CERTAIN ASPECTS OF QUANTUM AND CLASSICAL INTEGRABLE SYSTEMS

by

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A Dissertation

Submitted to the Faculty of Purdue University

In Partial Fulfillment of the Requirements for the degree of

Doctor of Philosophy

Department of Mathematical Sciences

Indianapolis, Indiana

August 2023

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Dedicated to my wife Inna, my parents Alexei and Olga, and my sister Kamilla.

ACKNOWLEDGMENTS

To my advisor, Vitaly Tarasov, I thank you for your patience, guidance, and support. I am proud to have learned from you and worked with you.

To my examining committee: Evgeny Mukhin, Alexander Its, Daniel Ramras, I thank you for the time and effort you spent on me. Throughout my graduate career you have been a constant source of motivation and inspiration for me.

I am grateful to my collaborators Alexander Its and Kenta Miyahara. Without your help I would not get that far in my research.

I would like to express my gratitude to my advisor in Saint-Petersburg University, Ludwig Faddeev, whose wide range of interest and philosophy had a huge impact on me. Special thanks are also due to Nikolai Filonov, who shared his passion for math and is a great ping-pong partner.

I would like to thank my fellow graduate students who had become my close friends: Andrei Prokhorov, Filipp Uvarov, Ahmad Barhoumi, and Joel Zirkle.

I would like to appreciate a friendly and productive environment at the Department of Mathematical Sciences at IUPUI. Especially thanks to Tina Carmichael and Cindy Ritchie. I was lucky to pursue my PhD degree at this place.

Finally, I appreciate all the support of my dear wife Inna.

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ABSTRACT

Kosmakov Maksim. Ph.D., Purdue University, May 2023. Certain aspects of quantum and classical integrable systems. Major Professor: Vitaly Tarasov.

We derive new combinatorail formulas for vector-valued weight functions for the evolution modules over the Yangians $Y(\mathfrak{gl}_n)$. We obtain them using the Nested Algebraic Bethe ansatz method.

We also describe the asymptotic behavior of the radial solutions of the negative tt^* equation via the Riemann-Hilbert problem and the Deift-Zhou nonlinear steepest descent method.

1. INTRODUCTION

Before sixties of XX century the list of problems of classical and quantum physics that admit exact solution in one or another form was very short and included just a few examples (some problems on tops in classical mechanics, Ising model, Heisenberg model). They seemed to be exotic exceptions.

In the sixties-seventies the situation drastically changed: large families of non-trivial exactly solvable models were discovered in that time and the basic principles underlying their construction and integrability were understood. Herewith it has appeared that many of them unexpectedly emerge in very different physical contexts and are directly related to the structure of our world.

There are several important types of integrable systems that are known:

- Models with a small number of degrees of freedom (integrable cases of tops).
- Systems of N interacting particles in one dimension: the Calogero-Moser model and all its relatives (classical and quantum).
- Nonlinear partial differential equations as well as difference equations: the Korteweg-de Vries equation (KdV), the nonlinear Schrodinger equation (NLS), the sine-Gordon equation (SG), the Toda chain and the 2D Toda lattice, the Benjamin-Ono equation (BO), the Kadomtsev-Petviashvili equation (KP) and many others.
- Models of statistical mechanics on 2D lattice: Ising model, six- and eight-vertex models and their generalizations.
- Integrable models of quantum physics on 1D lattice: spin chains of XXX, XXZ and XYZ type and their various generalizations.
- Integrable models of quantum field theory in $1 + 1$ dimensions: one-dimensional bose-gas with point-like interaction (quantum NLS equation), the Thirring model, the sine-Gordon model, ...

This division is rather conditional and the list is not complete. There are deep and beautiful connections between the different types of integrable models. For example, poles of singular solutions to integrable partial differential equations move as particles of integrable many-body systems of the Calogero-Moser type. Another example is that the connection between classical and quantum systems is not exhausted by their correspondence in the classical limit.

Here are several possible meanings of integrability:

- A possibility to integrate the equation (i.e. eliminate all derivatives).
- Existence of a complete set of integrals of motion in involution (the Liouville integrability).
- Presence of a large set of exact solutions and a possibility to express the answers through known elementary or special functions.
- A possibility to reduce the problem to a solution of a finite system of algebraic or integral equations.
- Presence of rich symmetries and interesting algebraic or analytic structures.

The different types of integrable systems mentioned above provide examples to all these meanings of the notion of integrability.

This dissertation consists of two parts. The first part, discussed in Chapter 2, is connected to the quantum integrable systems. The second part, discussed in Chapter 3, is connected to the classical integrable systems. In this common introduction we would like to recall the main concepts of the integrability. We will start with finite dimensional Hamiltonian systems, where we have the notion of integrability in the Liouville sense.

1.1 Liouville integrability

Consider a Hamiltonian dynamical system with a $2d$-dimensional phase space M parameterised by the canonical variables

$$(q_\mu, p_\mu), \qquad \mu = 1, ..., d.$$

Let the Hamiltonian function be $H(q_\mu, p_\mu)$, where q_μ, p_μ denotes the collection of variables (1.1). The Hamiltonian equations are

$$\begin{aligned} \frac{dq_\mu}{dt} &= \frac{\partial H}{\partial p_\mu} = \{q_\mu, H\}, \\ \frac{dp_\mu}{dt} &= -\frac{\partial H}{\partial q_\mu} = \{p_\mu, H\}, \qquad \mu = 1, ..., d, \end{aligned} \tag{1.1.1}$$

where $\{\cdot, \cdot\}$ are the Poisson brackets, that satisfy the following requirements :

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0, \quad \{q_\mu, p_\nu\} = \delta_{\mu\nu}, \qquad \mu, \nu = 1, \ldots, d. \tag{1.1.2}$$

One calls the system (1.1.1) *Liouville integrable* if one can find d independent conserved quantities F_μ, $\mu = 1, ..., d$, in involution, namely

$$\{F_\mu, F_\nu\} = 0, \qquad \forall \;\; \mu, \nu = 1, ..., d. \tag{1.1.3}$$

Independence here refers to the linear independence of the set of one-forms dF_μ. Note that, since d is the maximal number of such quantities, and since conservation of all the F_μ means $\{H, F_\mu\} = 0 \;\forall \, \mu = 1, ..., d$, then one concludes that

$$H = H(F_\mu),$$

i.e. the Hamiltonian itself is a function of the quantities F_μ.

Theorem (Liouville)**.** *The equations of motion of a Liouville-integrable system can be solved "by quadratures".*

Due to this theorem for a Liouville integrable system there always exists a change of canonical coordinates, where one of the new variables coincides with the conserved quantity F_i:

$$(p_\mu, q_\mu) \to (\varphi_\mu, F_\mu)\,,$$

and whose equations of motion can be described by

$$\begin{aligned} \frac{d\varphi_\mu}{dt} &= \frac{\partial H}{\partial F_\mu} = \Omega_\mu, \\ \frac{dF_\mu}{dt} &= -\frac{\partial H}{\partial \phi_\mu} = 0, \end{aligned} \tag{1.1.4}$$

with some constants Ω_μ, $\mu = 1, \dots, d$. In these new coordinates the dynamic is very simple:

$$F_\mu(t) = \alpha_\mu \quad \text{and} \quad \varphi_\mu(t) = \Omega_\mu t + \varphi_\mu(0), \quad \mu = 1, \dots, \mu,$$

where α_μ are constants. Such coordinates (F_μ, φ_μ) are called *action-angle coordinates*.

Example (1 dimensional Harmonic-oscillator)**.** The Hamiltonian for the 1D classical harmonic oscillator, with mass $m = 1$, is given by

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} = E, \quad \omega \in \mathbb{R}.$$

If we introduce new variables (F, φ) by

$$\begin{aligned} p &= F \cos\varphi, \\ q &= \frac{F}{\omega} \sin\varphi, \end{aligned}$$

then in new coordinates the Hamiltonian becomes

$$H = \frac{F^2}{2} = E = \text{const}.$$

Thus the variables (F, φ) are the actions-angle coordinates since they satisfy (1.1.4) with $\Omega = \sqrt{2E}$.

Liouville theorem is very powerful and plays an important role in classical integrability. But there is another formalism that possesses many advantages in comparison: the Lax pair and the classical YangBaxter equation. The main advantage of this procedure is that one can naturally generalize it to describe $(1 + 1)$ integrable field theories by using the so-called Lax connection.

1.2 Lax pair

Suppose one can find two matrices L, M such that Hamilton's equations of motion can be written in the following form:

$$\frac{dL}{dt} = [M, L], \tag{1.2.1}$$

where $[M, L]$ is a commutator of two matrices: $[M, L] \equiv ML - LM$.

Such two matrices L, M are said to form a *Lax pair*. From (1.2.1), we can immediately obtain a set of conserved quantities:

$$O_n \equiv \operatorname{tr} L^n,$$

since

$$\frac{dO_n}{dt} = \sum_{i=0}^{n-1} \operatorname{tr}\, L^i\, [M, L] L^{n-1-i} = 0, \qquad \forall\, n \in \mathbb{N}.$$

Of course not all of these conserved charges are independent, and in the next section we will discuss the required condition for the independence. Let us also point out that the Lax pair is not unique. For example, adding constant multiples of the identity to L and M preserves (1.2.1). Less trivial freedom is a *gauge transformation*

$$L \longrightarrow g\, L\, g^{-1}, \qquad M \longrightarrow g\, M\, g^{-1} + \frac{dg}{dt}\, g^{-1},$$

with g an invertible matrix depending on the phase-space variables.

