

On A Simpler and Faster Derivation of Single Use Reliability Mean and Variance for Model-Based Statistical Testing

Yufeng Xue, Lan Lin, Xin Sun
Department of Computer Science
Ball State University
Robert Bell Building, Room 455
Muncie, IN 47306, USA
{yxue2, llin4, xsun6}@bsu.edu

Fengguang Song
Department of Computer and Information Science
Indiana University-Purdue University Indianapolis
723 W. Michigan St., SL 280
Indianapolis, IN 46202, USA
fgsong@cs.iupui.edu

Abstract

Markov chain usage-based statistical testing has proved sound and effective in providing audit trails of evidence in certifying software-intensive systems. The system end-to-end reliability is derived analytically in closed form, following an arc-based Bayesian model. System reliability is represented by an important statistic called single use reliability, and defined as the probability of a randomly selected use being successful. This paper continues our earlier work on a simpler and faster derivation of the single use reliability mean, and proposes a new derivation of the single use reliability variance by applying a well-known theorem and eliminating the need to compute the second moments of arc failure probabilities. Our new results complete a new analysis that could be shown to be simpler, faster, and more direct while also rendering a more intuitive explanation. Our new theory is illustrated with three simple Markov chain usage models with manual derivations and experimental results.

1 Introduction

This paper re-examines the underlying reliability analysis for *statistical testing* based on a *Markov chain usage model*. This form of statistical testing, developed by the University of Tennessee Software Quality Research Laboratory (UTK SQRL), has been around for more than two decades [5, 4, 8, 6, 9, 12, 11]. With the software use being modeled as a finite-state, discrete parameter, time-homogeneous, and irreducible Markov chain, where the states represent “states of system use” and the arcs represent possible transitions between states of use, the method allows for quantitative certification of software using empirical test data by a statistical protocol. A public domain tool supporting statistical testing called the *JUMBL: J Us-*

age Model Builder Library, also developed by UTK SQRL, is freely available [7, 1].

In this paper we focus on the derivation of a system end-to-end reliability estimate, called *single use reliability* (both mean and variance), driven solely by the test data without any mathematical growth assumptions, given the Markov chain usage model, and improve on an earlier analytical solution described in [8]. Our new derivation of the mean was inspired earlier and published in [2], however, it was not until recently that we figured out a new derivation of the variance that is similarly simpler, faster, more direct, and more intuitive, which arrived through a rather convoluted path. The derivation in [8] follows from the definitions of mean and variance and only first principles, which could be a little counter-intuitive to understand. We demonstrate a new derivation and complete a new analysis. Through three examples we show the new derivation agrees with the old derivation (by implementation and experiments), as well as a direct application of the definition. The new theory is fully implemented in the latest version of the JUMBL.

2 Single Use Reliability Mean and Variance: The Old Derivation

Current reliability analysis underlying statistical testing follows the arc-based Bayesian model [8, 10]. Here one applies Miller’s Bayesian model [3] to individual arcs of the Markov chain, and compute for each arc a transition reliability (both mean and variance) from a posterior beta distribution. System end-to-end reliability is computed through the *single use reliability* estimate, defined as “the probability of a randomly selected use executing correctly relative to a specification of correct behavior,” [5, 8] either analytically [8] or through simulation [10]. The analytical solution in closed form [8], both faster and more precise than simulation, was implemented in the JUMBL. In this section we

summarize the major steps and results of this derivation.

Let $P = [p_{ij}]$ be the $n \times n$ transition matrix of a Markov chain usage model. The (i, j) -th entry p_{ij} of P is the conditional probability of the next state being state j given the current state being state i . State 1 is the source. State n is the sink and the only absorbing state (assuming a reasonable error recovery scheme). Given $P_{n \times n}$, $Q_{(n-1) \times n}$ denotes the submatrix of P omitting the last row, and $\dot{Q}_{(n-1) \times (n-1)}$ denotes the submatrix of P omitting the last row and the last column. \dot{Q} is the transition matrix of the Markov chain restricted to the transient states.

Let $r_{i,j}$ be a random variable for ‘‘transition reliability,’’ that is, the fraction of successful transitions from state i to state j . Let $f_{i,j}$ be another random variable for ‘‘transition failure probability,’’ that is, the fraction of unsuccessful transitions from state i to state j . Notice that $f_{i,j} = 1 - r_{i,j}$.

