}_{u_1,\ldots,u_n}(x)\right] \tag{7}$$

$$\hat{f}(x) \quad = \quad f(x) + \frac{d}{dx}\ \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(x)}{\mathcal{W}_{u_1,\ldots,u_n}(x)}\right]. \tag{8}$$

A second n-th order Darboux transformation can be constructed from the first one through the formulas

$$\hat{\psi}_{\mathcal{T}}(x) = \exp\left[\int^{-x} \hat{f}(t)\ dt\right] \hat{\psi}(-x), \qquad \hat{f}_{\mathcal{T}}(x) = f(-x), \qquad \hat{e}_{\mathcal{T}}(x) = e(-x) - f'_{\mathcal{T}}(x). \tag{9}$$

Here, the function $\hat{\psi}_{\mathcal{T}}$ is a solution of (5), where $\hat{e}$ and $\hat{f}$ must be replaced by $\hat{e}_{\mathcal{T}}$ and $\hat{f}_{\mathcal{T}}$, respectively. This completes our review of the Darboux transformations for the one-dimensional case. In two dimensions, our initial equation has the form

$$\frac{\partial^2\phi_1(x,z)}{\partial x^2} - \frac{\partial^2\phi_1(x,z)}{\partial z^2} - f(x,z)\ \frac{\partial\phi_1(x,z)}{\partial x} + f(x,z)\ \frac{\partial\phi_1(x,z)}{\partial z} + e(x,z)\ \phi_1(x,z) \ = \ 0. \tag{10}$$

First-order Darboux transformations were constructed for this equation [27]. In the following section we will show that the two-dimensional Dirac equation with diagonal matrix potential can be converted into the form (10). As such, higher-order Darboux transformations can be constructed for the Dirac equation.

3. The two-dimensional Dirac equation

The purpose of the Darboux transformations that we will construct here, is to provide a mapping between the solutions of two different Dirac equations. These two partner equations can be written in the form

$$\left[\sigma_x\ p_x + \sigma_y\ p_y + V(x,y)\right] \Phi(x,y) \quad = \quad 0 \tag{11}$$

$$\left[\sigma_x\ p_x + \sigma_y\ p_y + \hat{V}(x,y)\right] \hat{\Phi}(x,y) \quad = \quad 0, \tag{12}$$

where σ_x, σ_y stand for the Pauli matrices, p_x, p_y denote the momentum operators, $V = \text{diag}(V_{11}, V_{22})$, $\hat{V} = \text{diag}(\hat{V}_{11}, \hat{V}_{22})$, are diagonal 2×2 matrices that represent the potentials.

Furthermore, the two-component solutions are given by Φ and $\hat{\Phi}$, respectively. Starting out with the initial equation (11), in the first step we introduce the solution components by

$$\Phi(x,y) = [\phi_1(x,y), \phi_2(x,y)]^T . \tag{13}$$

Upon substitution of this setting into our Dirac equation along with the above definitions, we obtain the following system

$$-i\, \frac{\partial \phi_1(x,y)}{\partial x} + \frac{\partial \phi_1(x,y)}{\partial y} + V_{22}(x,y)\ \phi_2(x,y) = 0 \tag{14}$$

$$-i\, \frac{\partial \phi_2(x,y)}{\partial x} - \frac{\partial \phi_2(x,y)}{\partial y} + V_{11}(x,y)\ \phi_1(x,y) = 0. \tag{15}$$

Without restriction we will now make the standing assumption that $V_{11} \neq 0,\ V_{22} \neq 0$. We can solve the first equation (14) of the system with respect to ϕ_2. This gives

$$\phi_2(x,y) = \frac{1}{V_{22}(x,y)} \left[i\, \frac{\partial \phi_1(x,y)}{\partial x} - \frac{\partial \phi_1(x,y)}{\partial y} \right]. \tag{16}$$

Substitution of this function into the second component (15) renders the latter equation in the form

$$\frac{\partial^2 \phi_1(x,y)}{\partial x^2} + \frac{\partial^2 \phi_1(x,y)}{\partial y^2} + \frac{1}{V_{22}(x,y)} \left[-\frac{\partial V_{22}(x,y)}{\partial x} + i\, \frac{\partial V_{22}(x,y)}{\partial y} \right] \frac{\partial \phi_1(x,y)}{\partial x} -$$
$$- \frac{1}{V_{22}(x,y)} \left[i\, \frac{\partial V_{22}(x,y)}{\partial x} + \frac{\partial V_{22}(x,y)}{\partial y} \right] \frac{\partial \phi_1(x,y)}{\partial y} + V_{11}(x,y)\ V_{22}(x,y)\ \phi_1(x,y) = 0. \tag{17}$$

In order to prepare for our Darboux transformations, let us now assume that u is a solution to this equation. Next, we define the following functions e and f from the entries V_{11}, V_{22} of our initial potential matrix:

$$e(x,y) = V_{11}(x,y)\ V_{22}(x,y) \tag{18}$$

$$f(x,y) = \frac{1}{V_{22}(x,y)} \left[\frac{\partial V_{22}(x,y)}{\partial x} - i\, \frac{\partial V_{22}(x,y)}{\partial y} \right]. \tag{19}$$