Example. A Lax pair for the harmonic oscillator can be written down as follows:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix}.$$

Example (KdV equation)**.** The Korteweg-de Vries equation (KdV) was one of the first examples of the integrable systems that has infinitely many integrals of motion [54]. It was suggested in 1895 for description of waves on shallow water. Propagation of waves in nonlinear media with dispersion in the case of general position is also described by the KdV equation.

$$4u_t = 6uu_x + u_{xxx}.$$

The corresponding Lax pair has the following form [36]:

$$\partial_t L = [A, L],$$

where

$$L = \partial^2 + u, \quad A = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x, \quad \partial := \partial/\partial x.$$

Although A and L are not matrices in this case, one can still use the algebraic structure to construct the integrals of motions.

Example (Toda lattice)**.** The first and most prominent example of a discrete integrable system is the Toda lattice discovered half a century ago [94, 95]. It describes the system of n points $q_1, q_2, \ldots, q_n$ on the real line interacting with the potential

$$U(q_1, \ldots, q_n) = \sum_{i=1}^{n-1} e^{-(q_{i+1}-q_i)}.$$

The Hamiltonian equations of motion of the Toda lattice imply

$$(q_n)_{tt} = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}, \quad n \in \mathbb{N}. \tag{1.2.2}$$

In 1974 Flaschka [37, 38] and Manakov [62] separately showed that the equation (1.2.2) with periodic condition $q_n = q_1$, can be written in Lax form through a change of variables, and thus the periodic Toda lattice Hamiltonian system is completely integrable. Six years later, Moser [69] showed that the (real, finite) nonperiodic Toda lattice is completely integrable as well.

1.3 Classical r-matrix and involutivity

Suppose we have found a Lax pair for a 2d-dimensional Hamiltonian dynamical system, and suppose that we can obtain d independent conserved quantities. This does not yet guarantee their *involutivity* (1.1.3). Hence, we have not yet secured integrability. For that, we need an extra ingredient.

We will regard L and M as elements of some matrix algebra $\mathfrak{g}$, with the matrix entries being functions on phase space. We introduce the following tensor notation:

$$X_1 \equiv X \otimes 1, \qquad X_2 \equiv 1 \otimes X$$

as elements of $\mathfrak{g} \otimes \mathfrak{g}$. Then, one has the following

Theorem. *The eigenvalues of L are in involution if and only if there exists an element $r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$, function of the phase-space variables, such that*

$$\{L_1, L_2\} \;=\; [r_{12}, L_1] - [r_{21}, L_2], \tag{1.3.1}$$

where $r_{21} = P \circ r_{12}$, P being the permutation operator acting on the two copies of $\mathfrak{g} \otimes \mathfrak{g}$.

Assume that r is a constant matrix independent of the dynamical variables and suppose $r_{12} = -r_{21}$. In order for the Jacobi identity to hold for the Poisson bracket (1.3.1) one needs to impose the following condition on r:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \;=\; 0. \tag{1.3.2}$$

We call such an r a *constant classical r-matrix*, and (1.3.2) the *classical Yang-Baxter equation (CYBE)*.

The most interesting case for our purposes will be when the Lax pair depends on an additional complex variable u, called the *spectral parameter*. This means that in some cases we can find a family of Lax pairs, parameterised by u, such that the equations of motion are equivalent to the condition (1.2.1) for all values of u.

1.4 Zero curvature representation and transfer matrix

An important step we need to take is to generalise what we have reviewed so far for classical finite-dimensional dynamical systems, to the case of classical *field theories.* We will restrict ourselves to two-dimensional field theories, meaning one spatial dimension x and one time t. This means that we will now have equations of motion obtained from a classical field theory Lagrangian (Euler-Lagrange equations).

The notion of integrability we gave earlier, based on the Liouville theorem, is inadequate when the number of degrees of freedom becomes infinite, as it is the case for field theories. What we will do is to adopt the idea of a Lax pair, suitably modifying its definition, as a starting point to define an integrable field theory.

Suppose you can find two (spectral-parameter dependent) matrices L, M such that the Euler-Lagrange equations of motion can be recasted in the following form:

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = [M, L]. \tag{1.4.1}$$

We will call such field theories *classically integrable.*

Example. The Painlevé equations, which we discuss in the second part of the dissertation, are examples of the classically integrable systems.

The condition (1.4.1), known as the *zero curvature representation*, is also the compatibility condition for the following *auxiliary linear problem*:

$$(\partial_x - L)\Psi = 0,$$
$$(\partial_t - M)\Psi = 0.$$

Such two matrices L, M in (1.4.1) are also said to form a *Lax pair*, and one can in principle obtain a sequence of conserved quantities for the field theory by following a well-defined procedure.

Let us introduce the so-called *monodromy matrix* $T(u)$ by

$$T(u) = P \exp \left[\int_{s_-}^{s_+} L(x,t,u) dx \right],$$

where P denotes a path-ordering with greater x to the left, s_- and s_+ are two points on the spatial line, and u is the spectral parameter. This object can be thought of as implelementing a parallel transport along the segment $[s_-, s_+]$, in accordance with the fact that the Lax pair can be thought of as a connection.

Assuming that our spatial domain is $[0, 2\pi]$ with periodic boundary condition on the fields, one can show that

$$\partial_t T \;=\; [M(0,t,u), T].$$

This implies that the trace of T, called the *transfer matrix*

$$\mathfrak{t}(u) \;\equiv \operatorname{tr} T(u),$$

is conserved *for all* u. By expanding in u, one obtains a family of conserved charges, which are the coefficients of the expansion:

$$\mathfrak{t}(u) = \sum_{n \geq 0} Q_n \, u^n, \qquad \partial_t Q_n = 0, \; \forall \, n \geq 0. \tag{1.4.2}$$

This forms the starting point for the construction of the integrable structure.

1.5 Sklyanin Exchange Relations

In the previous section we have constructed family of the conserved charges coming from the transfer matrix $\mathfrak{t}(u)$. But we don't know yet if the are independent. Motivated by (1.3.1), we suppose that the canonical Poisson brackets imposed on the fields have the following consequence for L:

$$\{L_1(x,t,u), L_2\,(y,t,u')\} = [r_{12}\,(u-u')\,, L_1(x,t,u) + L_2\,(y,t,u')]\,\delta(x-y), \tag{1.5.1}$$

Let us also assume that the r-matrix $r_{12}(u-u')$ does not itself depends on the fields, and satisfies

$$r_{12}(u-u') = -r_{21}(u'-u) .$$

Theorem (Sklyanin Exchange Relations)**.** *Given* (1.5.1)*, the Poisson brackets of the monodromy matrix satisfy*

$$\{T_1(u), T_2(u')\} \;=\; [r_{12}(u-u'), T_1(u)T_2(u')]. \tag{1.5.2}$$

From this, one can immediately conclude that the conserved charges generated by the transfer matrix are all in involution. Indeed, tracing by $\mathrm{tr}_1 \otimes \mathrm{tr}_2$ both sides of (1.5.2), one obtains

$$\{\mathfrak{t}(u), \mathfrak{t}(u')\} = 0, \tag{1.5.3}$$

where we have used cyclicity of $\mathrm{tr}_1 \otimes \mathrm{tr}_2$ which is the natural trace operation on $\mathfrak{g} \otimes \mathfrak{g}$. By expanding (1.5.3) we obtain the desired involution property of the charges (1.4.2).

Moreover, under the assumptions described after (1.5.1), the Jacobi identity for Sklyanin's exchange relations (1.5.2) admits all matrices r, that satisfy the *classical Yang-Baxter equation with spectral parameter*, namely

$$[r_{12}(u_1-u_2), r_{13}(u_1-u_3)] + [r_{12}(u_1-u_2), r_{23}(u_2-u_3)] + [r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0.$$

Summarizing, we have seen that the variables L and T can be thought of as the most convenient variables to display the integrable structure of the classical models. It will not come as a surprise then that quantisation, that we discuss in the next sections, best proceeds from the Sklyanin relations.

1.6 Quantum harmonic oscillator

For the Quantum systems by the integrability we mean the presence of rich symmetries and interesting algebraic or analytic structures. Let us illustrate it on the simple example of quantum harmonic oscillator.

The Hamiltonian operator for the quantum harmonic oscillator has form

$$\hat{H} = \frac{1}{2}(\hat{P}^2 + \hat{Q}^2).$$

The operators $\hat{P}$ and $\hat{Q}$ satisfy the following commutation relations

$$[\hat{Q}, \hat{Q}] = [\hat{P}, \hat{P}] = 0, \qquad [\hat{Q}, \hat{P}] = i,$$

where $[\hat{Q}, \hat{P}]$ is a commutator of two operators. These relations can be thought as a *quantization* of the relations (1.1.2).