With the arc-based Bayesian model [10], each arc (transition) reliability $r_{i,j}$ has a standard beta distribution $B(\alpha_{i,j}, \beta_{i,j})$ with two parameters $\alpha_{i,j}$ (for total successes on transitions from state i to state j) and $\beta_{i,j}$ (for total failures on transitions from state i to state j), where $\alpha_{i,j} = a_{i,j} + s_{i,j}$ and $\beta_{i,j} = b_{i,j} + f_{i,j}$ with $a_{i,j}, s_{i,j}, b_{i,j}, f_{i,j}$ representing prior successes, observed successes, prior failures, and observed failures, respectively, on transitions from state i to state j . In case no prior information is available, $a_{i,j} = b_{i,j} = 1$. Each executed test case is mapped to the usage model and each executed step is marked as successful or failing. The observed success and failure counts are summed for each individual arc in the usage model.

From the posterior (beta) distribution $B(\alpha_{i,j}, \beta_{i,j})$ for $r_{i,j}$ we may compute the mean and variance of $r_{i,j}$:

$$E[r_{i,j}] = \frac{\alpha_{i,j}}{\alpha_{i,j} + \beta_{i,j}} = \frac{a_{i,j} + s_{i,j}}{a_{i,j} + s_{i,j} + b_{i,j} + f_{i,j}}, \quad (1)$$

$$\begin{aligned} \text{Var}[r_{i,j}] &= \frac{\alpha_{i,j} \beta_{i,j}}{(\alpha_{i,j} + \beta_{i,j})^2 (\alpha_{i,j} + \beta_{i,j} + 1)} \\ &= \frac{(a_{i,j} + s_{i,j})(b_{i,j} + f_{i,j})}{(a_{i,j} + s_{i,j} + b_{i,j} + f_{i,j})^2 (a_{i,j} + s_{i,j} + b_{i,j} + f_{i,j} + 1)}. \end{aligned} \quad (2)$$

Since $\text{Var}[r_{i,j}] = E[r_{i,j}^2] - E^2[r_{i,j}]$, we have $E[r_{i,j}^2] = E^2[r_{i,j}] + \text{Var}[r_{i,j}]$.

Given $f_{i,j} = 1 - r_{i,j}$, we can compute the mean and variance of $f_{i,j}$ as $E[f_{i,j}] = E[1 - r_{i,j}] = 1 - E[r_{i,j}]$ and $\text{Var}[f_{i,j}] = \text{Var}[1 - r_{i,j}] = \text{Var}[r_{i,j}]$. Similarly we have $E[f_{i,j}^2] = E^2[f_{i,j}] + \text{Var}[f_{i,j}]$.

By our assumption state n (the sink) is the only absorbing state of the Markov chain. A test case ends when the sink is first encountered, therefore, we are only interested in transitions from any state other than the sink (any transient state). In the matrices defined below ($A, B, S, F, R_1, R_2, F_1$, and F_2), i is any integer from 1 to $n - 1$ inclusive, and j is any integer from 1 to n inclusive.

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be two matrices of size $(n - 1) \times n$ whose entries are prior arc successes and fail-

ures, respectively, obtained from prior testing experience. Let $S = [s_{i,j}]$ and $F = [f_{i,j}]$ be two matrices of size $(n - 1) \times n$ whose entries are observed arc successes and failures, respectively, obtained through testing.

Let $R_1 = [E[r_{i,j}]]$ be an $(n - 1) \times n$ matrix whose (i, j) -th entry is the expected arc reliability of going from state i to state j , and $R_2 = [E[r_{i,j}^2]]$ be an $(n - 1) \times n$ matrix whose (i, j) -th entry is the expected value of $r_{i,j}^2$. Let \dot{R}_1 and \dot{R}_2 denote respectively the submatrices of R_1 and R_2 omitting the last columns.

Similarly we define $F_1 = [E[f_{i,j}]]$ as an $(n - 1) \times n$ matrix whose (i, j) -th entry is the expected arc failure probability of going from state i to state j , and $F_2 = [E[f_{i,j}^2]]$ as an $(n - 1) \times n$ matrix whose (i, j) -th entry is the expected value of $f_{i,j}^2$.