Upon inserting these functions, our equation (17) takes the form

$$\frac{\partial^2 \phi_1(x,y)}{\partial x^2} + \frac{\partial^2 \phi_1(x,y)}{\partial y^2} - f(x,y)\ \frac{\partial \phi_1(x,y)}{\partial x} - i\ f(x,y)\ \frac{\partial \phi_1(x,y)}{\partial y} + e(x,y)\ \phi_1(x,y) = 0. \tag{20}$$

In the next step we rewrite this equation, switching our coordinate y for another coordinate z according to $y = -iz$. This gives

$$\frac{\partial^2 \phi_1(x,z)}{\partial x^2} - \frac{\partial^2 \phi_1(x,z)}{\partial z^2} - f(x,z)\ \frac{\partial \phi_1(x,z)}{\partial x} +$$
$$+ f(x,z)\ \frac{\partial \phi_1(x,z)}{\partial z} + e(x,z)\ \phi_1(x,z) = 0. \tag{21}$$

Note that for the sake of brevity we did not rename the functions entering in (21), even though their second argument changed. We observe that equation (21) coincides with (10). Consequently, we can define an arbitrary-order Darboux transformation that connects solutions

Figure 1. Construction of the Darboux transformation for the Dirac equation.

and potentials of our Dirac equations (11) and (12) as the diagram in figure 1 visualizes. In the first step, we decouple our Dirac equation (11) and switch the y-coordinate in order to match the form (21) that was considered in [27]. Next, we perform a Darboux transformation that generalizes the single-variable version reviewed in section 2. It then remains to reverse the decoupling process and the coordinate change in order to reinstate Dirac form. These three steps define a mapping between our initial Dirac equation (11) and its transformed counterpart (12). The explicit form of this mapping will be determined in the following section.

4. Higher-order Darboux transformations

As can be seen from the diagram in figure 1, the construction of our Darboux transformations is based on their counterparts for equation (21). Let us remark that the first-order case of the Darboux transformations for the latter two-dimensional equation is explicitly shown in [27], while the higher-order case is not shown. Now, as a starting point, we observe that the one-dimensional equation (4) can be written by means of the operators (1) as

$$\left[D_{+\lambda} D_{-\lambda} - f(x)\ D_{-\lambda} + e(x)\right] \psi(x) \quad = \quad 0. \tag{22}$$

In the same way we can express our two-dimensional equation (21) through the operators (1) if we formally replace the parameter λ in the latter operators by the partial derivative with respect to z. We have

$$\left[D_{+\frac{\partial}{\partial z}} D_{-\frac{\partial}{\partial z}} - f(x,z)\ D_{-\frac{\partial}{\partial z}} + e(x,z)\right] \phi_1(x,z) \quad = \quad 0. \tag{23}$$

Evaluation shows that this equation coincides with (21). Consequently, we must also be able to express (20) in an analogous manner, since it is related to (23) by a coordinate change. Let us first generalize the operators (1) by defining

$$D_\pm \quad = \quad \frac{\partial}{\partial x} \pm i\ \frac{\partial}{\partial y}, \tag{24}$$

which we can use to write equation (20) in the following form

$$\left[D_- D_+ - f(x,y)\ D_+ + e(x,y)\right] \phi_1(x,y) \quad = \quad 0.$$

We observe that this equation is expressed through operators (24) in the same way as the one-dimensional case (22), except for differences in the notation. More precisely, the operator $D_{+\lambda_j}$ was replaced by its two-dimensional counterpart D_-. Hence, we will construct our Darboux transformations by using the same approach that was taken in the one-dimensional scenario, see section 2. To this end, we modify the definition of the generalized Wronskians (2) by switching the operators (1) for their two-dimensional counterparts (24). Upon introducing functions u_j, $j = 1, ..., n$, our generalized Wronskian reads

$$\mathcal{W}_{u_1,...,u_n}(x,y) = \det \begin{pmatrix} u_1(x,y) & u_2(x,y) & \cdots & u_n(x,y) \\ D_- u_1(x,y) & D_- u_2(x,y) & \cdots & D_- u_n(x,y) \\ D_-^2 u_1(x,y) & D_-^2 u_2(x,y) & \cdots & D_-^2 u_n(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ D_-^{n-1} u_1(x,y) & D_-^{n-1} u_2(x,y) & \cdots & D_-^{n-1} u_n(x,y) \end{pmatrix}. \tag{25}$$

In the next step we proceed to rewrite the functions (3) that can be expressed by means of the operators (1) as

$$\chi_j(x) = \left[D_{+\lambda_j} - f(x)\right] u_j(x), \quad j = 1, ..., n.$$

Consequently, we can introduce the two-dimensional version of these functions in the following way

$$\begin{aligned} \chi_j(x,y) &= \left[D_- - f(x,y)\right] u_j(x,y) \\ &= \frac{\partial u_j(x,y)}{\partial x} - i\,\frac{\partial u_j(x,y)}{\partial y} - f(x,y)\; u_j(x,y), \quad j = 1, ..., n. \end{aligned} \tag{26}$$

At this point we must distinguish two Darboux transformations, the one-dimensional versions of which are shown in section 2. We will assume that the two-dimensional functions u_j, $j = 1, ..., n$, are solutions to equation (20).