There is a *canonical representation* of these operators $\hat{Q}$ and $\hat{P}$ acting on the functions $\psi(x)$ from the Schwartz space:

$$\hat{Q}\psi(x) = x\psi(x), \qquad \hat{P}\psi(x) = -i\frac{d}{dx}\psi(x).$$

For such $\hat{P}$ and $\hat{Q}$ the Hamiltonian operator $\hat{H}$ becomes differential operator :

$$\hat{H}\psi = -\frac{d^2}{dx^2}\psi(x) + x^2\psi(x).$$

Suppose we are interested in the spectrum of this operator, i.e levels of energies E:

$$\hat{H}\psi = E\psi.$$

We could study this equation using the standard tools of ODE but instead we will use some algebraic structure of this model. First we define the *annihilation operator* by

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}),$$

and its adjoint, the *creation operator*, by

$$\hat{a}^* = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}).$$

Together with the *number operator* $\hat{N} = \hat{a}^*\hat{a}$ these operators will satisfy the following commutation relations:

$$[\hat{a}, \hat{a}^*] = 1, \quad [\hat{N}, a^*] = a^*, \quad [\hat{N}, a] = -a. \tag{1.6.1}$$

The Hamiltonian $\hat{H}$ in terms of $\hat{a}, \hat{a}^*$ becomes

$$\hat{H} = \frac{1}{2}(\hat{N} + 1),$$

so the eigenvectors of $\hat{N}$ are also the eigenvectors of $\hat{H}$. Suppose we have found such an eigenvector ψ of $\hat{N}$:

$$\hat{N}\psi = \lambda\psi,$$

then one can show, using commutation relations (1.6.1), that $a\psi$ is also an eigenvector of $\hat{N}$ with energy $\lambda - 1$. In other words, given any energy eigenvector, we can act on it with the annihilation operator $\hat{a}$, to produce another eigenvector with less energy. By repeated application of the operator $\hat{a}$, it seems that we can produce energy eigenstates down to $\lambda = -\infty$. However, since

$$\lambda = (\psi, \hat{N}\psi) = (\psi, \hat{a}^*\hat{a}\psi) = (\hat{a}\psi, \hat{a}\psi) \geq 0$$

the smallest eigenvalue of $\hat{N}$ is 0. The corresponding eigenvector $|0\rangle$, called *vacuum vector*, is also the eigenvector of $\hat{H}$ with eigenvalue $\frac{1}{2}$, in other words we have

$$\hat{H}|0\rangle = \frac{1}{2}|0\rangle.$$

By the same reasoning, one can show that the vectors

$$|n\rangle = \frac{(\hat{a}^*)^n}{\sqrt{n!}}|0\rangle \tag{1.6.2}$$

are the eigenvectors of the Hamiltonian $\hat{H}$ with the eigenvalues $E_n = n + \frac{1}{2}$. Finally, one can easily find the vector $|0\rangle$ as a solution of the equation:

$$\hat{a}|0\rangle = 0 \iff \psi(x)' + x\psi(x) = 0,$$

and using formula (1.6.2) obtain the following expression for the all eigenvectors $|n\rangle$:

$$|n\rangle = \frac{1}{\sqrt{2^n n!}} \exp(-x^2/2)\; H_n(x),$$

where $H_n(x)$ are the Hermite polynomials, $H_0 \equiv 1$.

Summarizing, we were able to find eigenvalues and eigenvectors of the Hamiltonian $\hat{H}$ algebraically, using the annihilation and creation operators. In other quantum integrable systems the algebraic structure is much more complicated, but nevertheless the idea of the creation and annihilation operators sometimes is very helpful (see Section 2.1.2).

1.7 Quantisation

We have seen in Section 1.4 that the matrices $T(t,u)$ and $L(x,t,u)$ are the most convenient variables to describe the integrability in classical case. One can try to describe the quantum analog of these matrices, in other words, try to *quantize* them.

Mathematically, the quantisation procedure involves the concept of *Lie bialgebras* and the so-called *Manin triples* (see for instance [29] and references therein). The term *quantisation* incorporates the meaning of completing the classical algebraic structure to a quantum group, or, equivalently, obtaining from a classical r-matrix a solution to the *quantum Yang-Baxter Equation*

$$R_{12}\, R_{13}\, R_{23} \;=\; R_{23}\, R_{13}\, R_{12}, \qquad R_{ij} \sim 1 \otimes 1 \,+ i\, \hbar\, r_{ij} + \mathcal{O}(\hbar^2).$$

The quantisation of the Sklyanin exchange relations is attained by simply *"completing the $\hbar$ series"* into the famous RTT relations:

$$\widehat{T}_1(u)\widehat{T}_2(u')R(u-u') \;=\; R(u-u')\widehat{T}_2(u')\widehat{T}_1(u), \qquad \widehat{T}(u) = T(u) + \mathcal{O}(\hbar), \qquad (1.7.1)$$

where the quantum monodromy $\widehat{T}$ is now understood as the normal-ordering of the classical product integral expression. We can see that (1.7.1) tends to (1.5.2) for $\hbar \to 0$.

This RTT relation will be the starting point of our discussion in Section 2.1.1.

2. COMBINATORIAL FORMULAE FOR RATIONAL WEIGHT FUNCTIONS

One of the quantum integrable models that we mentioned in the introduction was the Heisenberg model. The problem to find eigenvalues and eigenvectors of the Hamiltonian for the Heisenberg model was first addressed by H.Bethe [5] who looked for the eigenvectors as values of a certain rational function (Bethe vector) on solutions of some system of algebraic equations (Bethe ansatz equations).

The algebraic interpretation of the approach proposed by H.Bethe and other authors [101, 86, 87] is called *Algebraic Bethe ansatz* (ABA). It was developed as a part of the Quantum Inverse Scattering Method (QISM), that emerged in the late 70's in the works of the Leningrad School [32, 33]. The *Nested Algebraic Bethe ansatz* (Nested ABA) is a generalization of the Algebraic Bethe ansatz for the integrable models associated with higher rank Lie algebras. It was developed almost simultaneously with ABA in [56, 58, 57, 78].

Later, the nested Bethe vectors (also called vector-valued weight functions) were used to construct integral representations for solutions of the quantized Knizhnik-Zamolodchikov (qKZ) equations [92]. The results of [57] has been extended to higher transfer matrices in [70]. Combinatorial formulae for the vector-valued weight functions associated with the differential Knizhnik–Zamolodchikov equations were developed in [63, 81, 82, 79, 34].

In the rank one case combinatorial formulae for vector-valued weight function are important in various areas from computation of correlation functions in integrable models, see [53], to evaluation of some multidimensional generalizations of the Vandermonde determinant [90]. In the $\mathfrak{gl}_n$ case combinatorial formulae, in particular, clarify analytic properties of the vector-valued weight function, which is important for constructing hypergeometric solutions of the qKZ equations associated with $\mathfrak{gl}_n$.

Let us finish this paragraph mentioning that the physical applications of Nested ABA are very wide as well. It turns out that Nested ABA models provide a more realistic description of strongly interacting systems. The reason is that in the Nested ABA we are dealing with several creation operators. This allows us to consider systems where several degrees of freedom of fundamental particles interact, for example, the spin and the charge of the electrons. Therefore, the Nested ABA solvable models have found wide application primarily in the physics of strongly correlated electronic systems (Yang–Gaudin model [64, 65, 101, 45], t-J model and Hubbard model [83, 59, 60, 28]). We can also consider systems consisting of several types of particles, such as systems with impurities (Kondo model) [1, 97, 98]. For a more detailed description of the application of Nested ABA to Fermi gases and ultracold atom systems, we refer the reader to review [46]. It is also worth mentioning that the Hamiltonians of integrable systems with a large number of degrees of freedom arise in supersymmetric gauge theories [67].

The goals of this chapter.

Quantum integrable models are frequently connected with Lie algebras $\mathfrak{g}$. For instance the Heisenberg chain is connected with the Lie algebra $\mathfrak{sl}_2$. Furthermore, quantum integrable systems usually have a chain structure. The whole space of states of a quantum system is a tensor product of the space of states of each node of this chain. The space of the states for one node of the chain is an irreducible finite dimensional representation of the Lie algebra $\mathfrak{g}$. The Bethe vector for the whole chain can be constructed from the Bethe vector of each node. This part is well understood. My research concerns obtaining formulas for the weight function for one node in the case of the Lie algebra $\mathfrak{gl}_n$. The technique that we used is the Nested Algebraic Bethe ansatz.

The main idea of the Nested Algebraic Bethe ansatz is to consider a sequence of subalgebras and construct weight functions for the larger subalgebras using weight functions for the smaller ones. There are two standard choices of subalgebras in $\mathfrak{gl}_n$-case

$$\mathfrak{gl}_n \supset \mathfrak{gl}_1 \oplus \mathfrak{gl}_{n-1} \qquad \text{and} \qquad \mathfrak{gl}_n \supset \mathfrak{gl}_{n-1} \oplus \mathfrak{gl}_1. \tag{2.0.1}$$

On the matrix level it means to proceed from $n \times n$ matrices to $(n-1) \times (n-1)$ matrices by either cutting out the first row and the first column or the last row and the last column. There is a natural question whether one can make a cut in the middle and consider a subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_{n-m} \subset \mathfrak{gl}_n$, where $1 < m < n-1$.