Given two matrices X and Y of the same size (dimension), $X \otimes Y$ denotes the entry-wise (or component-wise) product of X and Y . $X \otimes Y$ has the same size as X and Y .

We define four entry-wise products as follows. Two of them are of size $(n - 1) \times n$: $\mathcal{F}_1 = Q \otimes F_1$ and $\mathcal{F}_2 = Q \otimes F_2$. The other two are square matrices of order $n - 1$: $\mathcal{R}_1 = \dot{Q} \otimes \dot{R}_1$ and $\mathcal{R}_2 = \dot{Q} \otimes \dot{R}_2$.

Let I be an $(n - 1) \times (n - 1)$ identity matrix, and U be a column vector of ones of size n . It is established in [8] that F^* in (3) computes the expected *single use failure probability* (or *single use unreliability*) from any starting state.

$$F^* = (I - \mathcal{R}_1)^{-1} \mathcal{F}_1 U \quad (3)$$

Observe that F^* is a column vector of size $n - 1$. The i -th component of F^* is the computed probability of failure (the expected value) for an arbitrary use of the system from a particular usage state, state i , to the sink (i runs from 1 to $n - 1$ inclusive; the starting state could be any transient state).

Therefore, the expected single use reliability of the system (starting from the source) is one minus the first component of F^* computed by (3).

For an intuitive understanding of (3), consider all the paths in the usage model that originate from state i and have all but the last step successful; the last step on the path is the only failure step. The probability of taking one of such paths gives the failure probability from state i , and is computed in three steps. First, it is shown in [8] that $(I - \mathcal{R}_1)^{-1} = \mathcal{R}_1^0 + \mathcal{R}_1^1 + \mathcal{R}_1^2 + \dots$, hence the (i, j) -th entry in the inverse matrix computes the probability of successfully moving from state i to state j in any finite and arbitrary number of steps (starting from 0 step). State j must be transient because only the last failure step could lead to the sink. Second, the inverse matrix is multiplied by the single-step failure matrix \mathcal{F}_1 to give the probability of moving from any transient state to any state in the model with all but the last step successful. Here the last transition is made to either a transient state or the sink. And last, the

product is multiplied by the vector of ones of appropriate size to sum up the probabilities of taking paths with a fixed starting state, all successful prior steps before encountering the last failure step, and an arbitrary ending state. The sum is the failure probability from the particular starting state.

An equation is also given in [8] for computing the variance associated with the single use reliability (or equivalently, the variance associated with the single use failure probability) from any starting state.

$$V^* = (I - \hat{\mathcal{R}}_2)^{-1} \mathcal{F}_2 U + 2(I - \hat{\mathcal{R}}_2)^{-1} (\hat{\mathcal{R}}_1 - \hat{\mathcal{R}}_2) F^* - F^* \otimes F^* \quad (4)$$

In (4) I is an $(n - 1) \times (n - 1)$ identity matrix, and U is a column vector of ones of size n . V^* as computed is a column vector of size $n - 1$. The i -th component of V^* is the computed variance associated with the single use reliability (or with the single use failure probability) starting from state i (i runs from 1 to $n - 1$ inclusive).

Therefore, the single use reliability variance (when starting from the source) is the first component of V^* computed by (4).

3 A Simpler, Faster, and More Intuitive Derivation

In this section we illustrate a new derivation of single use reliability mean and variance that is simpler, faster, and more intuitive than the old derivation. The new derivation of the mean was published in [2], however, back then it was unclear if there existed an alternative and new derivation of the variance that is similarly simple and intuitive. This is the major contribution of this paper. The solution was found through a rather convoluted path. What prompted us to look for an alternative derivation was the observation that the old derivation of the variance follows its definition and first principles, and therefore could be counter-intuitive to understand. The new derivation presented here completes a new analytical solution to compute the system reliability (both mean and variance) based on testing experience observed at the arc level taking into account the usage model structure.

We are able to compute the single use reliability mean (expected value) directly, and not through the single use failure probability (or single use unreliability) as follows.

We define another entry-wise product of size $(n - 1) \times n$: $\mathcal{R}'_1 = Q \otimes R_1$. Let W be \mathcal{R}'_1 restricted to the last column. W is a column vector of size $n - 1$.