- **First Darboux transformation:** We will now generalize the Darboux transformation given in (6)-(8) to the scenario of two dimensions. Starting out with the transformed solution (6), we must replace the generalized Wronskians (2) by their two-dimensional counterparts (25). This yields

$$\hat{\phi}_1(x,y) = \frac{\mathcal{W}_{u_1,...,u_n,\phi_1}(x,y)}{\mathcal{W}_{u_1,...,u_n}(x,y)}. \tag{27}$$

  Next, we need to determine the transformed functions $\hat{e}$ and $\hat{f}$. These are given by the following expressions

$$\hat{e}(x,y) = e(x,y) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log\left[\mathcal{W}_{u_1,...,u_n}(x,y)\right] \tag{28}$$

$$\hat{f}(x,y) = f(x,y) + \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \log\left[\frac{\mathcal{W}_{\chi_1,...,\chi_n}(x,y)}{\mathcal{W}_{u_1,...,u_n}(x,y)}\right]. \tag{29}$$

  Thus, the functions (27)-(29) enter in the transformed partner equation of (20). This equation is given by

$$\frac{\partial^2 \hat{\phi}_1(x,y)}{\partial x^2} + \frac{\partial^2 \hat{\phi}_1(x,y)}{\partial y^2} - \hat{f}(x,y)\;\frac{\partial \hat{\phi}_1(x,y)}{\partial x} - i\;\hat{f}(x,y)\;\frac{\partial \hat{\phi}_1(x,y)}{\partial y} + \hat{e}(x,y)\;\hat{\phi}_1(x,y) = 0.$$

(30)

In the final step we must find the solution of our transformed Dirac equation (12) and the associated transformed matrix potential. Before we do so, let us construct the second Darboux transformation.

- **Second Darboux transformation:** As described in section 2, the two Darboux transformations in the one-dimensional case are interrelated by the mapping (9). In two dimensions, there is also such a mapping that can be written as [27]

$$\hat{\phi}_{1\mathcal{T}}(x,y) = \left\{ \exp\left[-D_-^{-1}\hat{f}(s,y)\right]\hat{\phi}_1(s,y)\right\}_{|s=-x} \tag{31}$$

$$\hat{e}_{\mathcal{T}}(x,y) = \hat{e}(-x,y) - D_-\hat{f}_{\mathcal{T}}(x,y) \tag{32}$$

$$\hat{f}_{\mathcal{T}}(x,y) = \left[D_+ D_-^{-1}\hat{f}(s,y)\right]_{|s=-x}, \tag{33}$$

note that D_-^{-1} represents the inverse operator of D_-. We will now calculate the explicit form of the three functions (31)-(33). Starting out with the first of them, we must determine the action of the inverse operator D_-^{-1} on the function $\hat{f}$. We find

$$D_-^{-1}\hat{f}(s,y) = \int^s \hat{f}(t, i\ s + y - i\ t)\ dt. \tag{34}$$

Substitution into (31) leads to

$$\begin{aligned}\hat{\phi}_{1\mathcal{T}}(x,y) &= \left\{ \exp\left[-\int^s f(t, i\ s + y - i\ t)\ dt\right]\hat{\phi}_1(s,y)\right\}_{|s=-x} \\ &= \exp\left[-\int^{-x} f(t, -i\ x + y - i\ t)\ dt\right]\hat{\phi}_1(-x,y) \\ &= \exp\left[-\int^{-x} f(t, -i\ x + y - i\ t)\ dt\right]\frac{\mathcal{W}_{u_1,\ldots,u_n,\phi_1}(-x,y)}{\mathcal{W}_{\chi_1,\ldots,\chi_n}(-x,y)},\end{aligned} \tag{35}$$

where in the last step we used (27). Next, we continue by calculating (33), since we will need the result for (32). Upon taking into account the form (34) of the inverse operator D_-^{-1} and (29), we obtain

$$\begin{aligned}\hat{f}_{\mathcal{T}}(x,y) &= \left\{\left(\frac{\partial}{\partial s} + i\ \frac{\partial}{\partial y}\right)\int^s \hat{f}(t, i\ s + y - i\ t)\ dt + \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(x,y)}{\mathcal{W}_{u_1,\ldots,u_n}(x,y)}\right]\right\}_{s=-x} \\ &= \left(-\frac{\partial}{\partial x} + i\ \frac{\partial}{\partial y}\right)\int^{-x} \hat{f}(t, -i\ x + y - i\ t)\ dt + \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(-x,y)}{\mathcal{W}_{u_1,\ldots,u_n}(-x,y)}\right] \\ &= \hat{f}(-x,y) + 2\ i\int^{-x} \frac{\partial \hat{f}(t,s)}{\partial s}_{|s=-i\ x+y-i\ t}\ dt + \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(-x,y)}{\mathcal{W}_{u_1,\ldots,u_n}(-x,y)}\right]\end{aligned} \tag{36}$$

