LOCALLY COMPACT PROPERTY A GROUPS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Amanda M. Harsy Ramsay

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2014

Purdue University

West Lafayette, Indiana

ACKNOWLEDGMENTS

I want to thank my advisor, Dr. Ji, for his guidance and support during my time at IUPUI. I also want to thank my husband, Benjamin, and my parents for their loving support and encouragement throughout my years of schooling.

TABLE OF CONTENTS

SYMBOLS

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
E^*	the dual space of E
$B\Gamma$	the classifying space of Γ, the quotient of a topological space for which all its homotopy groups are trivial by a free action of Γ
$C(X)$	the space of complex valued, continuous functions on$X$
$C_c(X)$	the space of complex valued, continuous functions on X with compact support
$C(X)^+$	the space of positive, continuous functions on X
$C_0(X)$	the space of continuous functions on X vanishing at infinity
$C_r^*(G)$	the reduced C^* algebra of G
$G(x)$	the stabilizer of x
$G^{(2)}$	the set of composable pairs of groupoid G
$G^{(0)}$	the unit space of groupoid G
$H_b^*(G, E^*)$	the bounded group cohomology of G with coefficients in E^*
$\mathscr{I}(C(X)W)$	the space of integral operators with values in W
[J]	the Johnson class
$K_*(X)$	K-group of X
$L^\infty(G)$	the set of all essentially bounded functions on G
$N >> 1$	for large enough $N \in \mathbb{N}$
$P(G)$	the set of all probability measures on G
$P_d(X)$	Rips Complex
V	the space of all functions $f : G \to C(X)$ with the norm $\|\|f\|\|_V = \sup_{x\in X} \sum_{g\in G} \|f_g(x)\|$

$W_{00}(G, X)$	subspace of all functions from X to $C(X)$ with finite support and such that for some constant $c = c(f) \in \mathbb{R}$, $\sum_{g \in G} f_g = c1_X$
$W_0(G, X)$	the closure of $W_{00}(G, X)$ in the V norm.
$B_\beta(G, X)^+$	the linear space of positive Borel functions on G such that $\beta\|f\|$ is bounded given β, a Borel system of measures β for a Borel map $f : G \to X$
βG	the Stone-Cech compactification
$\beta^\mu(G)$	the universal compact Hausdorff left G-space
χ_A	the characteristic function on A
1_X	the constant function 1 on X
$A \Delta B$	$(A \cap B^c) \cup (B \cap A^c)$

ABSTRACT

Harsy Ramsay, Amanda M. Ph.D., Purdue University, May 2014. Locally Compact Property A Groups. Major Professor: Ronghui Ji.

In 1970, Serge Novikov made a statement which is now called, "The Novikov Conjecture" and is considered to be one of the major open problems in topology. This statement was motivated by the endeavor to understand manifolds of arbitrary dimensions by relating the surgery map with the homology of the fundamental group of the manifold, which becomes difficult for manifolds of dimension greater than two. The Novikov Conjecture is interesting because it comes up in problems in many different branches of mathematics like algebra, analysis, K-theory, differential geometry, operator algebras and representation theory. Yu later proved the Novikov Conjecture holds for all closed manifolds with discrete fundamental groups that are coarsely embeddable into a Hilbert space. The class of groups that are uniformly embeddable into Hilbert Spaces includes groups of Property A which were introduced by Yu.

In fact, Property A is generally a property of metric spaces and is stable under quasi-isometry. In this thesis, a new version of Yu's Property A in the case of locally compact groups is introduced. This new notion of Property A coincides with Yu's Property A in the case of discrete groups, but is different in the case of general locally compact groups. In particular, Gromov's locally compact hyperbolic groups is of Property A.

1. INTRODUCTION

1.1 Motivation

In 1970, Serge Novikov made a statement which is now considered to be one of the major open problems in topology and is called, "The Novikov Conjecture" [1]. The motivation for this statement derived from the endeavor to "classify" manifolds of arbitrary dimensions. More precisely, the Novikov conjecture states that the higher signatures of manifolds with fundamental group G are homotopy invariant. This "classification" becomes difficult for manifolds of dimension greater than two. This partly has to do with the fact that there are too many possibilities for the fundamental group. The Novikov Conjecture helps with classification by relating the surgery map with the homology of the fundamental group of the manifold. This conjecture is interesting because it appears in problems in many different branches of mathematics like algebra, C*-Algebras, K-theory, analysis, differential geometry, operator algebras and representation theory [1]. We include the precise statement of the Novikov Conjecture below.

Conjecture 1 (The Novikov Conjecture:) *The higher signatures determined by a discrete group* Γ *are homotopy invariant. That is, for every rational cohomology class* $x \in H^*(B\Gamma;\mathbb{Q})$*, for every orientation preserving homotopy equivalence of closed oriented manifolds* $f : N \to M$*, and for every map* $g : M \to B\Gamma$*,*

$$\textit{signature}_x(M, g) = \textit{signature}_x(N, g \circ f) \in \mathbb{Q}.$$

One example of the interesting connection between the Novikov Conjecture and K-theory shows up in The Baum-Connes Conjecture. The Baum-Connes Conjecture connects the K-homology of the corresponding classifying space of proper actions of a group and the K-theory of the reduced C*-algebra of that group. The main result of the Baum-Connes Conjecture is that it implies the Novikov Conjecture.

Conjecture 2 (Baum-Connes Conjecture:) *The assembly map* μ $:K_*^\Gamma(\mathcal{E}\Gamma) \to K_*(C_r^*(\Gamma))$ *is an isomorphism for any discrete group* Γ*, where* $\mathcal{E}\Gamma$ *denotes the universal proper* Γ*-space [2].*

We also have a coarse version of the Baum-Connes Conjecture that involves discrete metric spaces called the Coarse Baum-Connes Conjecture. This conjecture connects the the locally finite K-homology group of the Rips complex with the C^* algebra of the Rips Complex.

Conjecture 3 (The Coarse Baum-Connes Conjecture:) *If* X *is a discrete metric space with bounded geometry and* $P_d(X)$ *is the Rips Complex for* $d > 0$*, then the index map from* $\lim_{d\to\infty} K_*(P_d(X)) \to \lim_{d\to\infty} K_*(C^*P_d(X))$ *is an isomorphism.*

In this case, $K_*(P_d(X))$ is the locally finite K-homology group of $P_d(X)$. Recall for a metric space X, and real number $d > 0$, the Rips Complex, $P_d(X)$, is the simplicial complex formed in which a simplex $\sigma \in P_d(X)$ if and only if $d(x, y) \leq d$ for each pair of vertices of σ. The main result of the Coarse Baum-Connes Conjecture is that it implies the Novikov Conjecture.

Theorem 1.1.1 (Higson, Roe, & Yu) *The Coarse Baum-Connes Conjecture implies the Novikov Conjecture.*

Yu later proved the Coase Baum-Connes conjecture (and therefore the Novikov Conjecture) holds for any bounded geometry metric space which is coarsely embeddable into a Hilbert space. This is a part of coarse geometry, a geometry that looks at the large-scale properties of spaces. Thus, in coarse geometry, we just need our embedding to be close to structure preserving and isomorphic. We define coarse embedding in the following way:

Definition 1.1.1 ([3]) *Given metric spaces X and Y, we say $f : X \to Y$ is a coarse embedding between X and Y if there exists nondecreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying*

1. $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$, *and*

2. $\lim_{t\to\infty} \rho_1(t), \rho_2(t) = +\infty$.

We can think of this in the following way, a space is coarsely embeddable into a Hilbert space if, when we squint, our space looks like a Hilbert space. We now state the following result from Yu:

Theorem 1.1.2 (Yu) *If a discrete group Γ is coarsely embeddable into a Hilbert space, then the Coarse Baum-Connes Conjecture holds for Γ.*

Yu also introduced Property A for metric spaces that implies coarse embedding of the space. Yu's original definition for this property is for a discrete metric space X and we state it below.

Definition 1.1.2 *A discrete metric space X has Property A if for all positive R and ε, there exists a positive number S and a family of finite non-empty subsets A_x in $X \times \mathbb{N}$, with $x \in X$ as the index, such that*

1. *for all x and x' with $d(x, x') < R$,* $\dfrac{|A_x \triangle A'_x|}{|A_x|} < \varepsilon$, *and*

2. *for all (x', n) in A_x, $d(x, x') < S$.*

Property A is a metric space property, but if we can construct a metric on a group by defining a length function on the generators of the group, we can think of our group as a metric space. In fact, if a discrete group has Yu's Property A, it is coarsely embeddable into a Hilbert space and thus the Novikov Conjecture holds for all closed manifolds with that fundamental group.

Theorem 1.1.3 (Yu) *If discrete group Γ has Property A, then Γ can be coarsely embedded into a Hilbert space.*

Theorem 1.1.4 (Yu) *If discrete group Γ has Property A, then the Novikov Conjecture is true for all closed manifolds with fundamental group Γ.*

Basically Property A is coarse version of amenability. That is, in a large geometry sense, our space does not change much with translation. Most of the work with Property A so far has been with discrete metric spaces. In this thesis, we introduce the concept of locally compact Property A groups. Recall that Property A is a metric space property, but we can construct a metric on a group by defining a length function on the generators of the group. Since we don't have the discrete topology, we need a few more conditions to define Property A for locally compact groups.

Definition 1.1.3 *We say a locally compact group, G, has property A if there exists a compact space X such that G acts amenably and continuously on X.*

Our main example is a locally compact group G which acts continuously and properly on a locally compact hyperbolic space X. In this case, we can define the G action on the boundary of X, ∂X in the following way: for $x \in \partial X$, $g \in G$, $g \cdot x = \lim_{n\to\infty} g \cdot \alpha_n$ where α_n is a geodesic in X such that $\alpha_n \to x$. In order to prove our theorem, we first show that a continuous group action on a locally compact hyperbolic space can be continuously extended to a group action on the Gromov boundary of that space. We then show that G acts amenably on ∂X.

1.2 Outline

We start Chapter 2 with a basic introduction of amenable groups and include a few definitions and examples of such groups. In particular, we discuss amenable groupoids and transformation groups. We then introduce a non-equivariant generalization of amenability called Yu's Property A. Some basic definitions for metric spaces with Property A will be given along with other characterizations of Property A. We also discuss how we can define Property A for discrete groups. In particular, we discuss notions of amenable actions and groups that are amenable at infinity. We close Chapter 2 with Monod's cohomological characterizations for Property A.

Chapter 3 gives some basic definitions and properties of locally compact hyperbolic spaces and groups. We prove that if a group acts on a locally compact hyperbolic space continuously, we can extend this action to a continuous action on the Gromov boundary of the hyperbolic space. Chapter 3 closes with proofs of some geometric properties of hyperbolic groups which will be used later in Chapter 5.

In Chapter 4, we discuss fundamental domains for group actions and show how we can construct a fundamental domain for a locally compact group that acts continuously and properly on a space. We then show how we can use the fundamental domain of a G-space to define a G-invariant measure on the space.

We begin Chapter 5 with introducing our definition of Locally Compact Property A Groups. We then show that locally compact Gromov hyperbolic groups have Property A. To do this, we prove that a locally compact group which acts continuously and properly on a locally compact hyperbolic space, X, acts amenably on the Gromov boundary of X, ∂X. We close Chapter 5 by showing that our definition for Property A is equivalent to Deprez and Li's definition of property A groups and Delaroche's concept of amenable at infinity.

2. AMENABLE ACTIONS AND YU'S PROPERTY A

2.1 Amenability

2.1.1 Basics

Amenable groups were originally introduced by John von Neumann in the late 1920's in response to the Banach-Tarski paradox. This paradox states that given a solid ball in a three dimensional space, you can decompose the ball into a finite number of non-overlapping "pieces" (think disjoint subsets) so that you can reassemble the ball without expanding or stretching in such a way that it yields two identical copies of the ball. That is, we can find a countable subgroup of $SO(3)$, the group of rotation operators on $\mathbb{R}^3$, such that the subgroup can be dissembled into four disjoint subgroups that can be reassembled into two copies of itself after rotation [4]. John von Neumann wanted to explore the properties of groups which did not have this pathological property. He found that these non-pathological groups did not change that much under translation. He named these groups "amenable" which comes from the German word, "measurable." So we have that a group is amenable if and only if it is not paradoxical [5]. In other words, an amenable group is a locally compact group that does not change much under translation. During this same time, others were asking the question whether there exists a finitely additive measure that is preserved under group translations. We have a more precise definition of amenability is stated below in terms of a left invariant linear functional. Notice that this left invariant mean, M, can be defined on the characteristic functions of subgroups which allows us to have such a measure.

Definition 2.1.1 ([6]) *A locally compact group G is amenable if it admits a left-invariant mean on $L^\infty(G)$, the space of essentially bounded functions on G. That is, there exists a linear functional M such that for all f in $L^\infty(G)$,*

1. $M(\chi_G) = 1$,
2. $M(f) \geq 0$, *if* $f \geq 0$, *and*
3. $M(af) = M(f)\ \forall a \in G$.

There are many other ways of characterizing amenable groups and one of the most common characterizations is Følner's Condition which demonstrates that an amenable group does not change much under translation.

Definition 2.1.2 (Følner's Condition for Amenability) *G has Følner's Condition if for any compact subset $K \subset G$ and $\varepsilon > 0$, there exists a measurable subset $U \subset G$, such that $\frac{|aU \Delta U|}{|U|} < \varepsilon$ for all $a \in K$.*

Often people say that a group is amenable if it admits a Følner sequence. This means for any compact subset $K \subset G$, there exists a sequence $< U_n >$ of measurable subsets such that $0 < |U_n| < \infty$ for all n and $\lim_{n\to\infty} \frac{|KU_n \Delta U_n|}{|U_n|} = 0$.

Polynomial growth and exponentially boundedness also imply amenability. But a group with unbounded growth is not amenable.

Definition 2.1.3 *G has polynomial growth if for every compact neighborhood V of e, the identity element of G, there exists a $d \in \mathbb{N}$ such that $\limsup_{n\to\infty} \frac{|V^n|}{n^d} < \infty$.*

Definition 2.1.4 *G is exponentially bounded if for every compact neighborhood V of the identity element $e \in G$, and for all $t \in (1, \infty)$, $\limsup_{n\to\infty} \frac{|V^n|}{t^n} < \infty$.*

There are many examples of amenable groups. Any locally compact abelian group G is amenable. Every locally compact solvable group is amenable and all compact groups have polynomial growth and thus are amenable. The free product $\mathbb{Z}_2 * \mathbb{Z}_2$

is also amenable since it has polynomial growth, and closed subgroups of amenable groups are amenable. In addition to this, we have that if π is a continuous, surjective homomorphism from amenable group G into locally compact H, then H is amenable. Moreover, if H is a closed, normal subgroup of G and both H and $\frac{G}{H}$ are amenable, then G itself is amenable. We also have if a locally compact group with the discrete topology is amenable, then it is amenable for all weaker topologies. Yet the converse for this statement is not true. The compact (amenable) rotation group $SO(3,\mathbb{R})$ of the unit sphere in $\mathbb{R}^3$ is not amenable with the discrete topology because we can find a closed non-amenable subgroup [6].

Of course there are non-examples of amenable groups. The free group of two generators has unbounded growth and is not amenable. In fact, a discrete free group that is freely generated by $\{a_1, a_2, ..., a_n\} \in G$ where $n \geq 2$ of orders p_i for $i = 1, ..., n$ is not amenable unless $n = 2$ and $p_1 = p_2 = 2$. $SL(2,\mathbb{R})$ contains a free group of two generators as a closed subgroup and thus is not amenable. This also means that $GL(2,\mathbb{R}), SL(n,\mathbb{C})$, and $GL(n,\mathbb{C})$ are not amenable [6]. And still there are some groups in which we cannot yet determine whether they are amenable or not. For example, it is still an open question whether Thompson's Group F is amenable. We already know it is not elementary amenable and if it is no amenable, then it will give another counterexample to von Neumann's Conjecture. This conjecture, which was disproved in 1980, proposed that group is not amenable if and only if it contains the free group of two generators as a subgroup. It was shown that both the Tarski monster group and certain Burnside groups are non-amenable and do not have the free group of two generators as a subgroup.