We define R^* as follows:

$$R^* = (I - \hat{\mathcal{R}}_1)^{-1} W \quad (5)$$

(5) has an intuitive explanation. As explained above for (3), the (i, j) -th entry in the inverse matrix computes the probability of successfully moving from the transient state

i to the transient state j in any finite and arbitrary number of steps (starting from 0 step). When multiplied by the single-step success matrix \mathcal{R}_1 restricted to the last column (i.e., W), the last steps are successful steps leading to the sink, hence R^* gives the probability of successfully moving from any transient state to the sink in any finite and arbitrary number of steps (starting from 0 step).

Observe that R^* is a column vector of size $n - 1$. The i -th component of R^* is the expected single use reliability starting from state i (i runs from 1 to $n - 1$ inclusive).

Therefore, the expected single use reliability of the system (starting from the source) is the first component of R^* computed by (5).

We propose an alternative way to compute the single use reliability variance. Let r be a random variable denoting the single use reliability. Let p_i and r_i denote the probability and the reliability of the i -th distinct path starting with the source ending with the sink (representing a distinct arbitrary use), respectively. Note that r is a discrete random variable that takes the value $E(r_i)$ with probability p_i , hence $E(r) = \sum_i p_i E(r_i)$. The variance can be computed by $Var(r) = E(r^2) - E^2(r)$. The problem boils down to how to compute $E(r^2)$.

Note that r^2 is also a discrete random variable that takes the value $E(r_i^2)$ with probability p_i , hence $E(r^2) = \sum_i p_i E(r_i^2)$. We have shown how to compute $E(r)$ using (5). With the same Markov chain we are able to compute $E(r^2)$ similarly. Now the (i, j) -th arc is associated with a new random variable (i.e., $E[r_{i,j}^2]$) instead of $E[r_{i,j}]$. We can substitute R_1 for R_2 and compute $E(r^2)$ similarly as follows, with the reasonable assumption that all $r_{i,j}^2$ s are independent random variables.

We define an entry-wise product of size $(n - 1) \times n$: $\mathcal{R}'_1 = Q \otimes R_2$. Let W' be \mathcal{R}'_1 restricted to the last column. W' is a column vector of size $n - 1$.

We define R'^* as follows:

$$R'^* = (I - \hat{\mathcal{R}}'_1)^{-1} W' \quad (6)$$

We define V^* as:

$$V^* = R'^* - R^* \otimes R^* \quad (7)$$

V^* computes the single use reliability variance with each state being the starting state. The i -th component of V^* is the single use reliability variance starting from state i (i runs from 1 to $n - 1$ inclusive).

Therefore, the single use reliability variance starting from the source is the first component of V^* computed by (7).

To sum up, the following steps are needed to compute single use reliability mean and variance by our new derivation:

1. Determine Q and \hat{Q} from the usage model.
2. Determine A and B from prior success and failure counts for each arc in the usage model.

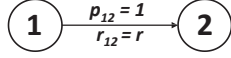


Figure 1. Example 1 of a Markov chain usage model. Arcs are annotated with transitional probabilities and arc reliabilities.

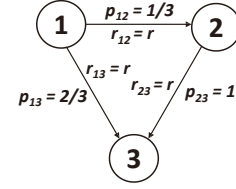


Figure 2. Example 2 of a Markov chain usage model. Arcs are annotated with transitional probabilities and arc reliabilities.

3. Determine S and F from observed success and failure counts for each arc in the usage model.
4. Compute R_1 and R_2 from A , B , S , and F .
5. Compute \mathcal{R}_1 and $\dot{\mathcal{R}}_1$, and W .
6. Compute R^* by (5).
7. Compute \mathcal{R}'_1 and $\dot{\mathcal{R}}'_1$, and W' .
8. Compute R'^* by (6).
9. Compute V^* by (7).
10. The expected value of the single use reliability is the first component of R^* .
11. The variance of the single use reliability is the first component of V^* .

Note that (1) the new derivation of the mean is simpler, faster, and more direct without the need to first compute the single use failure probability; (2) the new derivation of the variance is simpler and faster without the need to compute the second moments of arc failure probabilities; and (3) the new derivation of the variance is simpler, faster, and more intuitive by applying a well-known theorem to compute the variance (i.e., $Var(r) = E(r^2) - E^2(r)$), and by reusing the existing Markov chain and reusing with adaptation the formula we have derived for $E(r)$ to compute $E(r^2)$. We have implemented the new formulae in the JUMBL under a new analysis engine.