It remains to determine the explicit form of (32) that we obtain by combining (28), (34), and (36). This gives

$$\hat{e}_{\mathcal{T}}(x,y) = e(-x,y) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log\left[\mathcal{W}_{u_1,\ldots,u_n}(-x,y)\right] - $$
$$ - \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \left\{ \left(-\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \int^{-x} \hat{f}(t, -i\,x + y - i\,t)\,dt + \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(-x,y)}{\mathcal{W}_{u_1,\ldots,u_n}(-x,y)}\right] \right\}.$$

After applying the derivative operators, we arrive at the following result

$$\hat{e}_{\mathcal{T}}(x,y) = e(-x,y) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log\left[\mathcal{W}_{u_1,\ldots,u_n}(-x,y)\right] - $$
$$ - \frac{\partial \hat{f}(-x,y)}{\partial x} + 3\,i\,\frac{\partial \hat{f}(x,y)}{\partial y} - 4 \int^{-x} \left[\frac{\partial^2 \hat{f}(t,s)}{\partial s^2}_{|s=-i\,x+y-i\,t}\right] dt - $$
$$ - \left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \log\left[\frac{\mathcal{W}_{\chi_1,\ldots,\chi_n}(-x,y)}{\mathcal{W}_{u_1,\ldots,u_n}(-x,y)}\right]. \qquad (37)$$

Expressions (35)-(37) provide a representation of the second Darboux transformation. This means that the function $\hat{\phi}_{1\mathcal{T}}$ solves equation (30), if $\hat{e}$ and $\hat{f}$ are replaced by $\hat{e}_{\mathcal{T}}$ and $\hat{f}_{\mathcal{T}}$, respectively.

In order to determine the transformed potential matrix, we first recast our formulas (18) and (19) in terms of the quantities $\hat{e}$ and $\hat{f}$. This gives

$$\hat{e}(x,y) = \hat{V}_{11}(x,y)\,\hat{V}_{22}(x,y) \qquad (38)$$

$$\hat{f}(x,y) = \frac{1}{\hat{V}_{22}(x,y)} \left[\frac{\partial \hat{V}_{22}(x,y)}{\partial x} - i\,\frac{\partial \hat{V}_{22}(x,y)}{\partial y}\right], \qquad (39)$$

where the explicit form of $\hat{e}$, $\hat{f}$ can be found in (28), (29) for the first Darboux transformation and in (36), (37) for the second Darboux transformation, note that in the latter case they carry an index $\mathcal{T}$ that for simplicity we left out above. In the next step we solve equations (38) and (39) with respect to $\hat{V}_{11}$ and $\hat{V}_{22}$. We obtain

$$\hat{V}_{11} = \exp\left[-\int^{x} \hat{f}(t, -x + i\,y + t)\,dt\right] \frac{\hat{e}}{G(i\,x + y)} \qquad (40)$$

$$\hat{V}_{22} = \exp\left[\int^{x} \hat{f}(t, -x + i\,y + t)\,dt\right] G(i\,x + y), \qquad (41)$$

note that G is an arbitrary function of its argument. Also, recall that in case of the second Darboux transformation $\hat{e}$ and $\hat{f}$ must have an index $\mathcal{T}$. Now that we have determined the transformed Dirac potential, it remains to find the two-component solution of the Dirac equation (12). Since the function $\hat{\phi}_1$ or $\hat{\phi}_{1\mathcal{T}}$ represents the first component in the Dirac solution (13), according to (16) we can find the second component from

$$\hat{\phi}_{2\mathcal{T}}(x,y) = \frac{1}{\hat{V}_{22}(x,y)} \left[i\,\frac{\partial \hat{\phi}_{1\mathcal{T}}(x,y)}{\partial x} - \frac{\partial \hat{\phi}_{1\mathcal{T}}(x,y)}{\partial y}\right]. \qquad (42)$$

This result completes the construction of our higher-order Darboux transformations.

5. Applications

In this section we will present two examples of applying our Darboux transformations to Dirac systems. Our main focus is to keep calculations transparent by using functions that do not involve excessively long expressions. For the same reason we restrict ourselves to Darboux transformations of second order only.

5.1. Exponential-type potential

Let us consider our initial Dirac equation (11) for the following matrix potential

$$V(x,y) = \begin{pmatrix} 30\ \exp(y)\ \mathrm{sech}(x)^2 & 0 \\ 0 & \exp(-y) \end{pmatrix}, \tag{43}$$

note that numerical factors were chosen such as to simplify subsequent calculations. The diagonal entries of the potential can be read off as

$$V_{11}(x,y) = 30\ \exp(y)\ \mathrm{sech}(x)^2 \qquad V_{22}(x,y) = \exp(-y). \tag{44}$$

Upon substitution of these functions into the decoupled version (17) of our Dirac equation we find

$$\frac{\partial^2 \phi_1(x,y)}{\partial x^2} + \frac{\partial^2 \phi_1(x,y)}{\partial y^2} - i\ \frac{\partial \phi_1(x,y)}{\partial x} + \frac{\partial \phi_1(x,y)}{\partial y} + 30\ \mathrm{sech}(x)^2\ \phi_1(x,y) = 0. \tag{45}$$