There is a general formula that gives a procedure how to calculate the Bethe vector. It has form $\mathbb{B}(\boldsymbol{t})v$, where $\mathbb{B}(\boldsymbol{t})$ is some rational function of $\boldsymbol{t} = (t_1, \ldots, t_\xi)$ with values in the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, and v is a weight singular vector in some irreducible representation of $\mathfrak{g}$. For the case when $\mathfrak{g} = \mathfrak{gl}_n$, $\mathbb{B}(\boldsymbol{t})$ contains standard generators e_{ab}, a,$b = 1, \ldots, n$ and where vector v in this case should satisfy the following conditions: it should be annihilated by $e_{ab}, \quad 1 \leq a < b \leq n$, and it should be eigenvector of $e_{aa},\ 1 \leq a \leq n$, in other words,

$$e_{ab}v = 0, \quad 1 \leq a < b \leq n, \qquad e_{aa}v = \Lambda_a v, \quad 1 \leq a \leq n,$$

for some complex numbers Λ_a.

Our goal is to write down an explicit formula for $\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v$ for the irreducible representation V of $\mathfrak{gl}_n$, $v \in V$, in a given spanning set of this representation. There are spanning sets of V that have the form $\prod_{a>b} e_{ab}^{s_{ab}} v$. Since the generators e_{ab} do not commute, we have to chose some ordering in this product. If we have a chain of subalgebras of $\mathfrak{gl}_n$, then it provides some natural ordering of e_{ab}.

For the standard choice of subalgebras (2.0.1) such formulas for $\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v$ were obtained in [90]. Their result shows that for the chain of subalgebras that starts with $\mathfrak{gl}_n \supset \mathfrak{gl}_{n-1} \oplus \mathfrak{gl}_1$ one has

$$\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v = \sum_{\boldsymbol{s} \in \mathbb{Z}_{\geq 0}^{n-1}} F_{\boldsymbol{s}}(\boldsymbol{t}) \cdot e_{n,n-2}^{s_{n,n-1}} e_{n,n-2}^{s_{n,n-2}} \ldots e_{n,1}^{s_{n,1}}\ \mathbb{B}_{\mathfrak{gl}_{n-1}}(\boldsymbol{t})v,$$

where $F_{\boldsymbol{s}}(\boldsymbol{t})$ are some explicit rational functions and $\boldsymbol{s} = (s_{n,n-1}, s_{n,n-2}, \ldots, s_{n,2}, s_{n,1})$. Similarly, for $\mathfrak{gl}_n \supset \mathfrak{gl}_1 \oplus \mathfrak{gl}_{n-1}$ one has

$$\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v = \sum_{\boldsymbol{r} \in \mathbb{Z}_{\geq 0}^{n-1}} G_{\boldsymbol{r}}(\boldsymbol{t}) \cdot e_{21}^{r_{21}} e_{31}^{r_{31}} \ldots e_{n1}^{r_{n1}} \mathbb{B}_{\mathfrak{gl}_{n-1}}(\boldsymbol{t})v,$$

where $G_{\boldsymbol{r}}(\boldsymbol{t})$ are some explicit rational functions and $\boldsymbol{r} = (r_{21}, r_{31}, \ldots, r_{n1})$.
In our work we obtained new formulas for $\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v$ for a spanning set corresponding to the chain of subalgebras that starts with $\mathfrak{gl}_n \supset \mathfrak{gl}_m \oplus \mathfrak{gl}_{n-m}$. Our main result, stated in Section 2.4.3, can be formulated as follows :

Theorem. *For any m, $1 \le m \le n-1$, one has*

$$\mathbb{B}_{\mathfrak{gl}_n}(\boldsymbol{t})v = \sum_{\boldsymbol{q}} W_{\boldsymbol{q}}(\boldsymbol{t}) \prod_{m+1\le i\le n} \prod_{1\le j\le m} e_{ij}^{q_{ij}} \; \mathbb{B}_{\mathfrak{gl}_{n-m}}(\boldsymbol{t})\mathbb{B}_{\mathfrak{gl}_m}(\boldsymbol{t})v,$$

where the sum is taken over all collections of nonnegative numbers $\boldsymbol{q} = \{q_{ij}\}$, $m+1 \le i \le n$, $1 \le j \le m$. The function $W_{\boldsymbol{q}}(\boldsymbol{t})$ has the following structure

$$W_{\boldsymbol{q}}(\boldsymbol{t}) = \mathrm{Sym}_{\boldsymbol{t}^1} \dots \mathrm{Sym}_{\boldsymbol{t}^{n-1}} U_{\boldsymbol{q}}(\boldsymbol{t}),$$

where $\boldsymbol{t}$ consists of $n-1$ groups: $\boldsymbol{t} = (\boldsymbol{t}^1, \boldsymbol{t}^2, \dots, \boldsymbol{t}^{n-1})$, and the function $U_{\boldsymbol{q}}(\boldsymbol{t})$ is a product of ratios of linear functions in $\boldsymbol{t}$. Each group $\boldsymbol{t}^j$ corresponds to the j-th simple root α_j of $\mathfrak{gl}_n$ and the structure of the products in $U_{\boldsymbol{q}}(\boldsymbol{t})$ reflects the Dynkin diagram of $\mathfrak{gl}_n$.
For $m = 1$ and $m = n-1$ it gives the formulas from [90].

Our proof is based on the Nested ABA. In the next section we recall the main ideas of this method.

2.1 Algebraic Bethe Anstaz

2.1.1 *RTT*-relation and integrable models

The key relation for the Algebraic Bethe Ansatz (ABA) and the Nested Algebraic Bethe Ansatz (Nested ABA) is the RTT-relation

$$R(u,v)(T(u) \otimes I)(I \otimes T(v)) = (I \otimes T(v))(T(u) \otimes I)R(u,v) \tag{2.1.1}$$

Here $T(u)$ is a n by n matrix :

$$T(u) = \begin{pmatrix} T_{11}(u) & \dots & T_{1n}(u) \\ \dots\dots & \dots & \dots \\ T_{1n}(u) & \dots & T_{nn}(u) \end{pmatrix},$$

whose entries act in some vector space $\mathcal{H}$. We can think of $T(u)$ as an operator in $\mathbb{C}^n \otimes \mathcal{H}$. $T(u)$ is called the *monodromy* matrix and the space $\mathbb{C}^n$ is called the *auxiliary space*. The R-matrix is acting in $\mathbb{C}^n \otimes \mathbb{C}^n$.

Another commonly used form of equation (2.1.1) is

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v), \tag{2.1.2}$$

here the subscripts show in which of the two auxiliary spaces $\mathbb{C}^n$ the T-matrices act nontrivially. The R-matrix $R_{12}(u,v)$ acts in both spaces $\mathbb{C}^n$.

The first problem is to find an R-matrix acting in $\mathbb{C}^n \otimes \mathbb{C}^n$. The R-matrix should satisfy the Yang–Baxter equation

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2),$$

in order to provide compatibility of the RTT-relation. Our main example of the non-trivial R-matrix has the following form:

$$R(u,v) = \mathbb{I} + g(u,v)\mathbb{P}, \quad g(u,v) = \frac{c}{u-v}, \tag{2.1.3}$$

where $\mathbb{I}$ is the identity operator in $\mathbb{C}^n \otimes \mathbb{C}^n, \mathbb{P}$ is the permutation operator in the same space, and c is a constant. The permutation operator has the form

$$\mathbb{P} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji},$$

where E_{ij} is the $n \times n$ matrix with the unit at the intersection of i-th row and j-th column and zeros elsewhere:

$$(E_{ij})_{lk} = \delta_{il}\delta_{jk}, \qquad i,j,l,k = 1,\ldots,n.$$

The simplest example of the monodromy matrix $T(u)$ is $R(u,v)$ itself. The RTT-relation (2.1.2) in this case becomes the Yang-Baxter equation (2.2.3). Another example of $T(u)$ matrix, which is analogous to the previous one,

$$T(u) = R_{0L}(u,\xi_L)\dots R_{01}(u,\xi_1), \tag{2.1.4}$$

turned out to be connected to the $SU(n)$-invariant inhomogeneous XXX Heisenberg chain. The parameters ξ_i are inhomogeneities. Each R-matrix $R_{0i}(u,\xi_i)$ acts in the tensor product $V_0 \otimes V_i$, where every V_i is $\mathbb{C}^n$. The auxiliary space of the monodromy matrix is $V_0 \sim \mathbb{C}^n$. The quantum space is

$$\mathcal{H} = V_1 \otimes \dots \otimes V_L = \underbrace{\mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n}_{L \quad \text{times}}.$$

This example gives us the first evidence of the connection between the solutions of the RTT relation and integrable models. We can state this connection more precisely.

Consider the trace of the monodromy matrix $T(u)$ in the auxiliary space $\mathbb{C}^n$:

$$\operatorname{tr} T(u) = \sum_{i=1}^{n} T_{ii}(u).$$

The RTT-relation (2.1.2) yields

$$[\operatorname{tr} T(u), \operatorname{tr} T(v)] = 0.$$

Thus, if we consider $T(u)$ as a function of u and expand it near some point u_0, then $\operatorname{tr} T(u)$ will be a generating function of commuting operators:

$$\operatorname{tr} T(u) = \sum_k (u-u_0)^k I_k, \quad [I_k, I_n] = 0.$$

It turns out that in many cases the Hamiltonian H of the integrable model belongs to this commuting family. And thus it is a natural question to study the eigenvectors and eigenvalues of the transfer matrices $\operatorname{tr} T(u)$. Most of the tools to find eigenvalues and eigenvectors of transfer matrices are known under the name of Bethe ansatz. Its basic principles and techniques can be found in [53, 30, 84, 85]

In the next sections we will discuss the Bethe ansatz using the $\mathfrak{gl}_2$, $\mathfrak{gl}_3$, $\mathfrak{gl}_4$ cases as examples.