2.1.2 Amenable Groupoids and Transformation Groups

The notion of amenability has generalizations in many other branches of mathematics. We now can discuss amenable semi-groups, foliations, Banach algebras, C*-algebras, quantum groups and more. In [5], Claire Delaroche discusses amenable groupoids and their connection with the Novikov Conjecture for discrete groups. Recall the following definition of a groupoid:

Definition 2.1.5 (Groupoid [7]) *Let G be a set and $G^{(2)} \subset G \times G$, where $G^{(2)}$ is the set of composable pairs of G. We say G is a groupoid if there is a map from $G^{(2)} \to G$ with $(x, y) \to xy$ and an involution $G \to G$ with $x \to x^{-1}$ such that*

1. *Our map from $G^{(2)} \to G$ is associative, and*
2. *$\forall x \in G$, $(x^{-1}, x) \in G^{(2)}$ and $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.*

Corollary 2.1.1 (Amenable Groupoid Characterization [5]) *A locally compact groupoid, G, with a continuous Haar system λ and countable orbits is topologically amenable if and only if there exists a sequence $< f_n >$ of Borel functions in $B_\beta(G, \lambda)^+$ with $\lambda(f_n) > 0$ for all n and such that $\lim_n \frac{\int |f_n(g^{-1}g_1) - f_n(g_1)| d\lambda^{r(g)}(g_1)}{\lambda^{r(g)}(f_n)} = 0$ for all $g \in G$. Note that $B_\beta(G, \lambda)^+$ represents the linear space of positive Borel functions on G such that $\beta|f|$ is bounded given a Borel system of measures β. And we define a continuous Haar system λ as a family of measures $\{\lambda^x\}$ on G indexed by $x \in G^{(0)}$ where $G^{(0)}$ is the unit space such that $\mathrm{supp}\{\lambda^{\mathrm{x}}\} = \mathrm{G}^{\mathrm{x}}$ and for every $f \in C_c(G)$, the function $\lambda(f) : x \to \lambda^x(f)$ is continuous and invariant.*

Once we define amenable groupoids, it is a natural progression to discuss amenable transformation groups. Recall that given a countable discrete or locally compact group G, we can define a topological G-space to be a topological space X together with a continuous group action on it.

Definition 2.1.6 (Transformation Group [8]) *A transformation group (X, G) is a left $G-space$, where G is a locally compact group, X is a locally compact space, and $(x, g) \to g.x$ is a continuous left action from $X \times G$ to X.*

Definition 2.1.7 (Amenable Transformation Group [8]) *We have an amenable transformation group, (X, G), if the G-action on X is amenable. That is, if there exists a net $< f_i >_{i \in I}$ of continuous maps $x \to f_i^x$ from $X \to P(G)$, where $P(G)$ is* *the set of all probability measures on G equipped with the weak* topology, such that* $\lim_i ||g.f_i^x - f_i^{g.x}||_1 = 0$ *uniformly on compact subsets of* $X \times G$.

In [8], Delaroche proves several characterizations for amenable transformation groups. We add the following theorem for the convenience of the reader.

Theorem 2.1.2 (Delaroche) *The following are equivalent:*

1. (X, G) *is an amenable transformation group.*

2. *There exists a net* $< f_i > \in C_c(X \times G)$*, the space of complex valued continuous functions with compact support on* G*, such that*

   (a) $\lim_i \int_G |f_i(x,t)|^2 dt = 1$ *uniformly on compact subsets of* X*, and*

   (b) $\lim_i \int_G |f_i(sx, st) - f_i(x,t)|^2 dt = 0$ *uniformly on compact subsets of* $X \times G$.

3. *There exists a net* $< h_i >$ *of positive type functions in* $C_c(X \times G)$ *such that* $\lim_i h_i = 1$ *on compact subsets of* $X \times G$.

Ozawa has a similar definition for group acting amenably on a compact Hausdorff space and we state it below.

Definition 2.1.8 ([9]) *Given a compact Hausdorff space X, we say G acts amenably on X if there exists a sequence of continuous functions* $< \mu_n >$ *from* $X \to P_c(G)$ *where* $x \to \mu_n^x$ *such that for each* $g \in G$, $\lim_{n\to\infty} \sup_{x \in X} ||g.\mu_n^x - \mu_n^{g.x}|| = 0$*. Note* $P_c(G)$ *is the probability space of G equipped with the point-wise convergence topology which coincides with the norm topology.*

Notice that when X is just reduced to a point, Ozawa's definition coincides with saying our group G is amenable. Thus, if G is amenable, then every G-space is amenable. Furthermore if X is an amenable G-space that has an invariant Radon probability measure, then G is also amenable [9].

We can look at amenable actions in many different situations. We say a locally compact group G is amenable at infinity if it admits an amenable action on a compact space X. This would mean that the groupoid $X \rtimes G$ is amenable. An amenable locally compact group is amenable at infinity since we can take our compact space X to just be a point. In fact, there are cases of non-amenable groups which are amenable at infinity. For example, a discrete Gromov hyperbolic group is amenable at infinity since it acts amenably on its compact Gromov Boundary [8].

We can also connect amenability with exactness of a group. We say that a countable, discrete group is exact if there is a compact G-space X which is amenable. So in other words, the group, G, acts amenably on a compact space X. This is the definition of a group being amenable at infinity [8]. In fact, a finitely generated discrete group G is exact if and only if the Stone-Cech compactification βG is amenable [9].

2.2 Property A

2.2.1 Property A for Discrete Spaces

When exploring the Novikov Conjecture, Yu found that a bounded geometry metric space which is coarsely embeddable into a Hilbert space satisfies the Coarse Baum-Connes Conjecture. Yu introduced Property A which implies coarse embedding and therefore the Coarse Baum-Connes Conjecture. Yu's original definition for this property is for a discrete metric space X, but we will discuss how it can be generalized to groups with a length function as well.

Definition 2.2.1 (Property A for Discrete Spaces) *A discrete metric space X has Property A if for all positive R and ε, there exists a positive number S and a family of finite non-empty subsets A_x in $X \times \mathbb{N}$, with $x \in X$ as the index, such that*

1. *for all* x *and* x' *with* $d(x, x') < R$, $\dfrac{|A_x \triangle A_{x'}|}{|A_x|} < \varepsilon$, *and*

2. *for all* (x', n) *in* A_x, $d(x, x') < S$.

Property A is a metric space property, but if we can construct a metric on a group by defining a length function on the generators of the group, we can think of our group as a metric space. In fact, if a discrete group has Yu's Property A, as we mentioned in the introduction, it is coarsely embeddable into a Hilbert space and thus the Novikov Conjecture holds for all closed manifolds with that fundamental group. Like amenability, there are many equivalent characterizations of Property A. The following property gives an equivalent definition for Property A which is a little more useful than the original definition:

Proposition 2.2.1 *We say a metric space* (X, d) *has property A if for all* $\varepsilon > 0$ *and* $R > 0$, *there is a* $S > 0$ *and a map* $f : X \to P(X)$, *such that*

1. $||f_x - f_y|| \leq \varepsilon \ \forall x, y \in X$ *with* $d(x, y) < R$ *and*

2. *the support of* f_x *is a subset of* $\{y : d(x, y) < S\}$ *for every* $x \in X$.

Yu also found that a discrete metric space with bounded geometry and finite asymptotic dimension has Property A. Recall that a metric space has bounded geometry if there exists a map $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x \in X$, the number of elements in the R-ball, $B(x, R)$ is less than or equal to $N(R)$. In other words, every ball of radius R is uniformly bounded [10].

2.2.2 Property A for Discrete Groups

We would like to look at how we can define a metric on a group. On one hand, we can define a length function on G using a left translation invariant metric, d, on G. We will assume that any subset of G with finite diameter is finite and, in this sense, we say d is a proper metric. Now we can define a proper length function, ℓ, on G by putting $\ell(g) = d(g, e)$ where e is the identity element of G. On the other hand, we can define a left translation invariant metric on a group G if we have a proper length function ℓ. We can do this by defining $d(g, h) = \ell(h^{-1}g)$. Notice this metric is in fact left invariant under the group action since $d(g'g, g'h) = \ell(h^1 g'^{-1} g'g) = \ell(h^{-1}g) = d(g, h)$. For example, if G is generated by a finite number of elements in a symmetric set $S = S^{-1}$ we can define the length of $g \in G$ as $\ell(g) = \min_{k \in \mathbb{N}} \{g = s_{k_1} s_{k_2} \ldots s_{k_n},\ s_i \in S\}$ which basically means that we count the number of generators it takes to write g. This allows us to now think of our group G as a metric space.

Property A is a generalization of amenability. One can prove that amenability implies Property A using the Følner characterization of amenability. Recall G has Følner's Condition if for any compact subset $K \subset G$ and $\varepsilon > 0$, there exists a measurable subset $F \subset G$, such that $\dfrac{|aF \Delta F|}{|F|} < \varepsilon$ for all a in K. To show that an amenable group has property A, let $R > 0, \varepsilon > 0$. Let $\mathbb{N} = 1$ and consider our finite family of sets $\{A_g\}$ to be our Følner sets $\{F_g\}$.

Property A is also connected to the exactness of a group. Ozawa proved that Property A and exactness are equivalent for countable discrete groups with a proper left-invariant metric [11]. Recall, a locally compact G is exact if for every G-equivariant exact sequence $0 \to I \to A \to A/I \to 0$ of G-C* algebras, the sequence $0 \to C_r^*(G, I) \to C_r^*(G, A) \to C_r^*(G, A/I) \to 0$ is exact. Ozawa showed that Property A is equivalent to exactness of the reduced C* algebra of G, $C_r^*(G)$. We now state the following theorem from [9] which connects Property A with exactness.

Theorem 2.2.1 *For a countable discrete group G, TFAE*

1. *G is exact.*

2. *Given the left translation invariant metric d on G, the metric space (G, d) has Property A. That is $\forall \varepsilon > 0$ and $R > 0$, $\exists S > 0$ and a map $f : G \to P(G)$ such*

that $||f_x - f_y|| \leq \varepsilon \ \forall x, y \in G$ *with* $d(x, y) < R$ *and* $supp\{f_x\} \subset \{y : d(x, y) < S\}$ *for every* $x \in G$.

3. *For every* $\varepsilon > 0$ *and* $R > 0$, *there exists an* $S > 0$ *and a Hilbert space* H *and a map* $f : G \to H$ *such that* $|1- < f_x, f_y > | < \varepsilon$ *for every* $x, y \in G$ *with* $d(x, y) < R$, *and* $< f_x, f_y >= 0 \ \forall x, y \in G$ *with* $d(x, y) \geq S$.

We also have that if G is exact, then G is coarsely isomorphic to a subset of a Hilbert space. Not all groups are exact though. It is an open problem to determine whether Thompson's group, OUT(F_r), 3-manifold groups, groups of homeomorphisms or diffeomorphisms on the circle, or Burnside groups are exact [9].

Some examples of Property A groups are amenable groups, discrete Gromov hyperbolic groups, relative hyperbolic groups, and coxeter groups. Notice that although Gromov hyperbolic groups are not amenable, they are of Property A since they act amenably on their Gromov boundary. Free groups are also of Property A. For example the free group of two generators, F_2 is of Property A. In [12], they show how we can use the original definition from Yu and the Cayley graph of F_2 to show that F_2 is of Property A. Consider the Cayley graph of F_2, which is a tree, and fix a geodesic ray starting from e, the identity element of F_2. We can now define our family of sets in $F_2 \times \mathbb{N}$ in the following way. For a fixed $n \in \mathbb{N}$, we define A_x to be the unique geodesic segment of length $2n$ from x in the direction of our fixed ray. Then for each ε and $R > 0$, we pick a "good" n in $\mathbb{N}$ such that for every x and x' with $d(x, x') < R$, $\frac{|A_x \Delta A_{x'}|}{|A_x|} < \varepsilon$. We also have an S such that for every (x', n) in A_x, $d(x, x') < S$. In this case our S is just $2n$.

Nonexamples of Property A can be difficult to construct. Obviously, if a group does not embed into a Hilbert space, it is not of Property A. One example is Gromov's Monster Group. Another example is a residually finite, countably infinite discrete group with property T [13]. In [14], Arzhantseva, Guentner, and Spakula constructed the first example of a metric space with bounded geometry which does not have Property A even though it coarsely embeds into a Hilbert space. They start with a discrete group Γ and let $\Gamma^{(2)}$ represent the normal subgroup generate by all the squares of elements of Γ. They show in [14] that the box space of F_2 associated with the family (Γ_n), $X = \sqcup_{n=0}^{\infty} X_n$ coarsely embeds into a Hilbert space but is not of Property A. Here X_n is the Cayley graph of F_2/Γ_n with respect to the image of the canonical generators of the the free group of two generators, F_2 and $\Gamma_0 = F_2$, $\Gamma_n = \Gamma_{n-1}^{(2)}$ [14]. In [13] they also construct a group that embeds into a Hilbert space, but does not have Property A. They starts with a non-trivial, finite (hence amenable) group G and defines $\chi_G = \sqcup_{n=1}^{\infty} G^n$ with a metric, d, that restricts to the ℓ^1 metric on G^n. It turns out that χ_G coarsely embeds into ℓ^2, but does not have Property A [13].

2.2.3 Amenable Action Characterization for Property A

We can also use amenable actions to characterize Yu's Property A.

Definition 2.2.2 (Amenable Group Action) *The action of discrete group G on X is amenable if there exists an invariant mean for the action. That is, there is a function in the closure of the space of functions from $G \to C(X)$ with finite support such that $\sum_{g \in G} f_g = c1_x$ for some constant c and such that $f(g\phi) = f(\phi)$ $\forall g \in G$.*

Higson and Roe proved that a finitely generated discrete group has Property A if and only if the G action on its Stone-Cech compactification is topologically amenable. In [8], Delaroche defines the notion of "amenable at infinity" which describes a locally compact group acting amenably on a compact space.

Definition 2.2.3 (Amenable at Infinity) *A locally compact group G is said to be amenable at infinity, if there exists a compact space X such that G acts amenably on X.*

For example, we can see that a Gromov hyperbolic discrete group is amenable at infinity since it acts amenably on its Gromov boundary. From this notion of amenable at infinity, we can characterize groups that act on a compact space. Before we state Delaroche's theorem and include the proof, we first provide the definition of a tube.

Definition 2.2.4 *Let G be a locally compact group with compact subset K, then we have Tube(K)=$\{(g,h) \in G \times G : g^{-1}h \in K\}$. And we say that a subset $L \subseteq G \times G$ is a tube if $\{g^{-1}h : (g,h) \in L\}$ is precompact or if L is a subset of some other tube.*

Theorem 2.2.2 ([8]) *Given a locally compact group G, the following are equivalent:*

1. *G is amenable at infinity.*

2. *G acts amenably on $\beta^{\mu}(G)$. We define $\beta^{\mu}(G)$ in the following definition.*

3. *There exists a net $< f_i >$ of nonnegative functions in $C_{b,\theta}(G \times G)$ with support in a tube such that for each i and each h, $\int_G f_i(h,g)dg = 1$ and $\lim_i \int_G |f_i(h,g) - f_i(h,s)|dg = 0$ uniformly on tubes.*

4. *There exists a net $< \xi_i >$ of functions in $C_{b,\theta}(G \times G)$ with support in a tube such that for each i and h, $\int_G |\xi_i(h,g)|^2 dg = 1$ and $\lim_i \int_G |\xi(h,g) - \xi(s,g)|^2 dg = 0$ uniformly on tubes.*

5. *There exists a sequence $< h_i >$ of positive type kernels in $C_{b,\theta}(G \times G)$ with support in a tube such that $\lim_i h_i = 1$ uniformly on tubes.*

Before we provide the proof of this theorem, we first define some terminology.