4 Examples

In all the three examples (Figures 1 – 3) below the arcs are annotated with transitional probabilities (i.e., the p_{ij} s) and arc reliabilities (i.e., the r_{ij} s). For simplicity we assume all arc reliabilities have a uniform distribution with the means as r_{ij} s and the variances as 0s for the derivations in Sections 4.1 – 4.3. SUR is for single use reliability.

4.1 Example 1

By the new formula:

To compute the mean:

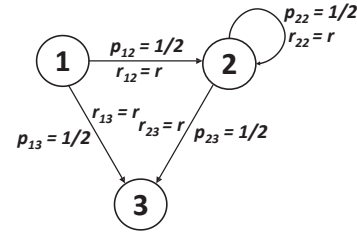


Figure 3. Example 3 of a Markov chain usage model. Arcs are annotated with transitional probabilities and arc reliabilities.

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Q = [0 \quad 1] \quad R_1 = [r_{11} \quad r]$$

$$\mathcal{R}_1 = Q \otimes R_1 = [0 \quad r] \quad \dot{\mathcal{R}}_1 = [0] \quad I = [1]$$

$$I - \dot{\mathcal{R}}_1 = [1] \quad (I - \dot{\mathcal{R}}_1)^{-1} = [1] \quad W = [r]$$

$$R^* = (I - \dot{\mathcal{R}}_1)^{-1} W = [r] \quad E(SUR) = r$$

To compute the variance:

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Q = [0 \quad 1] \quad R_1 = [r_{11} \quad r]$$

$$\mathcal{R}'_1 = Q \otimes R_2 = [0 \quad r^2] \quad \dot{\mathcal{R}}'_1 = [0] \quad I = [1]$$

$$I - \dot{\mathcal{R}}'_1 = [1] \quad (I - \dot{\mathcal{R}}'_1)^{-1} = [1] \quad W' = [r^2]$$

$$R'^* = (I - \dot{\mathcal{R}}'_1)^{-1} W' = [r^2] \quad E(SUR^2) = r^2$$

$$Var(SUR) = E(SUR^2) - (E(SUR))^2 = 0$$

We may also compute the single use reliability mean and variance directly based on its definition, given the Markov chain usage model. To compute the probability of a randomly chosen use (path) being successful, we compute the weighted sum of path reliabilities, with weights being the path probabilities.

By the definition of single use reliability:

To compute the mean:

$$E(SUR) = r * 1 = r$$

To compute the variance:

$$Var(SUR) = (r - r)^2 * 1 = 0$$

4.2 Example 2

By the new formula:

To compute the mean:

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 = \begin{bmatrix} r_{11} & r & r \\ r_{21} & r_{22} & r \end{bmatrix}$$

$$\mathcal{R}_1 = Q \otimes R_1 = \begin{bmatrix} 0 & \frac{r}{3} & \frac{2r}{3} \\ 0 & 0 & r \end{bmatrix} \quad \dot{\mathcal{R}}_1 = \begin{bmatrix} 0 & \frac{r}{3} \\ 0 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I - \dot{\mathcal{R}}_1 = \begin{bmatrix} 1 & -\frac{r}{3} \\ 0 & 1 \end{bmatrix} \quad (I - \dot{\mathcal{R}}_1)^{-1} = \begin{bmatrix} 1 & \frac{r}{3} \\ 0 & 1 \end{bmatrix} \quad W = \begin{bmatrix} \frac{2r}{3} \\ r \end{bmatrix}$$

$$R^* = (I - \dot{\mathcal{R}}_1)^{-1}W = \begin{bmatrix} \frac{2r+r^2}{3} \\ r \end{bmatrix} \quad E(SUR) = \frac{2r+r^2}{3}$$

To compute the variance:

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 = \begin{bmatrix} r_{11} & r & r \\ r_{21} & r_{22} & r \end{bmatrix}$$

$$\mathcal{R}'_1 = Q \otimes R_2 = \begin{bmatrix} 0 & \frac{r^2}{3} & \frac{2r^2}{3} \\ 0 & 0 & r^2 \end{bmatrix} \quad \dot{\mathcal{R}}'_1 = \begin{bmatrix} 0 & \frac{r^2}{3} \\ 0 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I - \dot{\mathcal{R}}'_1 = \begin{bmatrix} 1 & -\frac{r^2}{3} \\ 0 & 1 \end{bmatrix}$$

$$(I - \dot{\mathcal{R}}'_1)^{-1} = \begin{bmatrix} 1 & \frac{r^2}{3} \\ 0 & 1 \end{bmatrix} \quad W' = \begin{bmatrix} \frac{2r^2}{3} \\ r^2 \end{bmatrix}$$

$$R'^* = (I - \dot{\mathcal{R}}'_1)^{-1}W' = \begin{bmatrix} \frac{2r^2+r^4}{3} \\ r^2 \end{bmatrix} \quad E(SUR^2) = \frac{2r^2+r^4}{3}$$

$$\text{Var}(SUR) = E(SUR^2) - (E(SUR))^2 = \frac{2r^2+r^4}{3} - \left(\frac{2r+r^2}{3}\right)^2 = \frac{2r^2(r-1)^2}{9}$$

By the definition of single use reliability:

To compute the mean:

$$E(SUR) = r^2 * \frac{1}{3} + r * \frac{2}{3} = \frac{2r+r^2}{3}$$

To compute the variance:

$$\text{Var}(SUR) = (r - \frac{2r+r^2}{3})^2 * \frac{2}{3} + (r^2 - \frac{2r+r^2}{3})^2 * \frac{1}{3} = \frac{r^2(r-1)^2}{9} * \frac{2}{3} + \frac{4r^2(r-1)^2}{9} * \frac{1}{3} = \frac{2r^2(r-1)^2}{9}$$

4.3 Example 3

By the new formula:

To compute the mean:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad R_1 = \begin{bmatrix} r_{11} & r & r \\ r_{21} & r & r \end{bmatrix}$$

$$\mathcal{R}_1 = Q \otimes R_1 = \begin{bmatrix} 0 & \frac{r}{2} & \frac{r}{2} \\ 0 & \frac{r}{2} & \frac{r}{2} \end{bmatrix} \quad \dot{\mathcal{R}}_1 = \begin{bmatrix} 0 & \frac{r}{2} \\ 0 & \frac{r}{2} \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I - \dot{\mathcal{R}}_1 = \begin{bmatrix} 1 & -\frac{r}{2} \\ 0 & 1 - \frac{r}{2} \end{bmatrix}$$

$$(I - \dot{\mathcal{R}}_1)^{-1} = \frac{1}{1-\frac{r}{2}} \begin{bmatrix} 1 - \frac{r}{2} & \frac{r}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{r}{2-r} \\ 0 & \frac{2}{2-r} \end{bmatrix} \quad W = \begin{bmatrix} \frac{r}{2} \\ \frac{r}{2} \end{bmatrix}$$

$$R^* = (I - \dot{\mathcal{R}}_1)^{-1}W = \begin{bmatrix} \frac{r}{2} + \frac{r^2}{4-2r} \\ \frac{r}{2-r} \end{bmatrix} = \begin{bmatrix} \frac{r}{2-r} \\ \frac{r}{2-r} \end{bmatrix}$$

$$E(SUR) = \frac{r}{2-r}$$

To compute the variance:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad R_1 = \begin{bmatrix} r_{11} & r & r \\ r_{21} & r & r \end{bmatrix}$$

$$\mathcal{R}'_1 = Q \otimes R_2 = \begin{bmatrix} 0 & \frac{r^2}{2} & \frac{r^2}{2} \\ 0 & \frac{r^2}{2} & \frac{r^2}{2} \end{bmatrix} \quad \dot{\mathcal{R}}'_1 = \begin{bmatrix} 0 & \frac{r^2}{2} \\ 0 & \frac{r^2}{2} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I - \dot{\mathcal{R}}'_1 = \begin{bmatrix} 1 & -\frac{r^2}{2} \\ 0 & 1 - \frac{r^2}{2} \end{bmatrix}$$