2.1.2 $\mathfrak{gl}_2$ case

We start with the $\mathfrak{gl}_2$-case to demonstrate the method. In this case matrix $T(u)$ has the following form

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

And the transfer matrix is

$$\operatorname{tr} T(u) = A(u) + D(u).$$

In the method of ABA we assume that there exists a vector v, *called vacuum* or *weight singular vector*, that satisfy the following properties:

$$A(u) = a(u)v, \quad D(u)v = d(u)v, \quad C(u)v = 0.$$

Clearly, v is an eigenvector of $\operatorname{tr} T(u)$:

$$\operatorname{tr} T(u)v = (a(u) + d(u))\, v.$$

We look for other eigenvectors of $\operatorname{tr} T(u)$ in the following form:

$$\Psi = B(u_1) \ldots B(u_n)\, v.$$

It turns out that Ψ is an eigenvector of $\operatorname{tr} T(u)$ if the parameters $u_1, \ldots, u_n$ satisfy a system of **Bethe equations** (2.1.5). Those eigenvectors Ψ are called **Bethe vectors**. The operators B and C can be thought as the creation and annihilation operators (recall Section 1.6).

For example, consider $B(u)v$. Using RTT-relation we have

$$A(z)B(u) = f(u,z)B(u)A(z) + g(z,u)B(z)A(u),$$
$$D(z)B(u) = f(u,z)B(u)D(z) - g(z,u)B(z)D(u),$$
$$f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v},$$

and it gives

$$(A(z)+B(z))B(u)v = (f(u,z)a(z) + f(z,u)d(z))B(u)v + g(z,u)(a(u)-d(u))B(z)v.$$

So we see that $B(u)v$ is an eigenvector of the $\operatorname{tr} T(u)$ if $a(u) = d(u)$.

In general, one can prove that $\Psi = B(u_1)\dots B(u_n)v$ is an eigenvector of $\operatorname{tr} T(u)$ if

$$\frac{a(u_j)}{d(u_j)} = \prod_{k=1,k\neq j}^{n} \frac{f(u_j,u_k)}{f(u_k,u_j)}, \quad j = 1,\dots,n. \tag{2.1.5}$$

To conclude this short introduction, we would like to mention that there are two natural questions. The first on is whether the constructed vector Ψ is a non zero vector: $\Psi \neq 0$. The second one is whether the Bethe ansatz gives all the eigenvectors of the transfer matrix. The answers to these questions are not trivial, see [55], [75],[71], [72],[74].

2.1.3 $\mathfrak{gl}_3$ case

The problem of finding Bethe vectors in the $\mathfrak{gl}_3$ (and higher rank) based models is much more sophisticated than in the case considered above. For the $\mathfrak{gl}_3$ case matrix $T(u)$ has the following form

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix},$$

and

$$\operatorname{tr} T(u) = (T_{11}(u) + T_{22}(u) + T_{33}(u)).$$

The vacuum vector v should satisfy

$$T_{ii}(u)v = \Lambda_i(u)v, \qquad T_{ij}(u)v = 0, \quad \text{if } i > j.$$

Clearly, v is an eigenvector of $\operatorname{tr} T(u)$:

$$\operatorname{tr} T(u)v = (\Lambda_1(u) + \Lambda_2(u) + \Lambda_3(u))\, v.$$

Other examples of the Bethe vectors are

$$\Psi = T_{12}(u)v \text{ (if } \Lambda_1(u) = \Lambda_2(u)), \quad \Psi = T_{23}(v)v \text{ (if } \Lambda_2(v) = \Lambda_3(v)).$$

Less trivial example is

$$\Psi = T_{12}(u)T_{23}(v)v + g(v,u)\Lambda_2(v)T_{13}(u)v = T_{23}(v)T_{12}(u)v + g(v,u)\Lambda_2(u)T_{13}(v)v,$$

and the corresponding equations are

$$\frac{\Lambda_1(u)}{\Lambda_2(u)} = \frac{\Lambda_3(v)}{\Lambda_2(v)} = f(v,u).$$

One could try to generalize the construction from the $\mathfrak{gl}_2$-case, and consider a monomial

$$B_{\beta_1}(u_1)\dots B_{\beta_a}(u_a)\, v, \quad a = 0, 1, \dots \tag{2.1.6}$$

as a candidate for the transfer matrix eigenvector. Here every β_i is equal to either 1 or 2, $B_1(u) \equiv T_{12}(u)$ and $B_2(u) \equiv T_{13}(u)$. Generically, the monomial (2.1.6) is not invariant under the action of $\operatorname{tr} T(z)$. Therefore, it is quite natural to replace the monomial (2.1.6) by a polynomial

$$\Psi_a(\boldsymbol{u}) = \sum_{\beta_1,\dots,\beta_a} B_{\beta_1}(u_1)\dots B_{\beta_a}(u_a)\, F_{\beta_1,\dots,\beta_a} v, \quad a = 0, 1, \dots ,$$

where $F_{\beta_1,\dots,\beta_a}$ are some numerical coefficients, the sum is taken over every $\beta_i \in \{\beta_1,\dots,\beta_a\}$ and each β_i takes the values $\beta_i = 1,2$. This form gives the motivation for the Nested Algebraic Bethe Ansatz.

The Nested Algebraic Bethe Ansatz is a method that provides a systematic construction of the Bethe vectors for $\mathfrak{gl}_3$ and higher ranks. The idea is to reduce $\mathfrak{gl}_3$ case to $\mathfrak{gl}_2$ case. It can be achieved by considering the block structure for the matrix $T(u)$. For example, we can introduce the following block structure :

$$T(u) = \begin{pmatrix} A(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \mathbb{D}(u) \end{pmatrix} = \left(\begin{array}{c|cc} A(u) & B_1(u) & B_2(u) \\ \hline C_1(u) & D_{11}(u) & D_{12}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) \end{array}\right). \tag{2.1.7}$$

As we have seen it is natural to look for the Bethe vector in the following form:

$$\Psi_{a,b}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\beta_1,\dots,\beta_a} B_{\beta_1}(u_1)\dots B_{\beta_a}(u_a)\, F_{\beta_1,\dots,\beta_a}(\boldsymbol{u},\boldsymbol{v}), \tag{2.1.8}$$

$$\boldsymbol{u} = \{u_1,\dots,u_a\}, \quad \boldsymbol{v} = \{v_1,\dots v_b\},$$

where $\beta_i \in \{1,2\}$, $1 \le i \le a$, and $F_{\beta_1,\dots,\beta_a}(\boldsymbol{u},\boldsymbol{v})$ are some vectors. The method of Nested ABA gives an algorithm how to construct $F_{\beta_1,\dots,\beta_a}(\boldsymbol{u},\boldsymbol{v})$.

We start with the following $\mathfrak{gl}_2$ expression:

$$\widehat{\mathcal{T}}_0^{(a)}(z,\boldsymbol{u}) = \mathbb{D}_0(z)\mathcal{T}_0^{(a)}(z,\boldsymbol{u}), \quad \mathcal{T}_0^{(a)}(z,\boldsymbol{u}) = \boldsymbol{r}_{0a}(z,u_a)\dots\boldsymbol{r}_{01}(z,u_1), \tag{2.1.9}$$

where $\mathbb{D}_0(z)$ comes from the block structure of $T(u)$, see (2.1.7), and $\boldsymbol{r}(u,v)$ is the $\mathfrak{gl}_2$ R-matrix $\boldsymbol{r}$ that acts in $\mathbb{C}^2 \otimes \mathbb{C}^2$, see (2.1.3).

This $\widehat{\mathcal{T}}_0^{(a)}(z,\boldsymbol{u})$ acts in the space $V_0 \otimes \widehat{\mathcal{H}}$, where

$$V_0 \sim \mathbb{C}^2, \qquad \widehat{\mathcal{H}} = \mathcal{H} \otimes \left(\mathbb{C}^2\right)^{\otimes a},$$

where $\mathcal{H}$ is the original quantum space of the $\mathfrak{gl}_3$ model. The subscript 0 in $\widehat{\mathcal{T}}_0^{(a)}(z,\boldsymbol{u})$ stresses that the auxiliary space is V_0.

The matrix $\mathbb{D}(u)$, which acts in the space $V_0 \otimes \mathcal{H}$, can be treated as $\mathfrak{gl}_2$ monodromy matrix, since it satisfies the RTT-relation with the R-matrix $\boldsymbol{r}$: $\boldsymbol{r}_{12}(u,v)\mathbb{D}_1(u)\mathbb{D}_2(v) = \mathbb{D}_2(v)\mathbb{D}_1(u)\boldsymbol{r}_{12}(u,v)$. Moreover, the vector v is a singular weight vector of this monodromy matrix.