Definition 2.2.5 ([15]) *We let $\beta^{\mu}(G)$ denote the universal compact Hausdorff left G-space equipped with a continuous G-equivariant inclusion of G as an open dense subset with the following universal property. Any continuous G-equivariant map from G into a compact Hausdorff left G-space X has a unique extension to a continuous G-equivariant map from $\beta^{\mu}(G)$ into X.*

Definition 2.2.6 ([8]) *Given θ the homeomorphism of $G \times G$ such that $\theta(g,h) = (g^{-1}, g^{-1}h)$, we define $C_{b,\theta}(G \times G)$ to be the algebra of continuous, bounded functions f on $G \times G$ such that $f \circ \theta$ is the restriction of a continuous function on $\beta^{\mu}(G) \times G$.*

Definition 2.2.7 *A positive type kernel on $G \times G$ is a function k such that for every positive integer, n, and every $g_1, ..., g_n \in G$, the matrix $[k(g_i, g_j)]$ is positive.*

We include the following proof of 2.2.2 for the reader's convenience following [8]:

Proof (2)→ (1): By its definition, $\beta^\mu(G)$ is a compact space, so we have (1).

(1)→ (2): Assume G acts amenably on a compact space X, so there exists an amenable transformation group (X, G). Let $x_0 \in X$ and we know by the universal property of the compactification $B^\mu(G)$, that the map $f : G \to X$ where $g \to gx_0$ extends to a continuous map $p : B^\mu(G) \to X$. p is G-equivariant by definition 2.2.5. Recall by 2.1.7, a transformation group (X, G) is amenable if there exists a net $< f_i >_{i \in I}$ of continuous maps $x \to f_i^x$ from $X \to P(G)$ (where $P(G)$ is the set of all probability measure on G equipped with the weak* topology) such that $\lim_i ||g.f_i^x - f_i^{g.x}||_1 = 0$ uniformly on compact subsets of $X \times G$. We can use this net to define a net for the transformation group $(\beta^\mu(G), G)$ by defining a net $< \phi_i >$ of maps $y \to f_i^{p(y)}$ where $f_i^{p(y)}$ is the map $g \to f_i^{p(y)}(g) = f_i(p(y), g)$. Notice $\lim_i ||g.f_i^{p(y)} - f_i^{g.p(y)}||_1 = 0$ by the G-equivariance of p.

(4)→ (1): Suppose there exists a sequence $< h_i >$ of positive type kernels in $C_{b,\theta}(G \times G)$ with support in a tube such that $\lim_i h_i = 1$ uniformly on tubes. Since $h_i \in C_{b,\theta}(G \times G)$, there is a sequence of extensions to $\beta^\mu(G)$. For $(g, h) \in G \times G$, denote $k_i = h_i(g^{-1}, g^{-1}h)$ where k_i denotes its extension to $\beta^\mu(G) \times G$. For $x \in G$, $t_1, ..., t_n \in G$, we have by definition, $k_i(t_i^{-1}x, t_i^{-1}t_j) = h_i(x^{-1}t_i, x^{-1}t_j)$. Notice that $< k_i >$ is a net of continuous, positive type functions on $\beta^\mu(G) \times G$ with compact support. In fact, for any compact subset $K \subset G$, we have that $\sup_{(x,t) \in \beta^\mu(G) \times K} |k_i(x, t) - 1| = \sup_{(x,t) \in G \times K} |k_i(x, t) - 1| = \sup_{(x,t) \in G \times K} |h_i(x^{-1}, x^{-1}t) - 1| = \sup_{(s,t) \in G \times G, s^{-1}t \in K} |h_i(s, t) - 1| = 0$. Thus we have found a net of positive type functions with compact support which tend to 1 uniformly on compact subsets of $\beta^\mu(G) \times G$ and G is amenable at infinity.

(1)$\rightarrow$ (4): Suppose G is amenable at infinity, so by (2), $(\beta^\mu(G), G)$ is an amenable transformation group and by 2.1.2, we have that there exists a net $< f_i >$ of positive type functions in $C_c(\beta^\mu(G) \times G)$ which tend to 1 uniformly on compact subsets of $\beta^\mu(G) \times G$. Since $\beta^\mu(G)$ is compact, this means we have uniform convergence on G. Define a net $< h_i >$ on $G \times G$ by setting $h_i = f_i|_G \circ \theta^{-1}$. And thus we have (4).

(3) $\iff$ (1): We have similar arguments from 2.2.3 using similar methods of changing variables for (3) $\iff$ (1). ∎

Some examples of groups that are amenable at infinity include discrete hyperbolic groups, closed subgroups of connected Lie groups, almost connected groups, and, of course, amenable locally compact groups (reduce the space they act on to a point). Delaroche also found that every σ-compact locally compact group which is amenable at infinity is uniformly embeddable into Hilbert space [8]. Recall the following definition:

Definition 2.2.8 ([8]) *A locally compact group G is uniformly embeddable into a Hilbert space if there is a Hilbert space, H, and a map $f : G \rightarrow H$ such that*

1. *for all compact subset $K \subset G$, there exists $R > 0$ such that $g^{-1}h \in K \Rightarrow ||f(g) - f(h)|| \leq R$ and*
2. *for all $R > 0$, there exists a compact $K \subset G$ such that $||f(g) - f(h)|| \leq R \Rightarrow g^{-1}h \in K$.*

2.2.4 Cohomological Characterization

In [11], Brodzki, Niblo, Nowak, and Wright connect amenable actions and exactness of a countable discrete group with the bounded cohomology of the group. First we will go over a few definitions and notation before we discuss their cohomological characterization following the same notation as [11] and [16]. First we denote V as the space of all functions $f : G \to C(X)$ with the norm $||f||_V = \sup_{x\in X} \sum_{g\in G} |f_g(x)|$ where $f_g \in C(X)$ is the function obtained by evaluating f at g. From here we denote the subspace of V consisting of all functions with finite support and such that for some constant $c = c(f) \in \mathbb{R}$, $\sum_{g\in G} f_g = c1_X$, where 1_X is the constant function with value 1 on X, as $W_{00}(G, X)$. We let $W_0(G, X)$ denote the closure of this subspace in the V norm. $W_0(G, X)$ is basically an analogue of $\ell^1(G)$, which means that $W_0(G, X)^*$ and $W_0(G, X)^{**}$ are both analogous to $\ell^\infty(G)$ and $\ell^\infty(G)^*$. $\mu \in W_0(G, X)^*$ is said to be an invariant mean for the G action on X if $\mu(\pi) = 1$ and $\mu(gf) = \mu(f)$ for all $f \in W_0(G, X)^*$. We also introduce the submodule $N_0(G, X)$ of $W_0(G, X)^*$ which is analogous to the submodule $\ell^1_0(G)$ of all functions that sum up to 0. That is, we denote the kernel of the extension of the map $\pi : W_{00}(G, X) \to \mathbb{R}$ where $\sum_{g\in G} f_g = \pi(f)1_X$ as $N_0(G, X)$.

Now let G act isometrically on a Banach Space, E. Consider the cochain complex $C^m_b(G, E^*)$ which consists of the set of G-equivariant bounded cochains $\phi : G^{m+1} \to E^*$ equipped with the natural differential d as in the homogeneous bar resolution. We denote the bounded cohomology with coefficients in E^* by $H^*_b(G, E^*)$.

Definition 2.2.9 (Johnson Class) *Given a countable, discrete group G acting by homeomorphisms on a compact Hausdorff topological space X, consider the bounded cohain of degree 1 with values in $N_{00}(G, X)$: $J(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$ which is a cocycle and thus represents a class in $H^1_b(G, N_0(G, X))^{**}$. We denote this class by [J] and call it the Johnson class. (Note that $N_{00}(G, X)$ is considered to be a subspace of $N_0(G, X)^{**}$.)*

Theorem 2.2.3 (Brodzki, Niblo, Nowak, Wright [11]) *Let G be a countable discrete group acting by homeomorphisms on a compact Hausdorff topological space X. TFAE:*

1. *G acts on X topologically amenably*

2. *The Johnson class [J]* $\in H_b^1(G, N_0(G,X)^{**})$ *is trivial*

3. $H_b^n(G, E^*) = 0$ *for* $n \geq 1$ *and for every* ℓ^1*-geometric* $G - C(X)$ *module* E*.*

In [17], Monod also discusses some of the characterization of topologically amenable actions in terms of bounded cohomology. Again, first we have a few definitions.

Definition 2.2.10 *Let E be a Banach space, then E is a* $C(X)$*-module if it is equipped with a contractive unital representation of the Banach algebra* $C(X)$*. That means for* $f \in C(X)$ *and* $v \in E$*,* $||f \cdot v|| \leq ||f|| \cdot ||v||$*.*

We say that E is a (G,X)-module if it is both a G and $C(X)$ module (G acts on E by isometries) and the representation of $C(X)$ is G-equivariant. We need gfg^{-1} to correspond to the action of $g \in G$ on $f \in C(X)$. That is, we need the G action on $C(X)$ to relate to both the G action on E and the $C(X)$ action on E. We can do this by defining $g \cdot f =: \tilde{f}$ such that $\tilde{f} \cdot v = g(f(g^{-1}v))$. In general not all (G,X)-modules have a cross product algebra. We also denote $\mathscr{I}(C(X), W)$ as the space of integral operators with values in W.

Definition 2.2.11 *A* $C(X)$*-module E is of type M if for all* $u \in E$*, and* $f_i \in C(X)^+$*,* $\sum_{i=1}^{n} ||f_i u|| \leq ||\sum_{i=1}^{n} f_i|| \cdot ||u||$*. And we say that a* $C(X)$*-module E is of type C if for all* $u_i \in E$*, and* $f_i \geq 0$*,* $||\sum_{i=1}^{n} ||f_i u_i|| \leq ||\sum_{i=1}^{n} f_i|| \cdot \max_i ||u_i||$*.*

In [17], Monod proves that if G is a locally compact, second countable group acting topologically amenably on a compact space X, then every dual (G,X)-module of type C is a relatively injective Banach G-module. In Monod's paper, he says that G acts amenably on a compact space X if there is a net $\{u_j\}_{j\in J} \in C(X, \ell^1(G))$ such that every $u_j(x)$ is a probability measure on G and $\lim_{j\in J} ||gu_j - u_j||_{C(X,\ell^1(G))} = 0$ for all $g \in G$. Using Monod's definition, E is a C(X) module if for all $u \in E$ and $\phi_i \in C(X)$, $\phi_i \geq 0$, then $\sum_{i=1}^{n} ||\phi_i u|| \leq ||\sum_{i=1}^{n} \phi_i|| \cdot ||u||$. Monod also proves the converse of this statement for discrete groups.

Theorem 2.2.4 *[Monod [17]]*

Let G be a group acting on a compact space X. TFAE:

1. *The G action on X is topologically amenable. That is, there is a net $\{u_j\}_{j \in J} \in C(X, \ell^1(G))$ such that every $u_j(x)$ is a probability measure on G and $\lim_{j \in J} ||gu_j - u_j||_{C(X,\ell^1(G))} = 0$ for all $g \in G$.*

2. *$H_b^n(G, C(X,V)^{**}) = 0$ for every Banach G-module V and every $n \geq 1$.*

3. *$H_b^n(G, \mathscr{I}(C(X), W^*)) = 0$ for every Banach G-module W and every $n \geq 1$.*

4. *$H_b^n(G, E^*) = 0$ for every Banach G-module E of type M and every $n \geq 1$.*

5. *Any of the previous three points hold for $n = 1$.*

6. *$C(X,V)^{**}$ is relatively injective for every Banach G-module V.*

7. *$\mathscr{I}(C(X), W)^*$ is relatively injective for every Banach G-module W.*

8. *Every dual (G, X)-module of type C is a relatively injective Banach G-module.*

9. *There is a G-invariant element in $C(X, \ell^1(G))^{**}$ summing to 1_X.*

10. *There is a norm one positive G-invariant element in $C(X, \ell^1(G))^{**}$ summing to 1_X.*

Monod mentions that his proof from 1 to 8 still holds for locally compact, second countable groups [17].

3. HYPERBOLIC SPACES AND GROUPS

3.1 Discrete Hyperbolic Spaces and Groups

One of the interesting classes of spaces that is of Property A is the class of discrete hyperbolic spaces and groups. Often when dealing with hyperbolic groups, we consider groups which are finitely generated with a metric relative to a finite generating subset. In this section we will discuss such groups, but later we will generalize this notion to compactly generated groups. First we review some of the common notions for hyperbolic spaces. Recall that we define a geodesic between a and b in a hyperbolic space X to be an isometry $g : [0, d(a, b)] \to X$ with $g(0) = a$ and $g(d(a, b)) = b$. We denote $[[a, b]]$ as the set of all geodesics between a and b and we use the notation $]]a, b[[$ to denote the geodesic with minimal length that starts at a and ends at b.

Definition 3.1.1 *A space, X, is hyperbolic if there exists $\delta > 0$ such that every geodesic triangle is δ-thin. That is to say, each of the triangle's sides is contained in a δ-neighborhood of the union of the other two sides.*

Definition 3.1.2 *A discrete group Γ, is hyperbolic if it admits a finite generating set such that the associated word metric in its Cayley graph is Gromov hyperbolic with respect to the finite set of generators.*

So for a discrete hyperbolic group Γ, we define our word metric in terms of our finitely generating set. That is, for $x, y \in \Gamma$ we define $d(x, y)$ =length $(g^{-1}h)$, the minimum number of generators from our finite generating set to generate $g^{-1}h$. This word metric is hyperbolic if every geodesic triangle in Γ's Cayley graph is δ-thin. With the word metric, we can treat our discrete hyperbolic groups like metric spaces. Hyperbolic groups and spaces have some very nice properties which we will use later in this paper. One property is that each geodesic triangle has a number which measures the distance for which two sides of the triangle stay close together. This number is called the Gromov product.

Definition 3.1.3 (Gromov Product) *We define the Gromov product between points a and b, denoted as $< a, b >_x$, as $< a, b >_x = \frac{1}{2}[d(a, x) + d(b, x) - d(a, b)]$, where x is a fixed a basepoint in the Cayley graph.*

We now have an alternative definition for a Gromov hyperbolic group using the Gromov product.

Definition 3.1.4 (Gromov Hyperbolic Group) *A discrete group Γ is δ-hyperbolic if for all $x, a, b, \gamma \in X$, $< a, b >_x \geq \min\{< a, \gamma >_x, < b, \gamma >_x\} - \delta$.*

We also have that any δ-word hyperbolic space satisfies the following inequality:

Definition 3.1.5 (Hyperbolic Inequality) *Given $x, y, z, t \in X$, where X is a discrete δ-word hyperbolic space, then $d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\} + 2\delta$.*

We use the Gromov product to define the boundary of a hyperbolic space.