$$(I - \dot{\mathcal{R}}'_1)^{-1} = \frac{1}{1-\frac{r^2}{2}} \begin{bmatrix} 1 - \frac{r^2}{2} & \frac{r^2}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{r^2}{2-r^2} \\ 0 & \frac{2}{2-r^2} \end{bmatrix}$$

$$W' = \begin{bmatrix} \frac{r^2}{2} \\ \frac{r^2}{2} \end{bmatrix}$$

$$R'^* = (I - \dot{\mathcal{R}}'_1)^{-1}W' = \begin{bmatrix} \frac{r^2}{2} + \frac{r^4}{4-2r^2} \\ \frac{r^2}{2-r^2} \end{bmatrix} = \begin{bmatrix} \frac{r^2}{2-r^2} \\ \frac{r^2}{2-r^2} \end{bmatrix}$$

$$E(SUR^2) = \frac{r^2}{2-r^2} \quad \text{Var}(SUR) = E(SUR^2) - (E(SUR))^2 = \frac{r^2}{2-r^2} - \left(\frac{r}{2-r}\right)^2 = \frac{2r^2(r-1)^2}{(2-r^2)(2-r)^2}$$

By the definition of single use reliability:

To compute the mean:

$$E(SUR) = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i \cdot r^i \cdot \frac{r}{2} \cdot \frac{r}{2} + \frac{r}{2} = \frac{r^2}{4-2r} + \frac{r}{2} = \frac{r}{2-r}$$

To compute the variance:

$$\begin{aligned} \text{Var}(SUR) &= (r - \frac{r}{2-r})^2 \cdot \frac{1}{2} + \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (r^i \cdot r \cdot r - \frac{r}{2-r})^2 \\ &= \frac{r^2}{2} \cdot \frac{(1-r)^2}{(2-r)^2} + \frac{r^2}{4} \cdot \sum_{i=0}^{\infty} (r^2 \cdot \left(\frac{r^2}{2}\right)^i + \frac{1}{(2-r)^2} \cdot \left(\frac{1}{2}\right)^i - \frac{2r}{2-r} \cdot \left(\frac{r}{2}\right)^i) \\ &= \frac{r^2}{2} \cdot \frac{(1-r)^2}{(2-r)^2} + \frac{r^2}{4} \cdot \left(\frac{2r^2}{(2-r)^2} - \frac{4r}{(2-r)^2} + \frac{2}{(2-r)^2}\right) \\ &= \frac{r^2}{2} \cdot \frac{(r^2-4r+2)(2-r)^2 + r^2(2-r)^2}{(2-r)^2(2-r)^2} = \frac{r^2}{2} \cdot \frac{4r^2-8r+4}{(2-r)^2(2-r)^2} \\ &= \frac{2r^2(r-1)^2}{(2-r^2)(2-r)^2} \end{aligned}$$

4.4 Experiments

We input the three examples in the JUMBL, and computed the single use reliability (SUR) means and variances using the old analysis as well as our new analysis. The results are summarized in Table 1. For each Markov chain usage model, we carried out the following steps for the experiments:

1. Generate a test suite that consists of minimum coverage test cases that cover every arc and every node of the model. The generated test suite happened to cover each arc exactly once (see Table 1).

Table 1. Single use reliabilities (means and variances) by the old and the new analyses using the JUMBL for Examples 1 – 3

	Example 1	Example 2	Example 3
Test Cases	1, 2	1, 3 1, 2, 3	1, 3 1, 2, 2, 3
SUR Mean (Old Derivation)	0.666666667	0.592592593	0.5
SUR Variance (Old Derivation)	$55.5555556E - 3$	$65.5006859E - 3$	$83.3333333E - 3$
SUR Mean (New Derivation)	0.666666667	0.592592593	0.5
SUR Variance (New Derivation)	$55.5555556E - 3$	$65.5006859E - 3$	$83.3333333E - 3$

- Record all tests as successful in the test suite, and run a test case analysis using the old engine to get the single use reliability mean and variance by the old derivation.
- With the same recorded test results run a test case analysis using the new engine to get the single use reliability mean and variance by the new derivation.