It will be sufficient to work with a particular solution of this equation. Our solution is given by

$$\phi_1(x,y) = \left\{ c_1\ \exp\left[-\left(\frac{1}{2}+\frac{1}{2}\sqrt{4\ k-1}\right)y\right] + c_2\ \exp\left[-\left(\frac{1}{2}-\frac{1}{2}\sqrt{4\ k-1}\right)y\right]\right\} \times$$
$$\times\ P_5^{\frac{1}{2}\sqrt{4\ k-1}}[\tanh(x)] \left[-1-\tanh(x)\right]^{\frac{i}{4}} \left[-1+\tanh(x)\right]^{-\frac{i}{4}}, \tag{46}$$

where P stands for the Legendre function of the first kind [1] and c_1, c_2 are free constants. Note that this function along with its counterpart ϕ_2 from (16) forms a solution to our initial Dirac equation (11) for the matrix potential (43). For the sake of brevity we omit to state ϕ_2 here. Now, in order to perform our Darboux transformations, we need to determine the functions e and f, as defined in (18) and (19), respectively. Insertion of our potential entries (44) yields

$$e(x,y) = 30\ \mathrm{sech}(x)^2 \qquad f(x,y) = i. \tag{47}$$

Let us consider each of the two Darboux transformations separately.

- **First Darboux transformation:** Recall that this transformation is given by the formulas (27)-(29), the second-order case of which is obtained by setting $n = 2$. In the next step we need to define two transformation functions u_1, u_2 that solve equation (20). We take these functions as the following special cases of our solution (46)

$$u_1(x,y) = \phi_1(x,y)_{|c_1=1,c_2=0,k=101/4}$$
$$= -945\ \exp\left[\left(-\frac{1}{2}-5\ i\right)y\right]\mathrm{sech}(x)^5\left[-1-\tanh(x)\right]^{\frac{i}{4}}\left[-1+\tanh(x)\right]^{-\frac{i}{4}} \tag{48}$$

$$u_2(x,y) = \phi_1(x,y)_{|c_1=0,c_2=1,k=101/4}$$
$$= -945\ \exp\left[\left(-\frac{1}{2}+5\ i\right)y\right]\mathrm{sech}(x)^5\left[-1-\tanh(x)\right]^{\frac{i}{4}}\left[-1+\tanh(x)\right]^{-\frac{i}{4}} \tag{49}$$

Let us point out that the numerical value of k is chosen such that the transformation functions become elementary. More precisely, we observe that for $k = n^2 + 1/4$, $n = 0, ..., 5$, the Legendre function in (46) degenerates to a polynomial. We will now proceed by finding the results of the Darboux transformation. Starting out by evaluating (28), we insert our transformation functions (48), (49), which leads to the result

$$\hat{e}(x, y) \;=\; 20 \;\text{sech}(x). \tag{50}$$

Next, we calculate the functions χ_1 and χ_2 that enter in the result (29) of our Darboux transformation. Upon evaluating (26), we get

$$\chi_1(x, y) \;=\; -4725 \;\exp\left[\left(-\frac{1}{2} - 5\;i\right) y\right] \text{sech}(x)^5 \left[-1 - \tanh(x)\right]^{1+\frac{i}{4}} \left[-1 + \tanh(x)\right]^{-\frac{i}{4}} \tag{51}$$

$$\chi_2(x, y) \;=\; -4725 \;\exp\left[\left(-\frac{1}{2} + 5\;i\right) y\right] \text{sech}(x)^5 \left[-1 - \tanh(x)\right]^{\frac{i}{4}} \left[-1 + \tanh(x)\right]^{1-\frac{i}{4}}. \tag{52}$$

We substitute these functions along with (48), (49) into (29). This gives after simplification

$$\hat{f}(x, y) \;=\; i - 2\;\tanh(x). \tag{53}$$

Now that we have the functions $\hat{e}$ and $\hat{f}$ from (38) and (39), respectively, we find the diagonal entries of the transformed Dirac matrix potential through (40), (41). This gives

$$\hat{V}_{11}(x, y) \;=\; \frac{20\;\exp(-i\;x)}{G(i\;x + y)} \tag{54}$$

$$\hat{V}_{22}(x, y) \;=\; \exp(i\;x)\;\text{sech}(x)^2\;G(i\;x + y). \tag{55}$$

The complex exponentials can be absorbed by the arbitrary function G. Upon setting

$$G(x) \;=\; \exp(-x), \tag{56}$$

the potential entries (54) and (55) take the following real-valued form

$$\hat{V}_{11}(x, y) \;=\; 20\;\exp(y) \tag{57}$$

$$\hat{V}_{22}(x, y) \;=\; \exp(-y)\;\text{sech}(x)^2. \tag{58}$$

These are the diagonal components of the potential matrix $\hat{V}$ that enters in our transformed Dirac equation (12). It remains to determine a solution to the latter equation through (27). We insert our transformation functions (48), (49), along with the solution (46). For the sake of brevity, we implement the settings $c_1 = c_2 = 1, k = 5/4$ in the latter solution. Evaluation of (27) gives the result

$$\hat{\phi}_1(x, y) \;=\; 30\;\exp\left(-\frac{y}{2}\right) \text{sech}(x)^5 \Bigg\{ \cos(y) \Big[30 - 25\;\cosh(2\;x) + \cosh(4\;x)\Big] - \\ - i\;\sin(y)\;\Big[-5\;\sinh(2\;x) + \sinh(4\;x)\Big]\Bigg\} \Big[-1 - \tanh(x)\Big]^{\frac{i}{4}} \Big[-1 + \tanh(x)\Big]^{-\frac{i}{4}}. \tag{59}$$