Definition 3.1.6 (Boundary of Hyperbolic Space) *Given a locally compact hyperbolic space X, the boundary of X, ∂X, is the set of equivalence classes of infinite geodesics. Two infinite geodesics a_n and b_m are equivalent if $\liminf_{n,m\to\infty} < a_n, b_m >_x = \infty$, where x is a fixed base point. This equivalence is independent of the basepoint x [18].*

It is nice that we can also define a metrizable topology on X, ∂X, and $X \cup \partial X$. We first define a basis on X as $\{W(a, \varepsilon)\}_{\varepsilon>0}$. Note that this basis is just the set of ε -balls around $a \in X$. Defining a basis for ∂X is slightly tougher. We define a basis for ∂X, $\{V(x, R)\}_{R>0}$, in the following way. Let $V(x, R) = \{z \in \bar{X} | \liminf_{n,m\to\infty} < y_n, b_m >_o \geq R$ where o is a fixed basepoint in X and where y_n is a geodesic whose endpoint is z and b_m is an infinite geodesic (possibly repeated) whose endpoint is $x\}$. We can also define convergence of a sequence of geodesics.

Definition 3.1.7 (Converging Geodesics in X) *Let $< \alpha_n >$ be a sequence of geodesics in X, we say $< \alpha_n >$ converges to geodesic β, if for every $\varepsilon > 0$, there exists $N >> 1$ such that for $n > N$, every point along α_n is within ε of a point along the geodesic β. That is, for $n > N$, α_n is within an ε-tube of β.*

Definition 3.1.8 (Converging Geodesics in $\bar{X}$) *Let $< \alpha_n >$ be a sequence of geodesics with endpoints in ∂X, we say $< \alpha_n >$ converges to geodesic β with endpoint $z \in \partial X$, if for every $\varepsilon, R > 0$, there exists $N >> 1$ such that for $n > N$, every point along α_n is within ε of a point along the geodesic β and $\liminf_{k,m\to\infty} < \alpha_n, \beta_m >_o > R$ where o is a fixed basepoint in X and β_m is a geodesic converging to x.*

3.2 Locally Compact Hyperbolic Groups

We have gone over properties of discrete hyperbolic spaces and groups. Now we will discuss the more general notion of locally compact hyperbolic groups.

Definition 3.2.1 ([19]) *A locally compact group, G, is hyperbolic if it admits a compact generating set such that the associated word metric is Gromov hyperbolic. We say G is compactly generated if there exists a symmetric, compact neighborhood K of the identity element in G such that for every $g \in G$, $g \in K^n$ for some $n \in \mathbb{N}$.*

The word metric measures the distance between two group elements g and h in G which allows us to treat our group like a metric space. We define the word metric as follows:

Definition 3.2.2 (Word Metric) *Given $g, h \in G$, where G is a locally compact hyperbolic group, we define the word metric d on X as $d(g, h) = \ell(h^{-1}g)$ in the setting of the associated Cayley graph and compact generating set where $\ell(h^{-1}g)$ is the length of the geodesic $h^{-1}g$. Since we are in a compactly generated G, we define this length function as length$(g) = \min_{g\in K^n}\{n\}$.*

Some examples of locally compact hyperbolic groups are $SO(n,1)$, $SU(n,1)$, and $Sp(n,1)$. In addition to these, the class of groups of the form $H \rtimes_\alpha \mathbb{Z}$ and $H \rtimes_\alpha \mathbb{Z}$ are also locally compact hyperbolic groups [20]. Note $\alpha(1)$ is such that there is a compact subset of H such that for every x in H, there is a natural number $\tilde{n}$ such that for every n greater than $\tilde{n}$, $\alpha(n)(x)$ is in the compact set. Since a hyperbolic group has a metric, our properties for hyperbolic spaces in 3.1 also are defined in the same way for hyperbolic groups. In particular we have the same definitions for the δ-thin triangles, the Gromov product, ∂G, $\bar{G}$, geodesics, and convergence of geodesics.

Most of the properties of locally compact hyperbolic groups are in the same spirit as discrete hyperbolic groups. Yet with this more general setting of locally compact hyperbolic groups, we have more non-trivial results that generalize to these groups. For example, we know that non-elementary finitely generated hyperbolic groups are not amenable since they contain a closed non-amenable subset [20]. Yet Caprace, deCornulier, Monod, and Tessera prove that we do have amenable non-elementary locally compact hyperbolic groups [19] They also give a description of all locally compact hyperbolic groups with cocompact amenable subgroups and prove that a locally compact group is Gromov hyperbolic if and only if it admits a continuous proper cocompact isometric action on a Gromov hyperbolic proper geodesic metric space.

Corollary 3.2.1 (Caprace, Cornulier, Monod, Tessera [19]) *A locally compact group G is hyperbolic if and only if G has a continuous proper cocompact isometric action on a proper geodesic hyperbolic space.*

Another difference between discrete hyperbolic groups and locally compact hyperbolic groups is discussed by Dreesen in [20]. Dreesen shows that because non-elementary discrete hyperbolic groups are countable, they cannot act transitively on their boundary. Yet locally compact hyperbolic groups can act transitively on their boundaries. This is possible even if the boundary is infinite [20]. We already know that discrete Gromov hyperbolic groups are of Property A and that a discrete hyperbolic group acts amenably on its compact Gromov Boundary and thus is amenable at infinity. We ask whether this is also valid for locally compact hyperbolic groups?

3.3 Group Actions on Hyperbolic Spaces

We can now discuss group actions on hyperbolic spaces.

Definition 3.3.1 *If a locally compact group G acts continuously on a locally compact hyperbolic space X, we define the G action on the boundary of X, ∂X, as follows: For $x \in \partial X$, $g \cdot x = \lim_{n\to\infty} g \cdot \alpha_n$ where α_n is a geodesic in X such that α_n has an endpoint x. We will sometimes write $\alpha_n \to x$.*

Lemma 3.3.1 *Suppose a locally compact group G acts by isometries on a locally compact hyperbolic space X continuously. Then we can continuously extend the group action to the Gromov boundary of X, ∂X.*

Proof Let $g \in G$, $x \in \partial X$, and $\alpha_n \in X$ such that $\alpha_n \to x$. Let U be an R neighborhood of x. That is, $U(x, R) = \{z \in \bar{X} : \liminf_{n,m\to\infty} < y_n, \beta_m >_0 > R, \beta_m \to x, y_n \to z\}$. Recall $< y_n, \beta_m >_0$ is the Gromov product. Because G is a locally compact group, we just need to show continuity at the identity element of G, 1_G. Let $\varepsilon > 0$. Now if $< g_n >$ is a sequence in G and $g_n \to 1_G$, then there exists an N large enough such that $d(g_n, 1_G) < \frac{\varepsilon}{2}$ for $n > N$. Since $\alpha_n \to x$, α_n is in the equivalence class of x and there is a $\tilde{N}$ such that for any $\beta_m \to x$, $\liminf_{n,m\to\infty} < \alpha_n, \beta_m >_0 > R + 2\varepsilon$. Let $\bar{N} = \max\{N, \tilde{N}\}$ and let $n > \bar{N}$. For each n, let $U_n(\alpha_n, \frac{\varepsilon}{2}) := \{y \in X : d(y, \alpha_n) < \frac{\varepsilon}{2}\}$ be a neighborhood of α_n. We want to show that $d(g_n y, \alpha_n) < \varepsilon$ for $y \in U(\alpha_n, \frac{\varepsilon}{2})$. Let $y \in U(\alpha_n, \frac{\varepsilon}{2})$, so $d(y, \alpha_n) < \frac{\varepsilon}{2}$. We also have that $d(g_n, 1_G) < \frac{\varepsilon}{2}$, which implies $d(g\alpha_n, \alpha_n) < \frac{\varepsilon}{2}$. Thus, $d(g_n y, \alpha_n) < d(g_n\alpha_n, \alpha_n) + d(g_n y, \alpha_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Now we want to show for $y \in U_n(\alpha_n, \frac{\varepsilon}{2})$ and $\beta_m \to x$, $< g_n y, \beta_m >_0 > R$ for $n > \bar{N}$. In other words, we want to show that $g_n y \in U(x, R)$ for $n > \bar{N}$. By definition, $< g_n y, \beta_m >_0 = d(g_n y, 0) + d(\beta_m, 0) - d(g_n y, \beta_m)$. Now $d(g_n y, \alpha_n) < \varepsilon$ implies $|d(g_n y, 0) - d(\alpha_n, 0)| < \varepsilon$ which implies that $d(\alpha_n, 0) - \varepsilon < d(g_n y, 0) < d(\alpha_n, 0) + \varepsilon$. And $d(g_n y, \beta_m) < d(g_n y, \alpha_n) + d(\beta_m, \alpha_n)$. Thus, $d(g_n y, 0) + d(\beta_m, 0) - d(g_n y, \beta_m) > d(\alpha_n, 0) - \varepsilon + d(\beta_m, 0) - d(g_n y, \beta_m) > d(\alpha_n, 0) - \varepsilon + d(\beta_m, 0) - d(g_n y, \alpha_n) - d(\beta_m, \alpha_n) = < \alpha_n, \beta_m >_0 - \varepsilon - d(g_n y, \alpha_n)$.

From earlier, we have that $\liminf_{n,m\to\infty} < \alpha_n, \beta_m >_0 > R + 2\varepsilon$ and $d(g_n y, \alpha_n) < \varepsilon$. Thus $< g_n y, \beta_m >_0 > R + 2\varepsilon - \varepsilon - \varepsilon = R$. Which is what we wanted to show. Therefore if $g_n \to 1_G$, then $g_n x \to x$, and thus G acts continuously on ∂X. Thus, if a locally compact group, G, acts continuously on a locally compact hyperbolic space, X, then we can continuously extend the group action to the Gromov boundary of X, ∂X. ■

3.4 Geometric Properties of Hyperbolic Groups

Germain has proved some helpful geometric properties for hyperbolic groups which we will use later in 5.2. First, we should note that the hyperbolic properties of a hyperbolic group G are preserved on its compactification, $G \cup \partial G$. Specifically, any geodesic triangle which is extended to the boundary is 24-δ thin for the metric of the group and two geodesics between the same two points are in a 8-δ neighborhood of each other [5]. The following lemmas are from Appendix B in Delaroche's book *Amenable Groupoids* for a hyperbolic space with measure.

Lemma 3.4.1 *Let $K \in \mathbb{Z}$, then there exists an M, $0 < M \leq K + 48\delta$ such that for all $a, b \in X$ and $x \in X \cup \partial X$ with $d(a, b) < K$, we have $d(p, q) < M$ for all $p \in]]a, x[[$ and $q \in]]b, x[[$ with $d(a, p) = d(b, q)$.*

Proof Let p be a points in the geodesic from a to x and q be a geodesic from b to x with $d(a, p) = d(b, q)$. We have that all triangles are 24δ-thin so, without loss of generality, we have that either there exists a point $q_0 \in]]b, x[[$ such that $d(q_0, p) < 24\delta$ or both p and q are within 24δ from the geodesic between a and b. For the first case, we have $|d(b,q_0) - d(a,p)| = |d(b,q_0) - d(b,q)| = |d(q,q_0)| < d(p,q) + d(q_0,p) < M + 24\delta < K + 24\delta$. Thus $d(q,q_0) < K + 24\delta$ and then since $d(a,p) = d(b,q)$, we have that $d(p,q) < d(q,q_0) + d(p,q_0) < K + 48\delta$. For the latter case, $d(p,q) < d(p,a) + d(b,q) + d(a,b) \leq 48\delta + d(a,b)$ which is what we want. ■

Lemma 3.4.2 *Let $K, L \in \mathbb{Z}$ and $L > 3K + \delta$. Let a, b, e, f be points in X such that $d(a,b) < K$ and $d(e,f) < K$. Also assume that $d(a,e) > 3L$ and $d(b,f) > 3L$. Then for all geodesics g_0 between a and e and all geodesics g between b and f, any point p of the segment $g([L,2L])$ is at a distance at most 4δ of a point q in $g_0([L-K, 2L+K])$ such that $d(b,p) = d(b,q)$.*

Proof Let a, b, e, f, p, q be points in X with the properties listed above. Recall the hyperbolic inequality: given $x, y, z, t \in X$, then $d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\} + 2\delta$.

Step 1: To show $d(p,q) < 2\delta + 4K$:
Using our inequality, we have that $d(b,f) + d(p,q) < 2\delta + \max\{d(b,p) + d(q,f), d(b,q) + d(p,f)\}$. Since $d(b,p) = d(b,q)$ and $d(p,f) < d(b,f)$ we have that $d(p,q) < 2\delta + d(q,f) - d(p,f)$. It is clear that $d(q,f) \leq K + d(q,e)$ and $d(a,q) \leq d(b,a) + K$. And we have that $d(p,f) = d(b,f) - d(b,p) = d(b,f) - d(b,q)$ because $d(b,p) = d(b,q)$. Therefore, $d(p,f) \geq d(b,f) - d(a,q) - K$ since $d(b,q) \leq d(a,q) + K$ which implies that $d(q,f) - d(p,f) < 2K + d(a,e) - d(b,f)$. We have this inequality since $d(q,f) \leq K + d(q,e)$ and $d(p,f) \geq d(b,f) - d(a,q) - K$, and so we have $d(q,f) - d(p,f) < K + d(q,e) - (d(b,f) - d(a,q) - K) = 2K + d(q,e) - d(b,f) + d(a,q) = 2K + d(a,e) - d(b,f)$. Now $|d(a,e) - d(b,f)| < 2K$ by the triangle inequality and since both $d(e,f)$ and $d(a,b)$ are less than K. Now we have that $d(q,f) - d(p,f) < 2K + d(a,e) - d(b,f) < 4K$. Since from earlier, we have that $d(p,q) < 2\delta + d(q,f) - d(p,f)$, we have that $d(q,p) < 4K + 2\delta$ (*) and Step 1 is done.

We can rearrange our inequality in the following way: We have that $d(b,f) + d(p,q) < 2\delta + \max\{d(b,p) + d(q,f), d(b,q) + d(p,f)\}$ which implies $d(p,q) < 2\delta +$

$\max\{d(b,p) + d(q,f), d(b,q) + d(p,f)\} - d(b,f)$. This shows that $d(p,q) < 2\delta + \max\{d(b,p)+d(q,f)-d(b,f), d(b,q)+d(p,f)-d(b,f)\}$. Now $d(b,p)+d(q,f)-d(b,f) = d(q,f) - d(p,f)$ since $d(b,f) = d(b,p) + d(p,f)$. And similarly, $d(b,q) + d(p,f) - d(b,f) = d(b,q) - d(b,p)$. Thus, we now have another application of the inequality and we see that $d(p,q) < 2\delta + \max\{d(b,q) - d(b,p), d(q,f) - d(p,f)\}$.

By symmetry, we also have $d(p,q) < 2\delta + \max\{d(a,p) - d(a,q), d(p,e) - d(q,e)\}$. (Here we use the inequality $d(a,e)+d(q,p) < 2\delta+\max\{d(a,q)+d(p,e), d(a,p)+d(q,e)\}$ and rearrange the inequality and use the same justification as we did for the other inequality.) We also have $d(a,p)+d(b,q) < 2\delta+\max\{d(a,q)+d(b,p), d(a,b)+d(p,q)\}$. Moreover, $d(b,p) > L$ and because q is in the geodesic strip $[L-K, 2L+K]$, $d(a,q) > L-K$. Therefore $d(a,q) + d(b,p) > 2L-K$. Also since p is in the geodesic strip $[L, 2L]$, $d(b,p) > L$, thus $d(a,q) > 2L - K - d(b,p) > L - K$. Since $d(a,b) < K$ and since we have (*) from earlier, $d(q,p) < 4K + 2\delta$. And we now have $d(a,b) + d(q,p) > 2\delta + 5K$.