The matrix $\mathcal{T}_0^{(a)}(z,\boldsymbol{u})$ in turn can be considered as the monodromy matrix of the inhomogeneous $\mathfrak{gl}_2$-invariant XXX chain of the length a, see (2.1.4). The role of the inhomogeneity parameters is played by the parameters $\boldsymbol{u} = \{u_1,\dots,u_a\}$. Notice that $\mathcal{T}_0^{(a)}(z,\boldsymbol{u})$ acts in the $V_0 \otimes \mathcal{H}^a$, where the quantum space $\mathcal{H}^a$ is the tensor product $(\mathbb{C}^2)^{\otimes a}$. Moreover, this monodromy matrix $\mathcal{T}_0^{(a)}(z,\boldsymbol{u})$ has the vacuum vector $\Omega^{(a)}$, where

$$\Omega^{(a)} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{a}.$$

Thus, $\widehat{\mathcal{T}}_0^{(a)}(z,,\boldsymbol{u})$ defined by (2.1.9) is the monodromy matrix of the $\mathfrak{gl}_2$ algebra, being the product of two monodromy matrices whose entries act in different spaces. Moreover, it has the vacuum vector $v \otimes \Omega^{(a)}$.

The rest of the construction looks similar to the $\mathfrak{gl}_2$ case. We present $\widehat{\mathcal{T}}_0^{(a)}(z,\boldsymbol{u})$ as

$$\widehat{\mathcal{T}}_0^{(a)}(z,\boldsymbol{u}) = \begin{pmatrix} \widehat{\mathcal{A}}^{(a)}(z,\boldsymbol{u}) & \widehat{\mathcal{B}}^{(a)}(z,\boldsymbol{u}) \\ \widehat{\mathcal{C}}^{(a)}(z,\boldsymbol{u}) & \widehat{\mathcal{D}}^{(a)}(z,\boldsymbol{u}) \end{pmatrix},$$

and define

$$\mathbb{F}(\boldsymbol{u},\boldsymbol{v}) = \widehat{\mathcal{B}}^{(a)}\left(v_1,\boldsymbol{u}\right)\dots\widehat{\mathcal{B}}^{(a)}\left(v_b,\boldsymbol{u}\right)\ v\otimes\Omega^{(a)}$$

Next we fix the basis of $\mathbb{C}^2$: $w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and consider the following basis of $(\mathbb{C}^2)^{\otimes a}$:

$$w_{\beta_1} \otimes \dots \otimes w_{\beta_a}, \quad \beta_i \in \{1,2\},\ i = 1,\dots,a.$$

Finlay, the desired coefficients $F_{\beta_1,\ldots,\beta_a}$ from (2.1.8) can be obtained as the "coordinates" of $\mathbb{F}(\boldsymbol{u},\boldsymbol{v})$ in this basis:

$$\mathbb{F}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\beta_1,\ldots,\beta_a} F_{\beta_1,\ldots,\beta_a}(\boldsymbol{u},\boldsymbol{v}) \otimes w_{\beta_1} \otimes \ldots \otimes w_{\beta_a}.$$

One can show that the vector constructed above,

$$\Psi_{a,b}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\beta_1,\ldots,\beta_a} B_{\beta_1}(u_1)\ldots B_{\beta_a}(u_a)\, F_{\beta_1,\ldots,\beta_a}(\boldsymbol{u},\boldsymbol{v}),$$

is indeed an eigenvector of the transfer matrix $\operatorname{tr} T(u)$ if the following equations are satisfied:

$$\begin{aligned} \frac{\Lambda_1(u_k)}{\Lambda_2(u_k)} &= \frac{f(u_k,\hat{\boldsymbol{u}}_k)}{f(\hat{\boldsymbol{u}}_k,u_k)} f(\boldsymbol{v},u_k), \qquad && k=1,\ldots,a, \\ \frac{\Lambda_2(v_j)}{\Lambda_3(v_j)} &= \frac{f(v_j,\hat{\boldsymbol{v}}_j)}{f(\hat{\boldsymbol{v}}_j,v_j)} \frac{1}{f(v_j,\boldsymbol{u})}, \qquad && j=1,\ldots,b, \end{aligned} \tag{2.1.10}$$

where

$$f(z,\hat{\boldsymbol{w}}_i) = \prod_{\substack{w_j\in\boldsymbol{w}\\ w_j\neq w_i}} f(z,w_j)\,; \quad f(z,\boldsymbol{v}) = \prod_{v_k\in\boldsymbol{v}} f(z,v_k)\,.$$

We could start with a different splitting of the transfer matrix $T(v)$:

$$T(v) = \left(\begin{array}{cc|c} A_{11}(v) & A_{12}(v) & B_1(v) \\ A_{21}(v) & A_{22}(v) & B_2(v) \\ \hline C_1(v) & C_2(v) & D(v) \end{array}\right), \tag{2.1.11}$$

and get another eigenvector vector $\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v})$ of $\operatorname{tr} T(v)$. It would have form

$$\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v}) = \sum_{\beta_1,\ldots,\beta_a} B_{\beta_1}(u_1)\ldots B_{\beta_a}(u_a)\, \tilde{F}_{\beta_1,\ldots,\beta_a}(\boldsymbol{u},\boldsymbol{v}), \tag{2.1.12}$$

where $\beta_i \in \{1,2\}$, $1 \le i \le a$. It turns out that $\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v})$ is the eigenvector of the transfer matrix $\operatorname{tr} T(u)$ if the parameters $\boldsymbol{u},\boldsymbol{v}$ in (2.1.12) satisfy the same system of algebraic equations (2.1.10). One could ask whether these two vectors $\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v})$ and $\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v})$ coincide or not. The answer is positive and is given in the next section.

2.1.4 Trace formula

The formulas for the Bethe vectors $\Psi_{a,b}(\boldsymbol{u},\boldsymbol{v})$, given by (2.1.8), and $\tilde{\Psi}_{a,b}(\boldsymbol{u},\boldsymbol{v})$, given by (2.1.12), look very different. For instance they are based on two different embeddings, (2.1.7) and (2.1.11), and thus have different ordering of the operators $B_a(u)$. Moreover, some of those operators have different arguments. Nevertheless, these different representations describe the same Bethe vector. It follows from the fact that both of these representations can be obtained from one formula, called the trace formula [92, 89, 4].

The main advantage of the trace formula is that it can be easily generalized to the case of $\mathfrak{gl}_n$. Besides, the Bethe parameters $\boldsymbol{u}$ and $\boldsymbol{v}$ are included in this representation in a more symmetric way. Finally, by the construction $\Psi_{a,b}(\boldsymbol{u},\boldsymbol{v})$ is symmetric over the variables $\boldsymbol{v}$. It turns out that is also symmetric over the variables $\boldsymbol{u}$, and this symmetry is far from evident at this moment. The new formula for Bethe vectors will allow us to prove the symmetry of these vectors with respect to the parameters $\boldsymbol{u}$.

To formulate the trace formula in the $\mathfrak{gl}_3$ case we start with a tensor product $V_{k_1}\otimes\cdots\otimes V_{k_a}\otimes V_{n_1}\otimes\cdots\otimes V_{n_b}$, where each $V_j\sim\mathbb{C}^3$. Let

$$\mathbb{T}_{\bar{k}}(\boldsymbol{u}) = T_{k_1}(u_1)\dots T_{k_a}(u_a), \qquad \mathbb{T}_{\bar{n}}(\boldsymbol{v}) = T_{n_1}(v_1)\dots T_{n_b}(v_b),$$

and

$$\mathbb{R}_{\bar{n},\bar{k}}(\boldsymbol{v},\boldsymbol{u}) = \prod_{1\le i\le b}^{\rightarrow}\;\prod_{1\le j\le a}^{\leftarrow} R_{n_i,k_j}(v_i,u_j). \tag{2.1.13}$$

Here every T_j acts in $V_j\otimes\mathcal{H}$. Each R-matrix $R_{i,j}$ acts in $V_i\otimes V_j$. We would like to draw attention to the ordering of the R-matrices in the double product (2.1.13). There the index i changes in the standard increasing direction, while the index j changes in the decreasing direction. For example, for $a=b=2$, the product (2.1.13) reads

$$\mathbb{R}_{\bar{n},\bar{k}}(\boldsymbol{v},\boldsymbol{u}) = R_{n_1,k_2}(v_1,u_2)R_{n_1,k_1}(v_1,u_1)R_{n_2,k_2}(v_2,u_2)R_{n_2,k_1}(v_2,u_1).$$

Proposition. *The off-shell Bethe vectors of the $\mathfrak{gl}_3$-invariant models have the following form:*

$$\Psi_{a,b}(\boldsymbol{u};\boldsymbol{v}) = \mathrm{tr}_{\bar{k},\bar{n}}\Big(\mathbb{T}_{\bar{k}}(\boldsymbol{u})\mathbb{T}_{\bar{n}}(\boldsymbol{v})\mathbb{R}_{\bar{n},\bar{k}}(\boldsymbol{v},\boldsymbol{u})E^{21}_{k_1}\dots E^{21}_{k_a}E^{32}_{n_1}\dots E^{32}_{n_b}\Big)v. \tag{2.1.14}$$

The trace is taken over all the spaces $V_{k_1},\dots,V_{k_a},V_{n_1},\dots,V_{n_b}$. The matrices $E^{21}_{k_j}$ and $E^{32}_{n_j}$ are the standard basis matrices that respectively act in the spaces V_{k_j} and V_{n_j}.