The associated function $\hat{\phi}_2$ can be found from (42). We obtain

$$\hat{\phi}_2(x,y) = -1200\ i\ \exp\left(\frac{y}{2}\right)\cos(y)\ \mathrm{sech}(x)\left[-4+7\ \mathrm{sech}(x)^2\right]\ \tanh(x)\times$$
$$\times\ \left[-1-\tanh(x)\right]^{\frac{i}{4}}\left[-1+\tanh(x)\right]^{-\frac{i}{4}}. \tag{60}$$

In summary, (59) and (60) form the two-component solution of our transformed Dirac equation (12) for the diagonal matrix potential with entries (57), (58). The probability density associated with the transformed solution is visualized in figure 2.

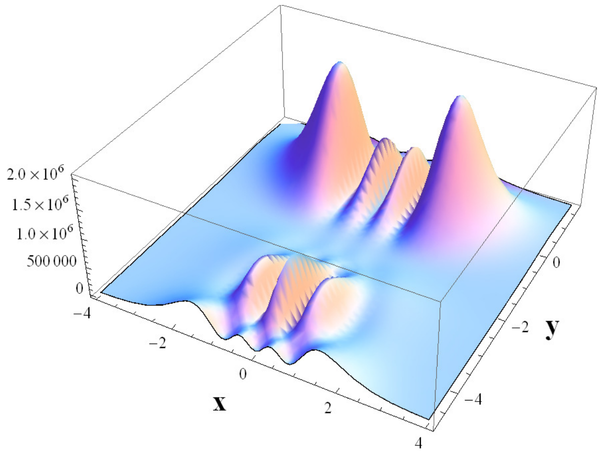

Figure 2. Graph of the probability density $|\hat{\Phi}|^2 = |\hat{\phi}_1|^2 + |\hat{\phi}_2|^2$ associated with the solution components (59) and (60).

- **Second Darboux transformation:** We will now apply formulas (35)-(37), where we implement the second-order case $n = 2$, and follow the same approach as in the previous case. To this end, we use the same transformation functions (48), (49) as in the previous case. As a consequence, the functions (51), (52) also remain the same. Substitution into (36) and (37) gives

$$\hat{e}_{\mathcal{T}}(x,y) = 18\ \mathrm{sech}(x)^2 \qquad \hat{f}_{\mathcal{T}}(x,y) = i+2\ \tanh(x).$$

In the next step we plug these findings into the entries (40) and (41) of the transformed potential matrix. Evaluation leads to

$$\hat{V}_{11}(x,y) = 18\ \exp(y)\ \mathrm{sech}(x)^4 \tag{61}$$
$$\hat{V}_{22}(x,y) = \exp(-y)\ \cosh(x)^2, \tag{62}$$

note that we used the setting (56). Now we calculate the first component (35) of the solution to our transformed Dirac equation (12). Since its general form is rather complicated, we implement the parameter settings $c_1 = c_2 = 1, k = 65/4$. After simplification we obtain

$$\hat{\phi}_{1\mathcal{T}}(x,y) = 15120\ \exp\left(i\ x-\frac{y}{2}\right)\mathrm{sech}(x)^3\ \sinh(x+4\ i\ y)\times$$
$$\times\ \left[-1-\tanh(x)\right]^{-\frac{i}{4}}\left[-1+\tanh(x)\right]^{\frac{i}{4}}. \tag{63}$$

The second component of our solution can be found through (42). We find

$$\hat{\phi}_{2\mathcal{T}}(x,y) = -45360\ i\ \exp\left(i\ x + \frac{y}{2}\right) \operatorname{sech}(x)^6\ \cosh(2\ x + 4\ i\ y) \times \times \left[-1 - \tanh(x)\right]^{-\frac{i}{4}} \left[-1 + \tanh(x)\right]^{\frac{i}{4}}. \quad (64)$$

Thus, (63) and (64) are components of the solution to our transformed Dirac equation (12) for the diagonal matrix potential with entries (61), (62). The probability density associated with the transformed solution is visualized in figure 3.

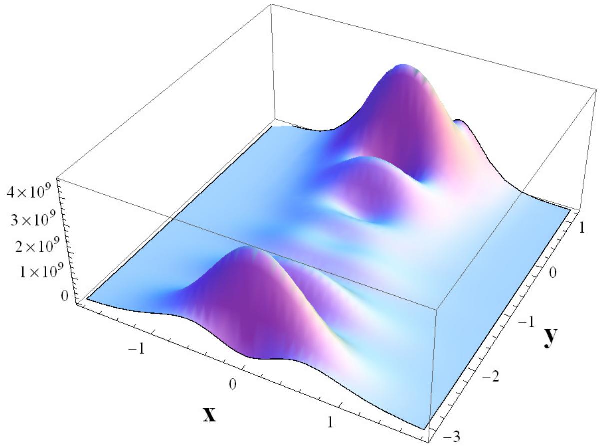

Figure 3. Graph of the probability density $|\hat{\Phi}|^2 = \left|\hat{\phi}_{1\mathcal{T}}\right|^2 + \left|\left(\hat{\phi}_2\right)_{\mathcal{T}}\right|^2$ associated with the solution components (63) and (64).