Since $L > \delta + 3K$, we have that $2L - K > 2\delta + 5K$ which implies that $d(a,q) + d(b,p) > 2L - K > 2\delta + 5K > d(a,b) + d(p,q)$. Thus $d(a,p) + d(b,q) < 2\delta + d(a,q) + d(b,p)$. Since $d(b,p) = d(a,q)$, we have $d(a,p) - d(a,q) < 2\delta$. Similarly $d(p,e) - d(q,e) < 2\delta + d(p,f) - d(q,f)$ and $d(p,q) < 2\delta + \max\{2\delta, d(p,f) - d(q,f)\}$. This means $d(p,q) < 2\delta$ or $d(p,q) < 4\delta$ (depending on the sign of $d(p,f) - d(q,f)$). ■

Lemma 3.4.3 *Let $K \in \mathbb{Z}$, and let $L > 3K + 150\delta$, then for any two points $a, b \in X$ with $d(a,b) < K$ and $x \in \partial X$ and for geodesic $g_0 =]]a, x[[$ and $g =]]b, x[[$, any point $p \in g([L, 2L])$ is at most 4δ from a point $q \in g_0([L-K, 2L+K])$.*

Proof Let $L > \delta + 3\sup\{K, M\}$. By 3.4.1, there exists an M such that $0 < M \leq K+48\delta$ such that 3.4.1 holds. That is, $d(p,q) < M$ for all $p \in]]a, x[[$ and $q \in]]b, x[[$ with $d(a,p) = d(b,q)$. Then we can define points e and f in 3.4.2 by setting $e = g_0(3L)$ and

$f = g(3L)$. Now we can apply 3.4.2 to these new points and thus we have the same conclusion as 3.4.2, $p \in g([L, 2L])$ is at most 4δ from a point $q \in g_0([L - K, 2L + K])$.

■

4. FUNDAMENTAL DOMAINS

4.1 Definition and Construction

Often when we talk about Property A, we require a space to have bounded geometry. That is, we need the number of elements in a ball to be uniformly bounded. We don't necessarily have this property when we have a locally compact space. We also don't necessarily have a G-invariant measure. Fortunately, we can construct a fundamental domain for a group action on a locally compact space and use this to construct a G-invariant measure on X.

Definition 4.1.1 (Fundamental Domain) *We say a set $F \subset X$ is a fundamental domain for the G-action on X if for every $x \in X$, there exists a $g \in G$ such that $x.g \in F$ and every orbit meets F once [21].*

A nice property of a fundamental domain F is that the projection map $\pi : X \to X/G$ restricts to an injective map on F and a surjection map on the closure of F, $\bar{F}$ [22]. Following [23], we construct a fundamental domain for locally compact G acting continuously and properly on a locally compact space X. Recall that a group action is proper if the map from $(x, g) \to (x, xg)$ is proper. And a function is proper if the inverse images of compact sets are compact. For any $x \in X$, let $G(x)$ be the stabilizer of x and define the set $F = \{z \in X | d(z, x) \leq d(z, xg), \forall g \in G - G(x)\}$. We let $G(A|B) =: \{g \in G | Bg \cap A \neq \emptyset\}$.

Proposition 4.1.1 *F has the following properties:*

1. *$G(F|F) = G(x)$, and*

2. *$FG(x) = F$.*

Proof (1): Let $y \in G(F|F)$, so $Fy \cap F \neq \emptyset$. Therefore there exists $f \in F$ such that $fy \in F$. Suppose $y \notin G(x)$, then $d(fy, xy) > d(fy, x) = d(f, xy^{-1}) > d(f, x)$. But by our isometric action, this cannot be so. Thus $G(F|F) \subset G(x)$. For the other containment, suppose $g \in G(x)$, so $xg = x$ and $x = xg^{-1}$. Let $z \in F$, then $d(zg, x) = d(z, xg^{-1}) = d(z, x) < d(z, xh)$ for $h \in G - G(x)$. Thus $zg \in F$. Therefore $G(x) \subset G(F|F)$ and thus $G(F|F) = G(x)$.

(2): Let $g \in G(x)$ and $f \in F$ then by (1), $g \in G(F|F)$ and therefore there exists a t such that $t = fg$ and $t \in F$. So we have that $FG(x) \subset F$. Now suppose $r \in F$. Since $G(x) = G(F|F)$, there exists $g \in G(x)$ such that $gF \cap F \neq \emptyset$. Therefore there is $\tilde{f} \in F$ such that $r = \tilde{f}g$. Thus $F \subset FG(x)$ and so $FG(x) = F$. ■

Note that since X is a locally compact space, $G(x)$ is closed and thus compact. Let $\lambda = \min_{g \in G - G(x)} d(xg, x)$. Because G acts continuously and properly, λ exists and $\lambda > 0$.

Proposition 4.1.2 *$F = \{z \in X | d(z, x) \leq d(z, xg), \forall g \in G - G(x)\}$ is a fundamental domain.*

Proof We want to check that every orbit meets our fundamental domain F. Suppose $z \in X$ and consider zG. Let $\tilde{g}$ be such that $d(z, x\tilde{g}) = \min_{g \in G - G(x)} d(z, xg)$. Then $d(z\tilde{g}^{-1}, x) = d(z, x\tilde{g}) < d(z, xg)$. Since $z\tilde{g}^{-1} \in zG$, zG intersects F. If zG intersects F more than once, we can just pick a representative for each orbit. ■

When F is the fundamental domain for the group action, $\bar{F}$ meets each G orbit of the action once. That is, the G translates of $\bar{F}$ cover X. In this case, we let F be open with $\bar{F}^{\circ} = F$ which means F has a relatively small boundary [24]. For F, a fundamental domain, we have that any point of X is equivalent with respect to the action of G to at least one point of the set $\bar{F}$. This means for the natural surjection $\pi : X \to X/G$, $\bar{F}$ is mapped to all of X/G [25]. Since $\bar{F}$ is a closed subset of a locally compact space X, $\bar{F}$ is compact.

4.2 G-Invariant Measure on Fundamental Domain

Given a locally compact group G acting on a locally compact space and corresponding fundamental domain F, we can construct a G-invariant measure on X.

Corollary 4.2.1 *There exists G-invariant measure on X given a fundamental domain $\bar{F}$.*

Proof $\bar{F}$ is compact so we can find a nice finite Borel measure, μ, on $\bar{F}$. Since the G translates of $\bar{F}$ cover X and $\mu(\partial F) = 0$, we can define the measure on $\bar{F} \times G$ as the product measure of μ and the Haar measure, dm_g, on locally compact G. Now consider $\tilde{X} = \{(x, g) : x \in X, g \in G, gx = x\}$. Since $\tilde{X} \subset \bar{F} \times G$, we have a measure $\tilde{\mu}$ on $\tilde{X}$. Now we define a measure, m, on X in the following way. If S is a subset of X and X is locally compact, we can assume S is bounded. Because of our proper action, we can lift S to a bounded subset in $\tilde{X}$. So our lifted set is in $\bar{F} \times K$ for a compact subset $K \subset G$. We now define $m(S) = \int_{g \in K} g\mu(S \cap g\bar{F}) dg_m$. Notice this measure is G-invariant since $M(\tilde{g}S) = \int_{g \in K} \tilde{g}g\mu(S \cap \tilde{g}g\bar{F}) dg_m$ and translates of $\bar{F}$ differ by a 0 measure set. ■

5. LOCALLY COMPACT PROPERTY A GROUPS

5.1 Definition

Most of the work with Property A has been with discrete topological spaces. Our work involves looking at the locally compact case. In this section, we introduce the concept of locally compact Property A groups. As we have mentioned before, Property A is a metric space property, but we can construct a metric on a group by defining a length function on the generators of the group. Since we don't have the discrete topology, we need a few more conditions to define Property A for locally compact groups.

Definition 5.1.1 *We say a locally compact group, G, has property A if there exists a compact space X such that G acts amenably and continuously on X.*

Definition 5.1.2 (Amenable Locally Compact Group Action) *We say that a locally compact group acts amenably on X, if there exists a sequence $< f_n >$ of nonnegative, compactly supported Borel functions on $G \times X$ such that $\forall x \in X$, $\int_G f_n(g,x)dg > 0$ and $\lim_{n\to\infty} \sup_{x \in X} \left(\frac{\int_G |f_n(g,x) - h.f_n(g,x)|dg}{\int_G f_n(g,x)dg}\right) = 0$ uniformly for all $h \in K$, where K is a compact subset of G. We define the group action as the action induced by the diagonal action of G on $G \times X$. That is, $(h.f)(g,x) = f(h^{-1}g, h^{-1}x)$.*

5.2 Groups Acting on Hyperbolic Spaces

Our main example is a locally compact group G which acts on a locally compact hyperbolic space X. Suppose G is a locally compact group that acts continuously and properly on a locally compact hyperbolic space X. Recall from 3.3.1, we can continuously extend this group action to the boundary of X, ∂X. We also know from

4.1, we can construct a fundamental domain for the G action on X. We will denote the closure of this fundamental domain as $\bar{F}$.

We now show that a locally compact group G which acts on a locally compact hyperbolic space X acts amenably on ∂X. Suppose $a \in X$, $x \in \partial X$, and $k \in \mathbb{N}$. For each $y \in \bar{F} \subset X$. We define $I_y(a,x,k) =: \{g : gy \in [[a_1, x[[, d(a,a_1) < k\}$. Let $\{y_i\}_{i \in I} \subset \bar{F}$ be the set of representatives of the fundamental domain and let $l > 0$. We define the following function on G, $F(a,x,k,l)(g) = \int_{y \in \bar{F}} \chi_A(gy)dm$, where A=$\{\cup_{a_1 \in B(a,k)} q^x_{a_1}[l, 2l]\}$ in which q is the geodesic from a_1 to x, and dm is the g-invariant measure on X, which exists by 4.2.1. Notice we only look at the strips from l to $2l$. Note also that since G is a locally compact group, G is equipped with the Haar measure. Now we define our averaging function $H(a,x,l)(g) = \frac{1}{\sqrt{l}} \sum_{k \leq \sqrt{l}} F(a,x,k,l)(g)$ and we give H the L^1 norm.

Proposition 5.2.1 $||H(a,x,l)|| \geq c \cdot l$ *for all* $a \in X$ *and* $x \in \partial X$ *where* c *is a constant.*

Proof For some $g \in G$ and $y \in \bar{F}$, $gy \cap A \neq 0$. Let U be a neighborhood of g of measure $c > 1$. Let T be the tube T around U of length l. Then $\int_{y \in \bar{F}} \chi_A(gy)dm \geq \int_{y \in \bar{F}} \chi_T(gy)dm$. And $\int_{y \in \bar{F}} \chi_T(gy)dm \geq l \cdot c$. ■

Proposition 5.2.2 *Given a compact subset K of G,* $\sup_{x \in \partial X, g \in K} ||H(ga,x,l) - H(a,x,l)|| = o(l)$ *for fixed* $g \in K$ *where* K *is a compact subset of* G *and* $a \in X$.

Before we prove this, we will state and prove the following lemma:

Lemma 5.2.1 *For every* $k \in \mathbb{N}$, $c \in \mathbb{Z}^+$, $a \in X$, $x \in \partial X$, $g \in K \subset G$ *(K a compact subset of G), we have* $\sup_{x \in \partial X} (\sum_{k < \sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)||) = O(l)$.

Proof If we increase the value of k, $F(a,x,k,l)(g)$ increases since we are increasing the size of the k-ball centered at a. Thus the map $k \mapsto F(a,x,k,l)$ is an increasing map and therefore we can write $\sum_{k<\sqrt{l}} ||F(a,x,k+c,l)-F(a,x,k,l)|| = \sum_{k<\sqrt{l}} ||F(a,x,k+c,l)|| - ||F(a,x,k,l)|| \leq \sum_{\sqrt{l}\leq k<\sqrt{l}+c} ||F(a,x,k,l)||$. Now, for l large enough, we will have $l > 3(\sqrt{l}+c) + 150\delta$. By 3.4.3 with $K = \sqrt{l}+c$ and $L = l$, we know that the set A in the definition of $F(a,x,k,l)$ is in a 4δ neighborhood of a geodesic with length $L + 2K = l + 2(\sqrt{l}+c)$. Now let $B(x_0, 4\delta)$ be a ball of radius 4δ and let $B = m(B(x_0, 4\delta))$, the measure of the 4δ ball. Since the measure on X is G-invariant, the measure of this ball will not change with translation. We also have that there exists a compact set $V \subset G$ such that $A \subset V \cdot \bar{F}$. Let $m_G(V)$ represent the measure of V. Thus, $||F(a,x,k,l)|| = \int_G |F(ax,k,l)(g)| dm_g \leq m_G(V) \cdot B \cdot (l + 2(\sqrt{l}+c))$. Therefore we have $\sum_{\sqrt{l}\leq k<\sqrt{l}+c} ||F(a,x,k,l)|| \leq c \cdot B \cdot (l + 2(\sqrt{l}+c)) \cdot m_G(V)$. Thus $\sup_{x\in\partial X} (\sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)||) = O(l)$. ■

Now we prove 5.2.2:

Proof Let $c = d(ga, a)$. Since $g \in K$ and K is compact, c is finite. Then $||H(ga,x,l) - H(a,x,l)|| = \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(ga,x,k,l) - F(a,x,k,l) + F(a,x,k+c,l) - F(a,x,k+c,l)||$ $\leq \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(ga,x,k,l) - F(a,x,k+c,l)|| + \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)|| =$ $\frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)|| + \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)||$.

Let A=: $\frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)||$ and B=:$\frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)||$. By 5.2.1, $\sqrt{l}B$ is $O(\sqrt{l})$ so B is $o(l)$. Now let's look at A. A= $\frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)|| = \frac{1}{\sqrt{l}} \sum_{0\leq k<c} ||F(a,x,k+c,l) - F(ga,x,c,l)||$ $+ \frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)|| \leq \frac{1}{\sqrt{l}} \sum_{0\leq k<c} ||F(a,x,2c,l) - F(ga,x,c,l)||$ $+ \frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)||$. The first term can be bounded because $k \mapsto F(a,x,k,l)$ is increasing and $k \leq c$ for the first part of the summation and thus we can bound the first term by $\frac{1}{\sqrt{l}} \sum_{0\leq k<c} ||F(a,x,k+c,l) - F(ga,x,k,l)|| \leq$

$\frac{1}{\sqrt{l}} \sum_{0 \le k < c} ||F(a,x,2c,l) - F(ga,x,c,l)|| = \frac{c}{\sqrt{l}}(||F(a,x,2c,l)|| + ||F(ga,x,c,l)||)$. Moreover, because $k > c$ for the second part of the summation and $c = d(ga,a)$, the second term $\frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)|| \le \frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k-c,l)||$ since $F(a,x,k-c,l) \le F(ga,x,k,l) \le F(a,x,k+c,l)$.

Now if we let $K = 2c$, we know by by 3.4.1 that there exists an M independent of x such that for $A = \cup_{a_1 \in B(a,2c)} q^x_{a_1}[l,2l]$, A is contained in a tube with radius M along a geodesic of length l. Let $\tilde{c}$ be the measure of a ball with radius M, $m(B(x_0, M))$, which does not change under translation by G. Like in our proof of our lemma, 5.2.1, we also have a compact subset $V \subset G$ such that $A \subset V \cdot \bar{F}$. Thus, there exists a constant $\tilde{c}$ such that for all $(a,x) \in X \times \partial X$, $||F(a,x,2c,l)|| = \int_G |F(a,x,2c,l)(g)| dm_g$ $\le m_g(V) \cdot \tilde{c} \cdot l = O(l)$. This implies that $\frac{c}{\sqrt{l}} \sup_{x \in \partial X}(||F(a,x,2c,l)|| + ||F(ga,x,c,l)||)$ is $o(l)$ as well since $||F(ga,x,c,l)|| \le ||F(a,x,2c,l)||$. And $\frac{1}{\sqrt{l}} \sup_{x \in \partial X} \sum_{c<k<\sqrt{l}} (||F(a,x,k+c,l) - F(a,x,k-c,l)||)$ is also $o(l)$ by 5.2.1 with our "k" in the lemma as $k-c$ and our "c" in the lemma as $2c$.