Equation (2.1.14) is known as a *trace formula*. Observe that using the *RTT* relation (2.1.2), formula (2.1.14) can be written in the following form:

$$\Psi_{a,b}(\boldsymbol{u};\boldsymbol{v}) = \mathrm{tr}_{\bar{k},\bar{n}}\Big(\mathbb{R}_{\bar{n},\bar{k}}\mathbb{T}_{\bar{k}}(\boldsymbol{u})\mathbb{T}_{\bar{n}}(\boldsymbol{v})(\boldsymbol{v},\boldsymbol{u})E^{21}_{k_1}\dots E^{21}_{k_a}E^{32}_{n_1}\dots E^{32}_{n_b}\Big)v. \tag{2.1.15}$$

It turns out that the vector $\Psi_{a,b}(\boldsymbol{u},\boldsymbol{v})$, given by (2.1.8), can be obtained by the expansion of formula (2.1.14), while the vector $\tilde{\Psi}_{a,b}$, given by (2.1.12), can be obtained by the expansion of the formula (2.1.15). But since they represent the same vector they have to coincide.

2.1.5 $\mathfrak{gl}_4$ case

We have seen in Section 2.1.3 that the idea of Nested ABA is to reduce the $\mathfrak{gl}_n$ case to the $\mathfrak{gl}_{n-1}$ using the block structure. In $\mathfrak{gl}_4$ case we have two standard options of splitting that reduce problem to the $\mathfrak{gl}_3$ case :

$$T(u) = \left(\begin{array}{ccc|c} A_{11}(u) & A_{12}(u) & A_{13}(u) & B_1(u) \\ A_{21}(u) & A_{22}(u) & A_{23}(u) & B_2(u) \\ A_{31}(u) & A_{32}(u) & A_{33}(u) & B_3(u) \\ \hline C_1(u) & C_2(u) & C_3(u) & D(u) \end{array}\right), \tag{2.1.16}$$

$$T(u) = \left(\begin{array}{c|ccc} A(u) & B_{12}(u) & B_{13}(u) & B_{14}(u) \\ \hline C_{21}(u) & D_{22}(u) & D_{23}(u) & D_{24}(u) \\ C_{31}(u) & D_{32}(u) & D_{33}(u) & D_{34}(u) \\ C_{41}(u) & D_{42}(u) & D_{43}(u) & D_{44}(u) \end{array}\right). \tag{2.1.17}$$

But there is also a natural splitting that decomposes $\mathfrak{gl}_4$ into two $\mathfrak{gl}_2$ cases:

$$T(u) = \left(\begin{array}{cc|cc} A_{11}(u) & A_{12}(u) & B_{13}(u) & B_{14}(u) \\ A_{21}(u) & A_{22}(u) & B_{23}(u) & B_{24}(u) \\ \hline C_{31}(u) & C_{32}(u) & D_{33}(u) & D_{34}(u) \\ C_{41}(u) & C_{42}(u) & D_{43}(u) & D_{44}(u) \end{array}\right). \tag{2.1.18}$$

For the standard splittings (2.1.16) and (2.1.17) the corresponding formulas for the Bethe vector can be obtained from the result of [90], but for the splitting (2.1.18) no formulas were known.

In the next sections we provide new formula for the Bethe vector that corresponds to this splitting (2.1.18) and generalize the result for the $\mathfrak{gl}_n$ case, where the splitting corresponds to the subalgebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_{n-m} \subset \mathfrak{gl}_n$, $1 \leq m \leq n-1$.

2.2 Notations

In this section we discus the notations that will be used in this chapter.

We will be using the standard superscript notation for embeddings of tensor factors into tensor products. For a tensor product of vector spaces $V_1 \otimes V_2 \otimes \ldots \otimes V_k$ and an operator $A \in \mathrm{End}(V_i)$, denote

$$A^{(i)} = 1^{\otimes(i-1)} \otimes A \otimes 1^{\otimes(k-i)} \in \mathrm{End}(V_1 \otimes V_2 \otimes \ldots \otimes V_k).$$

Also, if $B \in \mathrm{End}(V_j)$, $i \neq j$, denote $(A \otimes B)^{(ij)} = A^{(i)} B^{(j)}$, etc.

Fix a positive integer n. All over the paper we identify elements of End $\mathbb{C}^n$ with $n \times n$ matrices using the standard basis of $\mathbb{C}^n$. That is, for $L \in \mathrm{End}\, \mathbb{C}^n$ we have $L = (L^a_b)^n_{a,b=1}$, where L^a_b are the entries of L. Entries of matrices acting in the tensor products $(\mathbb{C}^n)^{\otimes k}$ are naturally labeled by multiindices. For instance, if $M \in \mathrm{End}\,(\mathbb{C}^n \otimes \mathbb{C}^n)$, then $M = \left(M^{ab}_{cd}\right)^n_{a,b,c,d=1}$.

The rational R-matrix is $R(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$,

$$R(u) = 1 + \frac{1}{u} \sum_{a,b=1}^{n} E_{ab} \otimes E_{ba}, \tag{2.2.1}$$

where $E_{ab} \in \mathrm{End}\,(\mathbb{C}^n)$ is the matrix with the only nonzero entry equal to 1 at the intersection of the a-th row and b-th column. The entries of $R(u)$ are

$$R^{ab}_{cd}(u) = \delta_{ac}\delta_{bd} + \frac{1}{u}\delta_{ad}\delta_{bc}. \tag{2.2.2}$$

The R-matrix satisfies the Yang-Baxter equation

$$R^{(12)}(u-v)R^{(13)}(u)R^{(23)}(v) = R^{(23)}(v)R^{(13)}(u)R^{(12)}(u-v). \tag{2.2.3}$$

The Yangian $Y(\mathfrak{gl}_n)$ is a unital associative algebra with generators $(T^a_b)^{\{s\}}$, $a, b = 1, \ldots, n$, and $s = 1, 2, \ldots$. Organize them into generating series:

$$T^a_b(u) = \delta_{ab} + \sum_{s=1}^{\infty} (T^a_b)^{\{s\}}\, u^{-s}, \quad a,b = 1,\dots,n. \tag{2.2.4}$$

The defining relations in $Y(\mathfrak{gl}_n)$ are

$$(u-v)\,[T^a_b(u), T^c_d(v)] = T^a_d(u)T^c_b(v) - T^a_d(v)T^c_b(u) \tag{2.2.5}$$

for all $a,b,c,d = 1,\dots,n$.

Combine series (2.2.4) into a matrix $T(u) = \sum_{a,b=1}^{n} E_{ab} \otimes T^a_b(u)$ with entries in $Y(\mathfrak{gl}_n)$. Then relations (2.2.5) amount to the following equality

$$R^{(12)}(u-v)T^{(1)}(u)T^{(2)}(v) = T^{(2)}(v)T^{(1)}(u)R^{(12)}(u-v),$$

where $T^{(1)}(u) = \sum_{a,b=1}^{n} E_{ab} \otimes 1 \otimes T^a_b(u)$ and $T^{(2)}(v) = \sum_{a,b=1}^{n} 1 \otimes E_{ab} \otimes T^a_b(v)$.

The Yangian $Y(\mathfrak{gl}_n)$ is a Hopf algebra. In terms of generating series (2.2.4), the coproduct $\Delta : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$ reads as follows:

$$\Delta\,(T^a_b(u)) = \sum_{c=1}^{n} T^c_b(u) \otimes T^a_c(u), \quad a,b = 1,\dots,n. \tag{2.2.6}$$

Denote by $\widetilde{\Delta} : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$ the opposite coproduct

$$\widetilde{\Delta}\,(T^a_b(u)) = \sum_{c=1}^{n} T^a_b(u) \otimes T^c_b(u), \quad a,b = 1,\dots,n. \tag{2.2.7}$$

Fix a collection of nonnegative integers $\xi_1, \xi_2, \dots, \xi_{n-1}$. Set $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_{n-1})$ and $\xi^a = \xi_1 + \ldots + \xi_a$, $a = 1,\dots,n-1$. Consider the variables t^a_i, $a = 1,\dots,n-1$, $i = 1,\dots \xi_a$. We will write

$$\boldsymbol{t}^a = (t^a_1, \dots, t^a_{\xi_a}), \qquad \boldsymbol{t} = (\boldsymbol{t}^1, \dots, \boldsymbol{t}^{n-1}). \tag{2.2.8}$$

We will use the ordered product notation for any noncommuting factors $X_1, \dots, X_k$,

$$\prod_{1\le i\le k}^{\rightarrow} X_i = X_1 X_2 \dots X_k, \qquad \prod_{1\le i\le k}^{\leftarrow} X_i = X_k X_{k-1} \dots X_1.$$