5.2. Polynomial-type potential

As in the previous application, our starting point is the initial Dirac equation (11) for a matrix potential that this time is given by

$$V(x,y) = \begin{pmatrix} k - x^2 - y^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (65)$$

where k is a parameter that will be discussed below. Our potential matrix is diagonal with entries

$$V_{11}(x,y) = k - x^2 - y^2 \qquad V_{22}(x,y) = 1. \quad (66)$$

In order to determine a solution to our Dirac equation for the potential (65), we plug the potential entries into the equation (17). We find

$$\frac{\partial^2 \phi_1(x,y)}{\partial x^2} + \frac{\partial^2 \phi_1(x,y)}{\partial y^2} + \left(k - x^2 - y^2\right)\ \phi_1(x,y) = 0. \quad (67)$$

This can be identified with a two-dimensional Schrödinger equation for stationary energy k and harmonic oscillator potential. A solution can be given in the form

$$\begin{aligned}\phi_1(x,y) &= \left[c_1 \exp\left(-\frac{x^2}{2}\right) H_{\frac{k_1-1}{2}}(x) + c_2 \exp\left(\frac{x^2}{2}\right) H_{\frac{-k_1-1}{2}}(i\,x)\right] \times \\ &\times \left[c_3 \exp\left(-\frac{y^2}{2}\right) H_{\frac{k_2-1}{2}}(y) + c_4 \exp\left(\frac{y^2}{2}\right) H_{\frac{-k_2-1}{2}}(i\,y)\right], \end{aligned} \tag{68}$$

where H stands for the Hermite function [1], and c_j, $j = 1, ..., 4$, are free constants. Furthermore, the parameters k_1 and k_2 must satisfy the constraint $k = k_1+k_2$. While ϕ_1 is the first component of the solution to our initial Dirac equation (11) for the potential (65), the associated second component ϕ_2 is found through formula (16). Now, upon substitution of the functions from (66) into (18) and (19), we obtain

$$e(x,y) = k - x^2 - y^2 \qquad\qquad f(x,y) = 0. \tag{69}$$

We are now ready to perform a Darboux transformation of second order. In the present example we will restrict ourselves to applying the first case, governed by (27)-(29) for $n = 2$. This case requires two transformation functions that we take from (68).

$$\begin{aligned} u_1(x,y) &= \phi_1(x,y)_{|c_1=c_3=1, c_2=c_4=0, k_1=1, k_2=3} \\ &= 2\,\exp\left(-\frac{x^2+y^2}{2}\right) y \end{aligned} \tag{70}$$

$$\begin{aligned} u_2(x,y) &= \phi_1(x,y)_{|c_1=c_4=0, c_2=c_3=1, k_1=-1, k_2=5} \\ &= \exp\left(-\frac{x^2-y^2}{2}\right)(4\,y^2-2). \end{aligned} \tag{71}$$

We observe that these functions are elementary because the parameters k_1 and k_2 were chosen accordingly, such that the Hermite functions degenerate to polynomials. Furthermore, note that the choice for the latter two parameters dictates $k = 4$. We now insert the transformation functions (70) and (71) into (28), which results in a complex-valued rational function, that we do not show here due to its length. Next, we calculate the functions χ_1 and χ_2 from (26). This gives

$$\chi_1(x,y) = \exp\left(-\frac{x^2+y^2}{2}\right)\left[-2\,x\,y + 2\,i\,(y^2-1)\right] \tag{72}$$

$$\chi_2(x,y) = 2\,\exp\left(-\frac{x^2-y^2}{2}\right)\left[i\,y\,(2\,y^2-5) + x\,(2\,y^2-1)\right]. \tag{73}$$

Upon substitution of these functions and (70), (71) into (29), we obtain $\hat{f}$. Since the explicit form of this function is very long, we omit to state it here. Instead, we will state the entries of the transformed Dirac potential matrix that are obtained from (40) and (41). The result reads

$$\hat{V}_{11}(x,y) = \frac{-2\,x\,y + 4\,x\,y^3 + i + 2\,i\,y^2}{-2\,x\,y + 4\,x\,y^3 - i - 2\,i\,y^2} \tag{74}$$

$$\hat{V}_{22}(x,y) = \frac{2\,x\,y\,[-6 + x^2 + (5-2\,x^2)\,y^2 - 2\,y^4] + 3\,i\,[2 + x^2 + (2\,x^2-3)\,y^2 + 2\,y^4]}{-2\,x\,y + 4\,x\,y^3 - i - 2\,i\,y^2}. \tag{75}$$