Notice our lemma gives that $\sup_{x \in \partial X} \sum_{c<k<\sqrt{l}} (||F(a,x,k+c,l) - F(a,x,k-c,l)||)$ is $O(l)$, so $\frac{1}{\sqrt{l}} \sup_{x \in \partial X} \sum_{c<k<\sqrt{l}} (||F(a,x,k+c,l) - F(a,x,k-c,l)||)$ is $o(l)$. Thus $||H(ga,x,l) - H(a,x,l)||$ is also $o(l)$ since both sets A and B are $o(l)$. ■

Proposition 5.2.3 *$(x,t) \mapsto H(a,x,l)(t)$ is Borel for $x \in \partial X$, $a \in X$, $t \in G$.*

Proof It is enough to show that $(x,t) \mapsto H(a,x,l)(t)$ is upper semi-continuous. That is, we want to show for every net $(x_n, t_m) \to (x,t)$, $\limsup_{n,m \to \infty} F(a,x_n,k,l)(t_m) \le F(a,x,k,l)(t)$. First we have a lemma.

Lemma 5.2.2 *Let $x_n \to x$, $x \in \partial X$, and $r \in \mathbb{N}$, $a \in X$, and $r_1, r_2 < r < 2l$, then there exists a neighborhood V of x such that for n large enough, $x_n \in V$ and $\cup_{g \in [[a,x_n[[} g([r_1,r_2]) \subset \cup_{g \in [[a,x[[} g([r_1,r_2])$.*

Proof Because we are just looking at geodesics, we just want to show for all $k \leq r$, there exists an $\tilde{n}$ large enough such that $\cup_{g \in [[a,x_n[[} g(k) \subset \cup_{g \in [[a,x[[} g(k)$ for $n \geq \tilde{n}$. This would give us $F(a, x_n, k, l)(t) \leq F(a, x, k, l)(t)$.

Assume this is not true. Suppose there exists a net of geodesics $< g_n >$ of geodesics starting at a with endpoints converging to x where this is not true. Consider the sequence $< g_n(k) >$ where $\{g_1 := [a, x_1], g_2 := [a, x_2], g_3 := [a, x_3], ..., g_n := [a, x_n], ...\}$, where $x_n \to x$ and such that $g_n(k) \notin \cup_{g \in [[a,x[[} g(k)$. We have that $k \leq r$ and so $\cup_{g \in [[a,x[[} g(k) \subset B(a, r)$. Since X is a locally compact hyperbolic space and $B(a, r)$, the open ball of radius r centered at a, is bounded, $\overline{B(a, r)}$ is compact. Thus $|\cup_{g \in [[a,x[[} g(k)|$ is bounded. By Arzela Ascoli's Theorem, if $< f_n >$ is a net of continuous functions on a compact space into a metric space and $|f_n(x)| \leq M$, that is, f_n is uniformly bounded and equicontinuous, then there exists a subsequence $< f_{n_k} >$ that converges uniformly [26].

Now for $k > 0$, the $g_n(k)$'s are continuous on a compact space with values in a metric space. We have that $|g_n(k)|$ is bounded because it is in an r-ball. That is, the length of each geodesic is bounded. So by Arzela Ascoli's theorem, there exists a subsequence $g_{n_m} \to g_\infty$ from a to x and $g_\infty(k) \notin \cup_{g \in [[a,x[[} g(k)$. But this is a contradiction since g_∞ would be a geodesic from a to x. Thus, our lemma is true, and so for $x_n \to x$, $F(a, x_n, k, l)(t) \leq F(a, x, k, l)(t)$. Therefore $\limsup_{n \to \infty} F(a, x_n, k, l)(t) \leq F(a, x, k, l)(t)$ and F is upper-semi-continuous in X.

■

We can now finish the proof of 5.2.3 by showing F is upper-semi-continuous in G. Suppose $t_n \to t$ in G. We already have a continuous group action which is uniformly continuous on compact sets, so F is continuous in G. Therefore, $\limsup_{n \to \infty} F(a, x, k, l)(t_n) \leq F(a, x, k, l)(t)$. therefore, if $(x_n, t_m) \to (x, t)$, then $\limsup_{n,m \to \infty} F(a, x_n, k, l)(t_m) \leq F(a, x, k, l)(t)$. And so, F is upper-semi-continuous in X and G and therefore is a Borel function. ■

Theorem 5.2.3 *A locally compact group that acts continuously on a locally compact hyperbolic space, X, acts amenably on ∂X.*

Proof Define f_n on $G \times \partial X$ by $f_n(g,x) = H(y_0, x, n)(g)$, where y_0 is fixed basepoint for X. Since H is a nonnegative Borel function that is compactly supported by $B(y_0, 3n)$, so is f_n. By 5.2.1, $||H(y_0, x, n)|| \geq n$. Therefore, $\int_G f_n(g,x)dg > 0$ for all $x \in \partial X$ and $n > 0$. Let $K \subset G$, be a compact subset of G. By our diagonal group action, for $h \in K \subset G$, $h.f_n(g,x) = f_n(h^{-1}g, h^{-1}x) = H(y_0, h^{-1}x, n)(h^{-1}g)$. By shifting the basepoint y_0, we get $H(y_0, h^{-1}x, n)(h^{-1}g) = H(hy_0, x, n)(g)$. Note we can do this by translating the geodesics from y_0 to $h^{-1}x$ by h. Since we are in a space with a G-invariant measure, this will not change the function H. Now,

$$\lim_{n\to\infty} \sup_{x\in\partial X} \left(\frac{\int_G |f_n(g,x) - h.f_n(g,x)|dg}{\int_G f_n(g,x)dg}\right) =$$

$$\lim_{n\to\infty} \sup_{x\in\partial X} \left(\frac{\int_G |H(y_0,x,n)(g) - H(y_0,h^{-1}x,n)(h^{-1}g)|dg}{\int_G H(y_0,x,n)(g)dg}\right) =$$

$$\lim_{n\to\infty} \sup_{x\in\partial X} \frac{||H(y_0,x,n)(g) - H(hy_0,x,n)(g)||}{||H(y_0,x,n)(g)||} \leq \lim_{n\to\infty} \frac{o(n)}{n} = 0 \text{ uniformly on } K \subset G.$$

Thus, G acts amenably on a compact space ∂X. ■

Our theorem shows that a group G that acts continuously on a locally compact hyperbolic space satisfies 2.2.3 and therefore is amenable at infinity. We also now have the following corollary since locally compact Gromov hyperbolic groups act continuously on their compact Gromov boundaries.

Corollary 5.2.4 *Locally compact Gromov hyperbolic groups have Property A.*

5.3 Other Definitions of Property A

In the fall of 2013, Steven Deprez and Kang Li introduced an equivalent notion of locally compact Property A Groups in [15]. In this section, we will show that our definition is equivalent.

Definition 5.3.1 *[Deprez, Li] Given a locally compact, second countable group G, we say G has Property A if for any compact subset $K \subset G$ and $\varepsilon > 0$, there exists* *a compact subset* $L \subset G$ *and a family of Borel sets* $\{A_g\}_{g\in G} \subseteq G \times \mathbb{N}$ *with finite measure* $(0 < \mu(A_g) < \infty)$ *such that*

1. *for all* $(g,t) \in$ *Tube(K),* $\dfrac{\mu(A_g \Delta A_h)}{\mu(A_g \cap A_h)} < \varepsilon$*, and*
2. *if* $(g', n) \in A_g$*, then* $(g, g') \in$ *Tube(L).*

Note that the measure on $G \times \mathbb{N}$ is the product measure of the Haar measure on G and the counting measure on $\mathbb{N}$. Also recall our definition of a tube from 2.2.4. For a compact subset K of G, Tube(K)=$\{(g,h) \in G \times G : g^{-1}h \in K\}$. And we say that a subset $L \subseteq G \times G$ is a tube if $\{g^{-1}h : (g,h) \in L\}$ is precompact or if L is a subset of some other tube.

Although we defined the following terms in 2.2.3, we will restate a few definitions and notations. Recall we let $\beta^{\mu}(G)$ represent the universal compact Hausdorff left G-space equipped with a continuous G-equivariant inclusion of G as an open dense subset. This space has the universal property that any continuous G-equivariant map from G into a compact Hausdorff left G-space X has a unique extension to a continuous G-equivariant map from $\beta^{\mu}(G)$ into X. This notion is basically analogous to the Stone Cech compactification.

We also recall that $C_{b,\theta}(G \times G)$ is the algebra of continuous, bounded functions f on $G \times G$ such that $f \circ \theta$ is the restriction of a continuous function on $\beta^{\mu}(G) \times G$, where θ is the homeomorphism of $G \times G$ such that $\theta(g,h) = (g^{-1}, g^{-1}h)$. We can also identify $C(\beta^{\mu}(G))$ with the C^*-algebra of bounded, left-uniform continuous functions on G. That is, $C(\beta^{\mu}(G))$ is the algebra of all bounded continuous functions on G such that $f(t^{-1}s) - f(s)$ uniformly goes to 0 as $t \to e \in G$, where e is the identity element of G [15]. And we have that a positive type kernel on $G \times G$ is a function k such that for every positive integer, n, and every $g_1, ..., g_n \in G$, the matrix $[k(g_i, g_j)]$ is positive. We now include the definition of a cut-off function for G and note that every locally compact second countable group has cut-off functions [15].

Definition 5.3.2 *f is a cut-off function for G if $f \in C_c(G)$ such that*

1. $f \geq 0$,
2. $f(g^{-1}) = f(g)\ \forall g \in G$,
3. supp{f} *is a compact neighborhood of the identity element of G, and*
4. $\displaystyle\int_G f(g)d\mu(g) = 1$.

Proposition 5.3.1 *[Deprez, Li [15]] Let f be a continuous function from $G \times G \to \mathbb{C}$. We have that $f \circ \theta$ extends to a continuous function on $\beta^\mu(G) \times G$ if and only if f satisfies the following two conditions:*

1. $\sup_{v \in G} |f(v, vt)| < \infty$ *for all $t \in G$, and*
2. $\sup_{v \in G} |f(vs_n, vt_n) - f(vs, vt)| \to 0$ *for all $s_n \to s$ and $t_n \to t$.*

Proof Assume we have a continuous function $f : G \times G \to \mathbb{C}$ such that we have conditions (1) and (2) from above. So by (1), we have that f is bounded. And for $s_n \to e$ and $t_n \to e$, where e is the identity element of G, we have that $\sup_{v \in G} |f(vs_n, vt_n) - f(ve, ve)| \to 0$ and thus $\sup_{v \in G} |f(vs_n, vt_n) - f(v, v)| \to 0$ by (2). Since θ is an homeomorphism, we have that $\sup_{v \in G} |f \circ \theta(vs_n, vt_n) - f \circ \theta(v, v)| \to 0$. This implies, by our definition of θ, that $\sup_{v \in G} |f \circ \theta(vs_n, vt_n) - f \circ \theta(v, v)| = \sup_{v \in G} |f(s_n^{-1}v^{-1}, s_n^{-1}v^{-1}vt_n) - f(v^{-1}, v^{-1}v)| = \sup_{v \in G} |f(s_n^{-1}v^{-1}, s_n^{-1}t_n) - f(v^{-1}, e)|$.

We want to show that f is in $C(\beta^\mu(G))$ is the algebra of all bounded continuous functions on G such that $f(t^{-1}s) - f(s)$ uniformly goes to 0 as $t \to e \in G$. In our case, we want to show that $f(t_1^{-1}s_1, t_2^{-1}s_2) - f(s_1.s_2)$ uniformly goes to 0 as $t_1, t_2 \to e$. So for $f(s_n^{-1}v^{-1}, s_n^{-1}t_n) - f(v^{-1}, e)$, we have $t_1 = s_n$, $s_1 = v^{-1}$, $s_2 = e$, and $t_2 = t_n^{-1}s_n$ which implies $t_2^{-1} = s_n^{-1}t_n$. Thus, we have that $f(t_1^{-1}s_1, t_2^{-1}s_2) - f(s_1, s_2)$ uniformly goes to 0 as $t_1, t_2 \to e$ and therefore $f \in C(\beta^\mu(G))$.

On the other hand, suppose $f \circ \theta$ extends to a continuous function on $\beta^{\mu}(G) \times G$, say F, so F is in $C(\beta^{\mu}(G) \times G)$. We look at the restriction of F on $G \times G$, which is $f \circ \theta$. Since F is bounded, so is the restriction, thus $\sup_{v \in G} |f \circ \theta(v^{-1}, t)| < \infty$. We also have that $\sup_{v \in G} |f \circ \theta(v^{-1}, t)| = \sup_{v \in G} |f(v, vt)|$ and therefore (1) is satisfied. Similar to the argument above, $F(t_1^{-1}s_1, t_2^{-1}s_2) - F(s_1, s_2)$ uniformly goes to 0 as $t_1, t_2 \to e \in G$ and so the restriction $f \circ \theta$ does as well. So we have $f \circ \theta(t_1^{-1}s_1, t_2^{-1}s_2) - f \circ \theta(s_1, s_2)$ uniformly goes to 0 as $t_1, t_2 \to e$. This gives us, $f(s_1^{-1}t_1, s_1^{-1}t_1t_2^{-1}s_2) - f(s_1^{-1}, s_1^{-1}s_2) = f(vs_n, vt_n) - f(vs, vt)$ where $v = s_1^{-1}$ and $t_1 = s_n \to e$ and $t_n = t_1t_2^{-1}s_2 \to s_2$ and thus we have (2).

■

Deprez and Li also prove the following characterization for their definition of Property A:

Theorem 5.3.1 *[Deprez, Li] Let G be a locally compact second countable group, then the following are equivalent:*

1. *G has Property A as in 5.3.1.*

2. *For $\varepsilon > 0$ and for any compact subset K of G, there exists a compact subset $L \subseteq G$ and a continuous map $f : G \to L^1(G)$ such that $||f_g|| = 1$, $\mathrm{supp}\{\mathrm{f_g}\} \subseteq \mathrm{gL}$ for every $g \in G$ and $\sup_{(g,h) \in Tube(K)} ||f_g - f_h||_1 < \varepsilon$.*

3. *For $\varepsilon > 0$ and for any compact subset K of G, there exists a compact subset $L \subseteq G$ and a weak* continuous map $v : G \to C_0(G)^*_+$ such that $||v_g|| = 1$, $\mathrm{supp}\{\mathrm{v_g}\} \subseteq \mathrm{gL}$ for every $g \in G$ and $\sup_{(g,h) \in Tube(K)} ||v_g - v_h|| < \varepsilon$.*

4. *For $\varepsilon > 0$ and for any compact subset K of G, there exists a compact subset $L \subseteq G$ and a continuous map $\phi : G \to L^2(G)$ such that $||\phi_g|| = 1$, $\mathrm{supp}\{\phi_{\mathrm{g}}\} \subseteq \mathrm{gL}$ for every $g \in G$ and $\sup_{(g,h) \in Tube(K)} ||\phi_g - \phi_h||_2 < \varepsilon$.*

5. *For $\varepsilon > 0$ and for any compact subset K of G, there exists a compact subset $L \subseteq G$ and a continuous type kernel $k : G \times G \to \mathbb{C}$ such that* $\mathrm{supp}\{\mathrm{K}\} \subseteq$ *Tube(L) and* $\sup_{(g,h)\in Tube(K)} |k(g,h) - 1| < \varepsilon$.