For each example, since each arc happened to be covered exactly once in the test suite, we have $s_{i,j} = 1, f_{ij} = 0$. Assuming no prior information $a_{i,j} = b_{i,j} = 1$. By (1) and (2) each arc reliability has a mean of $\frac{2}{3}$ and a variance of $\frac{1}{18}$. One can easily verify that if we plug in $r = \frac{2}{3}$ in the formulae we derived above for Examples 1 – 3, we get the same single use reliability means as shown in Table 1 (see the two rows for SUR mean). One can also verify for Example 1 that the single use reliability variance degenerates to the arc reliability variance (as there is only one arc in the path), i.e., $\frac{1}{18} = 55.5555556E - 3$.

We observe that for all the three examples (assuming an arc reliability mean of $\frac{2}{3}$ and an arc reliability variance of $\frac{1}{18}$ for every arc), our new derivation produces the same single use reliability mean and variance as the old derivation.

5 Conclusion

Statistical testing based on a Markov chain usage model has been well established in theory and proved sound and effective in practice [5, 4, 8, 6, 9, 12, 11], with tools available to support all the stages of testing and to automate the testing process [1, 7]. This paper presents a simpler, faster, more direct, and more intuitive derivation of the single use reliability mean and variance, following the arc-based Bayesian model [8, 10]. With our new theory single use reliability mean is obtained more directly without the need to first compute the single use failure probability. Single use reliability variance is obtained in a faster and simpler way applying a well-known theorem, without the need to compute the second moments of arc failure probabilities. We illustrate our new theory with three small Markov chain

usage models with manual derivations and experimental results.

Acknowledgements

This work was generously funded by Ontario Systems through the NSF Security and Software Engineering Research Center (S²ERC).

References

- [1] 2018. J Usage Model Builder Library (JUMBL). Software Quality Research Laboratory, The University of Tennessee. <http://jumbl.sourceforge.net/jumblTop.html>.
- [2] L. Lin, Y. Xue, and F. Song. A simpler and more direct derivation of system reliability using markov chain usage models. In *Proceedings of the 29th International Conference on Software Engineering and Knowledge Engineering*, pages 462–466, Pittsburgh, PA, 2017.
- [3] K. W. Miller, L. J. Morell, R. E. Noonan, S. K. Park, D. M. Nicol, B. W. Murrill, and J. M. Voas. Estimating the probability of failure when testing reveals no failures. *IEEE Transactions on Software Engineering*, 18(1):33–43, 1992.
- [4] J. H. Poore. Theory-practice-tools for automated statistical testing. *DoD Software Tech News: Model-Driven Development*, 12(4):20–24, 2010.
- [5] J. H. Poore, L. Lin, R. Eschbach, and T. Bauer. Automated statistical testing for embedded systems. In J. Zander, I. Schieferdecker, and P. J. Mosterman, editors, *Model-Based Testing for Embedded Systems in the Series on Computational Analysis and Synthesis, and Design of Dynamic Systems*. CRC Press-Taylor & Francis, 2011.
- [6] J. H. Poore and C. J. Trammell. Application of statistical science to testing and evaluating software intensive systems. In M. L. Cohen, D. L. Steffey, and J. E. Rolph, editors, *Statistics, Testing, and Defense Acquisition: Background Papers*. National Academies Press, 1999.
- [7] S. J. Prowell. JUMBL: A tool for model-based statistical testing. In *Proceedings of the 36th Annual Hawaii International Conference on System Sciences*, page 337c, Big Island, HI, 2003.
- [8] S. J. Prowell and J. H. Poore. Computing system reliability using Markov chain usage models. *Journal of Systems and Software*, 40(4):199–222, 2004.
- [9] S. J. Prowell, C. J. Trammell, R. C. Linger, and J. H. Poore. *Cleanroom Software Engineering: Technology and Process*. Addison-Wesley, Reading, MA, 1999.
- [10] K. Sayre and J. H. Poore. A reliability estimator for model based software testing. In *Proceedings of the 13th International Symposium on Software Reliability Engineering*, pages 53–63, Annapolis, MD, 2002.
- [11] J. A. Whittaker and J. H. Poore. Markov analysis of software specifications. *ACM Transactions on Software Engineering and Methodology*, 2(1):93–106, 1993.
- [12] J. A. Whittaker and M. G. Thomason. A Markov chain model for statistical software testing. *IEEE Transactions on Software Engineering*, 30(10):812–824, 1994.