Note that we chose the function G such as to absorb some complex terms. In the present case it is not possible to render the potential matrix entries in real-valued form. Now, in order to determine (27), we substitute the transformation functions (70), (71), and the solution (68) for the parameter settings $c_1 = c_3 = 1$, $c_2 = c_4 = 0$, $k_1 = 3, k_2 = 1$. We obtain

$$\hat{\phi}_1(x,y) = -4\, i\, \exp\left(-\frac{x^2+y^2}{2}\right) \frac{(x-i\,y)\,[-3+2\,y^2+x^2\,(4\,y^2-2)-8\,i\,x\,y]}{4\,x\,y^3-2\,x\,y-2\,i\,y^2-i} \tag{76}$$

We construct the remaining solution component $\hat{\phi}_2$ by means of (42). This gives

$$\hat{\phi}_2(x,y) = 4\, \exp\left(-\frac{x^2+y^2}{2}\right) \frac{1+2\,y^2+x^2\,(4\,y^2-2)}{4\,x\,y^3-2\,x\,y-2\,i\,y^2-i}. \tag{77}$$

Hence, the two component solution of our transformed Dirac equation (12) is provided by (76) and (77). The associated potential matrix entries are shown in (74), (75), and the probability density associated with the transformed solution is visualized in the left part of figure 4. The right part of that figure shows another example of a probability density obtained from a Darboux transformation with the transformation functions

$$u_1(x,y) = \phi_1(x,y)_{|c_1=c_3=1,c_2=c_4=0,k_1=1,k_2=5} \tag{78}$$

$$u_2(x,y) = \phi_1(x,y)_{|c_1=c_4=0,c_2=c_3=1,k_1=-1,k_2=7}, \tag{79}$$

and the overall settings

$$k = 6 \qquad k_1 = k_2 = 3. \tag{80}$$

For the sake of brevity we omit to state calculations and results associated with these parameter settings. Observe that the probability densities shown in figure 4 are bounded function. This

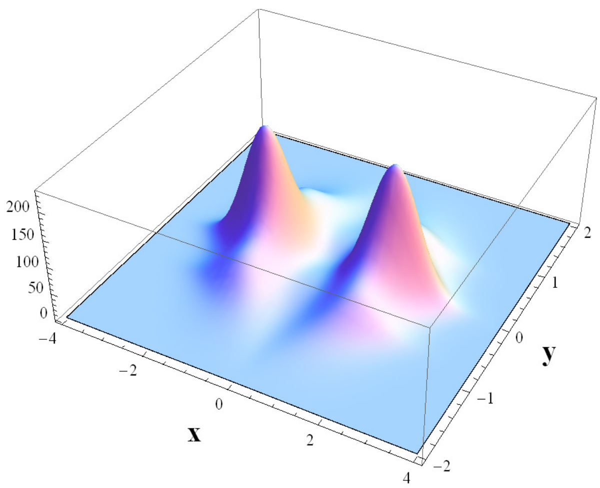

Figure 4. Graphs of probability densities $|\hat{\Phi}|^2 = \left|\hat{\phi}_1\right|^2 + \left|\hat{\phi}_2\right|^2$ associated with the solution components (76), (77).

is in contrast to the previous examples, where the probability densities behave like exponential functions in y-direction.

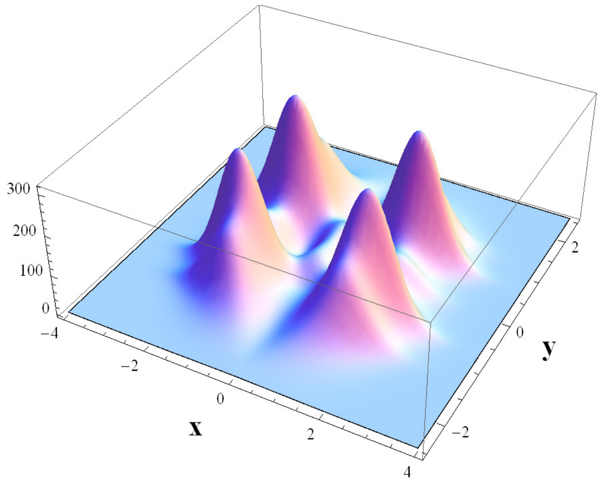

Figure 5. Graphs of probability densities $|\hat{\Phi}|^2 = \left|\hat{\phi}_1\right|^2 + \left|\hat{\phi}_2\right|^2$ associated with the settings (78)-(80).

6. Concluding remarks

In this work we have presented Wronskian representations of arbitrary-order Darboux transformations for the two-dimensional Dirac equation with diagonal matrix potential. While implementation is straightforward, technical difficulties can arise for a variety of reasons. First, in order to apply the Darboux transformations, a solution to the initial equation is needed. Finding such a solution can be tedious, given that the governing equation is two-dimensional. Furthermore, the process of determining the transformed Dirac potential requires the resolution of an integral that is difficult to do in a symbolic way, as the integrand is the result of the Darboux transformation, which can be of complicated form or contain special functions. In order to have better control over the Darboux transformations, it is desirable to have reality and regularity conditions for the transformed potential as well as for the associated solutions of the Dirac equation. These issues are subject to future research.

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