Continuous functions are standard for locally compact groups, but [15] shows that we can relax this condition. They prove the following two lemmas:

Lemma 5.3.2 *[Deprez, Li] Suppose G is a locally compact, second countable group such that if for any $\varepsilon > 0$ and compact subset K of G, there exists a compact subset $L \subseteq G$ and a map $f : G \to L^1(G)$ such that $||f_g|| = 1$,* $\mathrm{supp}\{\mathrm{f_g}\} \subseteq \mathrm{gL}$ *for every $g \in G$ and $\sup_{(g,h)\in Tube(K)} ||f_g - f_h||_1 < \varepsilon$, then the map $g \to f_g$ is continuous and thus G satisfies condition 2 of 5.3.1.*

Lemma 5.3.3 *[Deprez, Li] Given a locally compact, second countable group G, a measurable kernel $k_0 : G \times G \to \mathbb{C}$ which is bounded on every tube, and a cut-off function $f : G \to [0,\infty)$ for G (which exists for any locally compact second countable group [15]), we can now define a new kernel from $G \times G \to \mathbb{C}$ in the following way: let $k(s,t) = \int_G \int_G f(v)f(w)k_0(sv,tw)d\mu(v)d\mu(w)$ and we an show that k has the following properties:*

1. *k is bounded on every tube.*

2. *k is continuous and satisfies the following uniform continuity property: whenever $s_n \to s$ and $t_n \to t$, then $\sup_{v\in G} |k(vs_n, vt_n) - k(vs, vt)| \to 0$.*

3. *If* $\mathrm{supp}\{\mathrm{k_0}\}$ *is in a tube, so is the support of k.*

4. *If k_0 is a positive type kernel, then so is k.*

Now we are ready to provide the proof to 5.3.1 for the convenience of the reader.

Proof We prove $1 \iff 2$, $2 \iff 4$, and $4 \iff 5$ and we omit the proof of $2 \iff 3$ since we will not use property 3 below in 5.3.4.

1⇒2: Assume G has Property A in the sense of Deprez and Li. Let $K \subseteq G$ be a compact subset and let $\varepsilon > 0$. By 5.3.2, we don't necessarily need to find a continuous map. Since G has Property A, we can find a compact subset $L \subseteq G$ and a family of Borel subsets, $\{A_s\}_{s\in G}$ in $G\times\mathbb{N}$ with $0 < \mu(A_s) < \infty$. Recall μ is the product measure of the Haar measure, μ', and the counting measure on $\mathbb{N}$. We have the following properties now from our definition of Property A:

- for all $(x,t) \in$Tube(K), we have $\dfrac{\mu(A_s \Delta A_t)}{\mu(A_s \cap A_t)} < \dfrac{\varepsilon}{2}$, and
- if $(t,n) \in A_s$, then $(s,t) \in$Tube(L).

Given $s,t \in G$, put $A_{t,x} := (\{s\} \times \mathbb{N}) \cap A_t$. By Tonelli's theorem, we can determine that $\displaystyle\int_G |A_{t,s}| d\mu'(s) = \int_{G\times\mathbb{N}} \chi_A(s,n) d\mu(s,n) = \mu(A_t)$. We now define an almost everywhere measurable map for each $t \in G$, $\eta_t : G \to \mathbb{C}$ where $\eta_t(s) = \dfrac{|A_{t,s}|}{\mu(A_t)}$. This map is clearly nonnegative and in $L^1(G)$. Since $\displaystyle\int_G |\eta_t(s)| d\mu'(s) = \int_G \frac{|A_{t,s}|}{\mu(A_t)} d\mu'(s) = \frac{\mu(A_t)}{\mu(A_t)} = 1$, $||\eta_t||_1 = 1$ for every t. Also notice that $||\eta_s \cdot \mu(A_s) - \eta_t \cdot \mu(A_t)||_1 = \int_G ||A_{s,x}| - |A_{t,x}|| d\mu'(x)$. $||A_{s,x}| - |A_{t,x}|| = |(\{x\} \times \mathbb{N}) \cap A_s| - |(\{x\} \times \mathbb{N}) \cap A_t||$ which is less than or equal to $|(\{x\} \times \mathbb{N}) \cap (A_t \Delta A_s)|$ Thus again using Tonelli's theorem, $||\eta_s \cdot \mu(A_s) - \eta_t \cdot \mu(A_t)||_1 = \int_G ||A_{s,x}| - |A_{t,x}|| d\mu'(x) \leq \int_G |(\{x\} \times \mathbb{N}) \cap (A_t \Delta A_s)| d\mu'(x) = \mu(A_s \Delta A_t)$.

Now for all $(s,t) \in$Tube(K), $||\eta_s - \eta_t||_1 = ||\eta_s - \eta_t \cdot \dfrac{\mu(A_t)}{\mu(A_s)} + \eta_t \cdot \dfrac{\mu(A_t)}{\mu(A_s)} - \eta_t||_1 \leq ||\eta_s - \eta_t \cdot \dfrac{\mu(A_t)}{\mu(A_s)}||_1 + ||\eta_t \cdot \dfrac{\mu(A_t)}{\mu(A_s)} - \eta_t||_1 = \dfrac{\eta_s\mu(A_s) - \eta_t\mu(A_t)}{\mu(A_s)} + ||\eta_t||_1 |\dfrac{\mu(A_t)}{\mu(A_s)} - 1| \leq \dfrac{\mu(A_s \Delta A_t)}{\mu(A_s)} + |\dfrac{\mu(A_t)}{\mu(A_s)} - 1| = \dfrac{\mu(A_s \Delta A_t)}{\mu(A_s)} + |\dfrac{\mu(A_t) - \mu(A_s)}{\mu(A_s)}| \leq 2\dfrac{\mu(A_s \Delta A_t)}{\mu(A_s)} \leq 2\dfrac{\mu(A_s \Delta A_t)}{\mu(A_s \cap A_t)} < \varepsilon$. Furthermore, if $\eta_t(s) \neq 0$, then $(s,n) \in A_t$ for some n, which by definition of A_t means $(t,s) \in$Tube(L). Thus, supp$\{\eta_t\} \subseteq$ tL.

2⇒1: Let $K \subseteq G$ be a compact subset and let $\varepsilon > 0$. Choose $0 < \varepsilon' < 1$ such that $\dfrac{6\varepsilon'}{2 - 5\varepsilon'} < \varepsilon$. By assumption we have a compact subset $L \subseteq G$ and continuous map $f : G \to L^1(G)$ such that $||f||_1 = 1$, supp{f} $\subseteq$ tL for every $t \in G$ and

$\sup_{(s,t)\in Tube(K)} ||f_s - f_t||_1 < \varepsilon'$. We identify f_t with a representation function $f_t : G \to \mathbb{C}$ and we can see that support of this representation function is a subset of tL since $\{s \in G : f_t(s) \neq 0\} \subseteq tL$. Since $|||f_s| - |f_t|||_1 \leq ||f_s - f_t||_1$, we can assume f_t is nonnegative.

We have that $\mu'(L) > 0$ since otherwise, $||f_t||_1 = 0$ for every $t \in G$. Put $M := \dfrac{\mu(L)}{\varepsilon'} > 0$, and for each $t \in G$ put $A_t := \{(s,n) \in G \times \mathbb{N} : n \leq f_t(s) \cdot M\}$. We note A_t is a Borel subset of $G \times \mathbb{N}$ for each $t \in G$. For $s, t \in G$, we put $A_{t,s} := \{n \in \mathbb{N} : (s,n) \in A_t\}$ and for every $t \in G$, we define a measurable map $\theta_t : G \to [0, \infty)$ where $\theta_t(s) = \dfrac{|A_{t,s}|}{M} = \dfrac{|A_{t,s}|\varepsilon'}{\mu'(L)}$.

Notice θ_t satisfies the following two properties:

- $\mu(A_t) = M \cdot ||\theta_t||_1$ by the definition of A_t, θ_t, and $A_{t,s}$, and
- $||\theta_t - f_t||_1 < \dfrac{\mu'(L)}{M} = \varepsilon'$ for all $t \in G$.

Since $|||\theta_t| - |f_t|||_1 \leq ||\theta_t - f_t||_1 < \varepsilon$ and $||f_t||_1 = 1$, we have that $|\dfrac{\mu(A_t)}{M} - 1| < \varepsilon'$ which implies $1 - \varepsilon' < \dfrac{\mu(A_t)}{M} < 1 + \varepsilon'$ which implies $M(1-\varepsilon') < \mu(A_t) < M(1+\varepsilon')$. Thus, $\mu(A_t)$ is finite. In fact, $\mu(A_t \Delta A_s) = \int_G |A_{t,x} \Delta A_{s,x}| d\mu'(x) = \int_G ||A_{t,x}| - |A_{s,x}|| d\mu'(x) = M \cdot ||\theta_t - \theta_s||_1$. Therefore, $\dfrac{\mu(A_s \Delta A_t)}{\mu(A_s \cap A_t)} = \dfrac{2\mu(A_s \Delta A_t)}{2\mu(A_s \cap A_t)} = \dfrac{2\mu(A_s \Delta A_t)}{\mu(A_s) + \mu(A_t) - \mu(A_s \Delta A_t)}$ (since $\mu(A_s) + \mu(A_t) - \mu(A_s \Delta A_t) = 2\mu(A_s \cap A_t)$).

Moreover, we have $\dfrac{2\mu(A_s \Delta A_t)}{\mu(A_s) + \mu(A_t) - \mu(A_s \Delta A_t)} = \dfrac{2||\theta_s - \theta_t||_t}{||\theta_s||_1 + ||\theta_t||_1 - ||\theta_s - \theta_t||_1}$. We also have that $||\theta_t||_1 > (1 - \varepsilon')$ for each $t \in G$ and that $||\theta_s - \theta_t||_1 < 3\varepsilon'$ for every (s,t) ∈Tube(K). We can see that $\dfrac{\mu(A_s \Delta A_t)}{\mu(A_s \cap A_t)} < \dfrac{6\varepsilon'}{2(1-\varepsilon') - 3\varepsilon'} = \dfrac{6\varepsilon'}{2 - 5\varepsilon'} < \varepsilon$ for every (s,t) ∈Tube(K). And lastly, we have that (t,s) ∈Tube(L) since if $(s,n) \in A_t$, then $f_t(s) \neq 0$.

2⇒4: Let $f : g \to L^1(G)$ be the map we have in (2). For each $t \in G$, define $\phi_t = |f_t|^{\frac{1}{2}}$. Now we have $||\phi_t - \phi_s||_2^2 = \int_{x\in G} |\phi_t(x) - \phi_s(x)|^2 d\mu'(x) \leq \int_{x\in G} |\phi_t^2(x) - \phi_s^2(x)| d\mu'(x) = \int_{x\in G} ||\phi_t(x)| - |\phi_s(x)|| d\mu'(x) \leq ||\phi_t - \phi_s||_1$.

Now since $||f_t||_1$ and supp{$\mathrm{f_t}$} ⊆ tL and $\phi_t = |f_t|^{\frac{1}{2}}$, we have that $||\phi_t||_2 = 1$ and supp{ϕ_t} ⊆ tL. Furthermore, we see that $\sup_{(s,t)\in Tube(K)} ||f_s - f_t||_1 < \varepsilon$ which implies $\sup_{(s,t)\in Tube(K)} ||\phi_s - \phi_t||_2 < \varepsilon$ and gives us (4).

4⇒2: Let $\phi : G \to L^2(G)$ be the map in (4). For each $t \in G$, define $n_t = |\phi_t|^2$. By the Cauchy -Schwarz inequality, we have $||f_t - f_s||_1 = \int_G ||\phi_t(x)|^2 - |\phi_s(x)|^2| d\mu'(x) = \int_{x\in G} (|\phi_t(x)| + |\phi_s(x)|)||\phi_t(x)| - |\phi_s(x)|| d\mu'(x) \leq (|||\phi_t| + |\phi_s|||_2) \cdot (|||\phi_t| - |\phi_s|||_2) \leq 2||\phi_t - \phi_s||_2$. And thus we have (2).

5⇒4: Let K be a compact subset of G, $0 < \varepsilon < \frac{1}{2}$, and let f be a cut-off function for G. Since the support of cut-off functions are in compact sets, (supp{f} · (K ∪ {e}) · supp{f}) is a compact subset of G. By assumption, there exists a compact subset $L \subseteq G$ and a continuous positive type kernel k_0 on G such that supp{$\mathrm{k_0}$} ⊆Tube(L) and sup{$|k_0(s,t) - 1|$: $(x,y) \in$ Tube(supp{f} · (K ∪ {e}) · supp{f})}$< \varepsilon$. Since k_0 is a positive type kernel, k_0 is bounded. In fact $k_0(s,s) < (1+\varepsilon)$ for all $s \in G$, since, by our work above, we have $|k_0(s,s) - 1| < \varepsilon$. By 5.3.3, we can construct a continuous, bounded, positive type kernel supported in a tube,say L', $k : G \times G \to \mathbb{C}$ given by $k(s,t) = \int_G \int_G f(v)f(w)k_0(sv,tw)d\mu'(w)d\mu'(v)$. If $(s,t) \in$Tube($K \cup \{e\}$), then $|k(s,t) - 1| =| \int_G \int_G f(v)f(w)k_0(sv,tw)d\mu'(w)d\mu'(v) - \int_G f(v)d\mu'(v) \int_G f(w)d\mu'(w)| \leq \int_{\mathrm{supp\{f\}}} \int_{\mathrm{supp\{f\}}} f(w)f(v)|k_0(sv,tw) - 1| d\mu'(v)d\mu'(w) \leq$ sup{$|k_0(x,y) - 1|$: $(x,y) \in$Tube(supp{f} · (K ∪ {e}) · supp{f})}$< \varepsilon$.

Now let T_{k_0} be the integral operator induced by k_0 on $L^2(G)$ which is defined by $T_{k_0}(f)(s) = \int_G k_0(s,t)f(t)\mu'(t)$. T_{k_0} is bounded and positive and $k(s,t) =: < T_{k_0}f_t, f_s >$ in which $f_t(x) = f(t^{-1}x)$. And we consider a positive polynomial p such that $|p(t)-1| < \frac{\varepsilon}{||f||_2^2}$ for $t \in [0, ||T_{k_0}||]$. Let η be a continuous map from $G \to L^2(G)$ defined by $\eta_t = p(T_{k_0})f_t$. Now we have $| < \eta_t, \eta_s > -k(s,t)| = | < p(T_{k_0})f_t, p(T_{k_0})f_s > -k(s,t)| = | < p^2(T_{k_0})f_t, f_s > - < T_{k_0}f_t, f_s > |$ (since $k(s,t) =< T_{k_0}f_t, f_s >$). By our inner product properties, this is equal to $| < p^2(T_{k_0})f_t - T_{k_0}f_t, f_s > | \leq ||p^2(T_{k_0}) - T_{k_0}|| \cdot ||f_t||_2 \cdot ||f_s||_2 < \varepsilon$ for all $s, t \in G$.

Now we have $| < \eta_t, \eta_s > -1| = | < \eta_t, \eta_s > -k(s,t) + k(s,t) - 1| \leq | < \eta_t, \eta_s > -k(s,t)| + |k(s,t) - 1| < 2\varepsilon$ for every $(s,t) \in$Tube$(K \cup \{e\})$. This now implies $1 - 2\varepsilon < Re(< \eta_t, \eta_s >) < 2\varepsilon + 1$ for every $(s,t) \in$Tube$(K \cup \{e\})$. Since $\varepsilon < \frac{1}{2}$ and $|||\eta_t||_2^2 - 1| < 2\varepsilon$, then $0 < \sqrt{1-2\varepsilon} < ||\eta_t||_2 < \sqrt{1+2\varepsilon}$.

We now define a continuous map $\phi : G \to L^2(G)$ by $\phi_t = \frac{\eta_t}{||\eta_t||_2}$ which satisfies the following: $1 - Re(< \phi_t, \phi_s >) = 1 - \frac{Re(< \eta_t, \eta_s >)}{< \eta_t, \eta_t >^{\frac{1}{2}} < \eta_s, \eta_s >^{\frac{1}{2}}} \leq 1 - \frac{1-2\varepsilon}{1+2\varepsilon} = \frac{4\varepsilon}{1+2\varepsilon} < 4\varepsilon$ for all $(s,t) \in$Tube$(K \cup \{e\})$. And thus, we have that $||\phi_s - \phi_t|_2 = \sqrt{2 - 2Re(< \phi_t, \phi_s >)} < \sqrt{8\varepsilon}$ for all $(s,t) \in$Tube(K). Moreover, if p is of degree n, $\mathrm{supp}\{\phi_\mathrm{t}\} \subseteq t \cdot \mathrm{supp}\{\mathrm{f}\} \cdot ((\mathrm{L}^\mathrm{n})^{-1} \cup \cdots \cup \mathrm{L}^{-1} \cup \{\mathrm{e}\})$ which gives us our support requirement for (4). ■

In the same paper, Deprez and Li show that their definition of Property A is the same as Delaroche's notion of amenable at infinity groups. Recall our theorem in Section 2.2.2 in which Delaroche lists several characterizations for amenable at infinity. Since our definition of property A requires a group to act amenably on a compact set, our definition is equivalent to amenable at infinity. We now show that our definition is equivalent to Deprez and Li's definition for Property A groups.

Theorem 5.3.4 (H) *Given a locally compact, second countable group G, the following are equivalent:*

1. *There exists a compact space X such that G acts amenably and continuously on X.*

2. *G is amenable at infinity.*

3. *There exists a sequence $< h_i >$ of positive type kernels in $C_{b,\theta}(G \times G)$ with support in a tube such that $\lim_i h_i = 1$ uniformly on tubes.*

4. *For any compact subset $K \subset G$ and $\varepsilon > 0$, there exists a compact subset $L \subset G$ and a family of Borel sets $\{A_g\}_{g \in G} \subseteq G \times \mathbb{N}$ with finite measure $(0 < \mu(A_g) < \infty)$ such that*

   - *for all $(g,t) \in$ Tube(K), $\dfrac{\mu(A_g \Delta A_h)}{\mu(A_g \cap A_h)} < \varepsilon$, and*
   - *if $(g', n) \in A_g$, then $(g, g') \in$ Tube(L).*

5. *For $\varepsilon > 0$ and for any compact subset K of G, there exists a compact subset $L \subseteq G$ and a continuous type kernel $k : G \times G \to \mathbb{C}$ such that $\operatorname{supp}\{k\} \subseteq$ Tube(L) and $\sup_{(g,h) \in Tube(K)} |k(g,h) - 1| < \varepsilon$.*

Proof Both (2) $\iff$ (3) and (4) $\iff$ (5) were listed for convenience and proved in 2.2.2 and 5.3.1 respectively. Since G is locally compact, we must have a continuous group action. Thus we have (1) $\iff$ (2).

(3)$\Rightarrow$ (5): Let $\varepsilon > 0$ and let K be a compact subset of G, and suppose there exists a sequence $< h_i >$ of positive type kernels in $C_{b,\theta}(G \times G)$ with support in a tube such that $\lim_i h_i = 1$ uniformly on tubes. Since for any $i \in I$, h_i has its support in a tube, say L, and converges to 1 uniformly on tubes, we can find an $i \in I$ such that $\sup_{(g,h) \in Tube(K)} |h_i(g,h) - 1| < \varepsilon$. Therefore we have (5).

(5)$\Rightarrow$ (3): On the other hand, suppose for any $\varepsilon > 0$ and compact subset K of G, we have a continuous type kernel $k_0 : G \times G \to \mathbb{C}$ such that $\{K\} \subseteq$Tube(L) and $\sup_{(g,h) \in Tube(K)} |k_0(g,h) - 1| < \varepsilon$. By 5.3.3, we can define a kernel that is also of positive type, supported in a tube, bounded on every tube, and satisfied the uniform continuity property. That is, for any sequence $s_n \to s$ and $t_n \to t$ in G, then $\sup_{v \in G} |k(vs_n, vt_n) - k(vs, vt)| \to 0$. Notice that this uniform continuity is the same requirement needed in 5.3.1, and since k is bounded on every tube, we have the requirements of 5.3.1. Therefore our k is in $C_{b,\theta}(G \times G)$. Thus we have $\lim_i k_i = 1$ uniformly on tubes which is enough to show (3). ■

Deprez and Li show that a locally compact, second countable Hausdorff group which is of Property A satisfies the Baum Connes Conjecture. Thus, our locally compact Property A groups have the same result.

Corollary 5.3.5 (Deprez, Li) *If G is a locally compact, second countable, Hausdorff group which has Property A, then the Baum-Connes assembly map with coefficients for G is split-injective.*

6. CONCLUSION

6.1 Further Work

Since we have expanded the notion of Property A to locally compact groups, we can now ask whether we have similar results for properties we know to be true for discrete Property A groups. For example, we can ask whether we can generalize [16]'s work with relative Property A groups to the locally compact case. We can also connect our locally compact Property A groups with the cohomology of groups. As we mentioned in 2.2.4, we can connect amenable actions and Property A with the cohomology of a group. In [11], Brodzki, Niblo, Nowak, and Wright relate amenable actions and exactness of a countable discrete group with the bounded cohomology of the group. And in [17], Monod examines some of the characterization of topologically amenable actions in terms of bounded cohomology. The next step for our definition of locally compact Property A groups is to show similar results for the continuous bounded cohomology of a group.

Whereas [11] and [16] use the analogue of $\ell^1(G)$, $W_0(G,X)$, we should use an analogue of $L^1(G)$. We now denote by $\mathscr{V}$, the Banach space of all functions $f : G \to C(X)$ endowed with the sup-L^1 norm of f defined as $||f||_{\infty,1} = \sup_{x\in X} \int_G |f_g(x)|dg$. Note for a function $f : G \to C(X)$, f_g denotes the continuous function on X obtained by evaluating f at $g \in G$ where G is a locally compact group. We now define $\mathscr{W}_{00}(G,X)$ to be the subspace of V which contains all functions $f : G \to C(X)$ which have *compact* support and such that for some $c \in \mathbb{R}$ which depends on f, $\int_G f_g = c1_X$. Notice we have compact support instead of finite support like in [11] since we have a locally compact group. We denote the closure of $\mathscr{W}_{00}(G,X)$ in the sup-L^1 norm by

$\mathscr{W}_0(G,X)$. And we let $\pi : \mathscr{W}_{00}(G,X) \to \mathbb{R}$ be defined so that $\int_G f_g = \pi(f)1_X$. As with the finite support case, we can extend this map to the closure $\mathscr{W}_0(G,X)$ and we denote the kernal of this extension by $\mathscr{N}_0(G,X)$. And the continuous group action on X gives an isometric action of G on $C(X)$. That is, for $g \in G$ and $f \in C(X)$, we define the action $(g \cdot f)(x) = f(g^{-1}x)$. G also acts isometrically on V. For $g, h \in G$, $x \in X$, $f \in \mathscr{V}$, we have $(gf)_h(x) = f_{g^{-1}h}(g^{-1}x) = (g \cdot f_{g^{-1}h})(x)$.

In 2.2.4, we stated Monod's Theorem from [17] about the connection between a discrete group acting on a compact space X and the cohomology of the group. We will state it again for convenience.

Theorem 6.1.1 (Monod) *Let G be a group acting on a compact space X. TFAE:*

1. *The G action on X is topologically amenable. That is, there is a net* $\{u_j\}_{j\in J}$ $\in C(X, \ell^1(G))$ *such that every* $u_j(x)$ *is a probability measure on G and* $\lim_{j\in J} ||gu_j - u_j||_{C(X,\ell^1(G))} = 0$ *for all* $g \in G$.
2. $H^n_b(G, C(X,V)^{**}) = 0$ *for every Banach G-module V and every* $n \geq 1$.
3. $H^n_b(G, \mathscr{I}(C(X), W^*)) = 0$ *for every Banach G-module W and every* $n \geq 1$.
4. $H^n_b(G, E^*) = 0$ *for every Banach G-module E of type M and every* $n \geq 1$.
5. *Any of the previous three points hold for* $n = 1$.
6. $C(X,V)^{**}$ *is relatively injective for every Banach G-module V.*
7. $\mathscr{I}(C(X), W)^*$ *is relatively injective for every Banach G-module W.*
8. *Every dual* (G,X)*-module of type C is a relatively injective Banach G-module.*
9. *There is a G-invariant element in* $C(X, \ell^1(G))^{**}$ *summing to* 1_X.
10. *There is a norm one positive G-invariant element in* $C(X, \ell^1(G))^{**}$ *summing to* 1_X.

Monod mentions in [17] that the proof from 1 to 8 still holds for locally compact, second countable groups. Our next step in our research is to determine whether the whole proof works for locally compact, second countable groups. And thus locally compact Property A groups would follow under this characterization. In doing so, we would possibly have a locally compact version of Monod's theorem which may look something like the following:

Conjecture 4 *Let G be a group acting on a compact space X. TFAE:*

1. *The G action on X is topologically amenable.*
2. $H^n_{cb}(G, C(X,V)^{**}) = 0$ *for every Banach G-module V and every $n \geq 1$.*
3. $H^n_{cb}(G, \mathscr{I}(C(X), W^*)) = 0$ *for every Banach G-module W and every $n \geq 1$.*
4. $H^n_{cb}(G, E^*) = 0$ *for every Banach G-module E of type M and every $n \geq 1$.*
5. *Any of the previous three points hold for $n = 1$.*
6. $C(X,V)^{**}$ *is relatively injective for every Banach G-module V.*
7. $\mathscr{I}(C(X), W)^*$ *is relatively injective for every Banach G-module W.*
8. *Every dual (G, X)-module of type C is a relatively injective Banach G-module.*
9. *There is a G-invariant element $f \in C(X, L^1(G))^{**}$ such that $\int_G |f(g)| dg = 1_X$.*
10. *There is a norm one positive G-invariant element f in $C(X, L^1(G))^{**}$ such that $\int_G |f(g)| dg = 1_X$.*

6.2 Conclusion

In conclusion, locally compact Property A groups broadens the scope of Yu's Property A which in turn expands the scope of coarse embedding, the Baum-Connes Conjecture, and the Novikov Conjecture. Since these notions and conjectures appear in many different fields in mathematics including cohomology, differential geometry, and K-theory, our research could show up in a variety of applications.

LIST OF REFERENCES

LIST OF REFERENCES

[1] Matthias Kreck and Wolfgang Lück. The novikov conjecture oberwolfach-seminar january 2004. 2004.

[2] Steven C Ferry, Andrew Ranicki, and Jonathan Rosenberg. A history and survey of the novikov conjecture. *Novikov conjectures, index theorems and rigidity*, 1:7–66, 1995.

[3] Piotr Nowak. Coarse embeddings of metric spaces into banach spaces. *Proceedings of the American Mathematical Society*, 133(9):2589–2596, 2005.

[4] Terence Tao. *An introduction to measure theory*, volume 126. American Mathematical Soc., 2011.

[5] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[6] Jean-Paul Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1984. A Wiley-Interscience Publication.

[7] Paul S. Muhly, Jean N. Renault, and Dana P. Williams. Equivalence and isomorphism for groupoid C^*-algebras. *J. Operator Theory*, 17(1):3–22, 1987.

[8] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.*, 354(10):4153–4178 (electronic), 2002.

[9] Narutaka Ozawa. Amenable actions and applications. In *Proceedings oh the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures*, pages 1563–1580, 2006.

[10] Jean-Louis Tu. Remarks on yu's' property a'for discrete metric spaces and groups. *Bulletin de la societe mathematique de France*, 129(1):115–140, 2001.

[11] Jacek Brodzki, Graham A Niblo, Piotr W Nowak, and Nick Wright. Amenable actions, invariant means and bounded cohomology. *Journal of Topology and Analysis*, 4(03):321–334, 2012.

[12] Piotr Nowak and Guoliang Yu. What is property a. *Notices of the AMS*, 55(4):474–475, 2008.

[13] Rufus Willett. Some notes on property a. *arXiv preprint math/0612492*, 2006.

[14] Goulnara Arzhantseva, Erik Guentner, and Ján Špakula. Coarse non-amenability and coarse embeddings. *Geometric and Functional Analysis*, 22(1):22–36, 2012.

[15] Steven Deprez and Kang Li. Property a and uniform embedding for locally compact groups. *arXiv preprint arXiv:1309.7290*, 2013.

[16] Ronghui Ji, Crichton Ogle, and Bobby W. Ramsey. Relative property A and relative amenability for countable groups. *Adv. Math.*, 231(5):2734–2754, 2012.

[17] Nicolas Monod. A note on topological amenability. *arXiv preprint arXiv:1004.0199*, 2010.

[18] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[19] P.-E. Caprace, Y. de Cornulier, N. Monod, and R. Tessera. Amenable hyperbolic groups. *ArXiv e-prints*, February 2012.

[20] Dennis Dreesen. Embeddings of locally compact hyperbolic groups into lp-spaces. *arXiv preprint arXiv:1303.4250*, 2013.

[21] Mark Richard OBrien. *Right-Angled Coxeter Groups and CAT (0) Spaces A thesis submitted*. PhD thesis, TUFTS UNIVERSITY, 2008.

[22] John Kilgore. Fundamental domains of discrete groups acting on euclidean space. 2012.

[23] Jean Louis Koszul, RR Simha, and R Sridharan. *Lectures on groups of transformations*, volume 32. Tata institute of fundamental research Bombay, 1965.

[24] Athanase Papadopoulos. Introduction to Teichmüller theory, old and new, III. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 1–34. Eur. Math. Soc., Zürich, 2012.

[25] *Discrete geometry and topology*. American Mathematical Society, Providence, RI, 1992. On the 100th anniversary of the birth of Boris Nikolaevich Delone, A translation of Trudy Mat. Inst. Steklov. **1**96 (1991) [1 111 287], Proc. Steklov Inst. Math. **1**992, no. 4(196).

[26] James R Munkres. *Topology*, volume 2. Prentice Hall Upper Saddle River, 2000.

VITA

VITA

Amanda Harsy Ramsay was born in Munster, Indiana. After graduating salutatorian of Chesterton High School, she went to Taylor University in Upland, Indiana where she received a soccer scholorship and the President's Select Scholarship. During this time, Amanda completed her Bachelors of Science in Mathematics and finished a Coaching Certificate with a specialization in soccer. She then attended the University of Kentucky where she earned her Masters of Arts in Mathematics in 2009. Her thesis work, under the guidance of Dr. Edgar Enochs, involved exploring the Sylow P-Groups of Symmetric groups and General Linear Groups. In August 2009, Amanda entered the Graduate School of IUPUI, where she completed her Masters of Science in Mathematics and will finish her doctorate in May 2014. Her dissertation research involved generalizing Yu's Property A to locally compact groups. This work was done under the guidance of Dr. Ronghui Ji. During her time at IUPUI, Amanda distinguished herself by receiving the Mathematics Department's Outstanding Advanced Graduate Student Award in 2012 and the Mathematics Department's Outstanding Graduate Student Teaching Award in both 2013 and 2014. In 2014, the IUPUI School of Science named her the recipient of the School of Science Teaching Assistant Award. In August 2014, Amanda will join the mathematics faculty as a tenure-track assistant professor at Lewis University, a liberal arts school in Romeoville, Illinois.