Consider the vector space $(\mathbb{C}^n)^{\otimes \xi^{n-1}}$ and define

$$\overset{[j]}{\mathbb{T}}(\boldsymbol{t^j}) = \prod_{1\le k\le \xi_j}^{\rightarrow} T^{(\xi^{j-1}+k)}(t^j_k), \qquad \overset{[k,j]}{\mathbb{R}}(\boldsymbol{t^k},\boldsymbol{t^j}) = \prod_{1\le i\le \xi_k}^{\rightarrow}\left(\prod_{1\le l\le \xi_j}^{\leftarrow} R^{(\xi^{k-1}+i,\xi^{j-1}+l)}(t^k_i - t^j_l)\right),$$

where we view $T^{(\xi^{j-1}+k)}(t^j_k)$ as a matrix with entries in $Y(\mathfrak{gl}_n)$ acting on $(\xi^{j-1}+k)$-th copy of $\mathbb{C}^n$ in $(\mathbb{C}^n)^{\otimes \xi^{n-1}}$. For the expression

$$\widehat{\mathbb{T}}_\xi(\boldsymbol{t}) = \overset{[1]}{\mathbb{T}}(\boldsymbol{t}^1)\dots\overset{[n-1]}{\mathbb{T}}(\boldsymbol{t}^{n-1})\prod_{1\le i\le n-1}^{\leftarrow}\left(\prod_{1\le j<i}^{\leftarrow}\overset{[i,j]}{\mathbb{R}}(\boldsymbol{t^i},\boldsymbol{t^j})\right), \tag{2.2.9}$$

denote by $\mathbb{B}_\xi(\boldsymbol{t})$ the following entry

$$\mathbb{B}_\xi(\boldsymbol{t}) = \left(\widehat{\mathbb{T}}_\xi(\boldsymbol{t})\right)^{\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\dots,\boldsymbol{n-1}^{\xi_{n-1}}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_2},\dots,\boldsymbol{n}^{\xi_{n-1}}}, \tag{2.2.10}$$

where

$$\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\dots,\boldsymbol{(n-1)}^{\xi_{n-1}} = \underbrace{1,1,\dots 1}_{\xi_1},\ \underbrace{2,2,\dots 2}_{\xi_2},\ \dots\ ,\underbrace{n-1,\ n-1,\ \dots,\ n-1}_{\xi_{n-1}},$$

$$\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_2},\dots,\boldsymbol{n}^{\xi_{n-1}} = \underbrace{2,2,\dots,2}_{\xi_1},\ \underbrace{3,3,\dots,3}_{\xi_2},\ \dots,\ \underbrace{n,\ n,\ \dots,\ n}_{\xi_{n-1}}.$$

To indicate the dependence on n, if necessary, we will write $\mathbb{B}^{\langle n\rangle}_\xi(\boldsymbol{t})$. For a weight singular vector v with respect to the action of $Y(\mathfrak{gl}_n)$, we call the expression $\mathbb{B}_\xi(\boldsymbol{t})v$ the *(rational) vector-valued weight function* of weight $(\xi_1,\xi_2-\xi_1,\dots,\xi_{n-1}-\xi_{n-2},-\xi_{n-1})$ associated with v.

There is a one-parameter family of automorphisms $\rho_x : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)$ defined in terms of the series $T(u)$ by the rule $\rho_x T(u) = T(u-x)$, where in the right-hand side, each expression $(u-x)^{-s}$ has to be expanded as a power series in u^{-1}.

Denote by e_{ab}, a,$b = 1,\dots,n$, the standard generators of the Lie algebra $\mathfrak{gl}_n$. A vector v in a $\mathfrak{gl}_n$ -module is called singular of weight $(\Lambda^1,\dots,\Lambda^n)$ if $e_{ab}v = 0$ for all $a < b$ and $e_{aa}v = \Lambda_a v$ for all $a = 1,\dots,n$.

The Yangian $Y(\mathfrak{gl}_n)$ contains the universal enveloping algebra $U(\mathfrak{gl}_n)$ as a Hopf subalgebra. The embedding is given by the rule $e_{ab} \mapsto (T_a^b)^{\{1\}}$ for all $a, b = 1, \ldots, n$. We identify $U(\mathfrak{gl}_n)$ with its image in $Y(\mathfrak{gl}_n)$ under this embedding.

The evaluation homomorphism $\epsilon : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ is given by the rule $\epsilon : (T_b^a)(u) \mapsto \delta_{ab} + e_{ba}u^{-1}$ for all $a, b = 1, ..., n$. Both the automorphisms ρ_x and the homomorphism ϵ restricted to the subalgebra $U(\mathfrak{gl}_n)$ are the identity maps.

For a $\mathfrak{gl}_n$-module V denote by $V(x)$ the $Y(\mathfrak{gl}_n)$-module induced from V by the homomorphism $\epsilon \circ \rho_x$. The module $V(x)$ is called an evaluation module over $Y(\mathfrak{gl}_n)$.

A vector v in a $Y(\mathfrak{gl}_n)$-module is called singular with respect to the action of $Y(\mathfrak{gl}_n)$ if $T_b^a(u)v = 0$ for all $1 \le b < a \le n$. A singular vector v that is an eigenvector for the action of $T_1^1(u)$,...,$T_n^n(u)$ is called a weight singular vector, and the respective eigenvalues are denoted by $\langle T_1^1(u)v\rangle, \ldots, \langle T_n^n(u)v\rangle$.

Example. Let V be a $\mathfrak{gl}_n$-module and $v \in V$ be a $\mathfrak{gl}_n$-singular vector of weight $(\Lambda^1, \ldots, \Lambda^n)$. Then v is a weight singular vector with respect to the action of $Y(\mathfrak{gl}_n)$ in the evaluation module $V(x)$ and $\langle T_a^a(u)v\rangle = 1 + \Lambda^a(u - x)^{-1}$, $a = 1, \ldots, n$.

For $k < n$ we consider two embeddings of the algebra $Y(\mathfrak{gl}_k)$ into $Y(\mathfrak{gl}_n)$, called ϕ_k and ψ_k:

$$\phi_k(T^{\langle k\rangle}(u))_b^a = (T^{\langle n\rangle}(u))_b^a \qquad\qquad \psi_k(T^{\langle k\rangle}(u))_b^a = (T^{\langle n\rangle}(u))_{b+n-k}^{a+n-k}(u) \tag{2.2.11}$$

$a, b = 1, \ldots, k$. Here $(T^{\langle k\rangle}(u))_b^a$ and $((T^{\langle n\rangle}(u))_b^a$ are the series $T_b^a(u)$ for the algebras $Y(\mathfrak{gl}_k)$ and $Y(\mathfrak{gl}_n)$, respectively.

2.3 Combinatorial formulae for the $\mathfrak{gl}_4$ case

In this section we will focus on the $\mathfrak{gl}_4$ case. We are interested in writing down the following expansion for a weight function in a evaluation module over the $Y(\mathfrak{gl}_4)$:

$$\mathbb{B}_\xi(\boldsymbol{t})v = \sum_{\vec{m}\in\mathbb{Z}^6_{\ge 0}} F_{\vec{m}}(\boldsymbol{t}) \cdot e_{32}^{m_{32}} e_{31}^{m_{31}} e_{42}^{m_{42}} e_{41}^{m_{41}} e_{21}^{m_{21}} e_{43}^{m_{43}} v \tag{2.3.1}$$

with the functions $F_{\vec{m}}(\boldsymbol{t})$ given by explicit formulas. Various similar expansions for $\mathbb{B}_\xi(\boldsymbol{t})v$ were obtained in [89], however, expansion (2.3.1) is not covered there.

2.3.1 Splitting property of the weight functions

Let $T^{\langle 2\rangle}_{ab}(u)$ be series (2.2.4) for the algebra $Y(\mathfrak{gl}_2)$, and $R^{\langle 2\rangle}(u)$ be the corresponding rational R-matrix, see (2.2.1). Consider two $Y(\mathfrak{gl}_2)$-module structures on the vector space $\mathbb{C}^2$. The first one, called $L(x)$, is given by the rule

$$\pi(x) : T^{\langle 2\rangle}(u) \mapsto (u-x)^{-1} R^{\langle 2\rangle}(u-x),$$

and the second one, called $\bar{L}(x)$, is given by the rule

$$\varpi(x) : T^{\langle 2\rangle}(u) \mapsto (x-u)^{-1}\Big(\Big(R^{\langle 2\rangle}(x-u)\Big)^{(21)}\Big)^{t_2},$$

where the superscript t_2 stands for the matrix transposition in the second tensor factor.

Let $\mathbf{w}_1, \mathbf{w}_2$ be the standard basis of the space $\mathbb{C}^2$. The module $L(x)$ is a highest weight evaluation module with $\mathfrak{gl}_2$ highest weight $(1,0)$ and highest weight vector $\mathbf{w}_1$. The module $\bar{L}(x)$ is a highest weight evaluation module with $\mathfrak{gl}_2$ highest weight $(0,-1)$ and highest weight vector $\mathbf{w}_2$. For any $X \in \mathrm{End}(\mathbb{C}^2)$, set $\nu(X) = X\mathbf{w}_1$ and $\bar{\nu}(X) = X\mathbf{w}_2$.

Recall the coproducts Δ and $\widetilde{\Delta}$, see (2.2.6) and (2.2.7), and the embeddings $\psi_2 : Y(\mathfrak{gl}_2) \to Y(\mathfrak{gl}_4)$ and $\phi_2 : Y(\mathfrak{gl}_2) \to Y(\mathfrak{gl}_4)$ given by (2.2.11). For any k, denote by $\Delta^{(k)} : Y(\mathfrak{gl}_2) \to (Y(\mathfrak{gl}_2))^{\otimes(k+1)}$ and $\widetilde{\Delta^{(k)}} : Y(\mathfrak{gl}_2) \to (Y(\mathfrak{gl}_2))^{\otimes(k+1)}$ the iterated coproduct and opposite coproduct. Consider the maps

$$\psi_2\,(x_1,\dots,x_k) : Y(\mathfrak{gl}_2) \to \Big(\mathbb{C}^2\Big)^{\otimes k} \otimes Y(\mathfrak{gl}_4),$$

$$\psi_2(x_1,\dots,x_k) = (\nu^{\otimes k} \otimes \mathrm{id}) \circ (\pi\,(x_1) \otimes \cdots \otimes \pi\,(x_k) \otimes \psi_2) \circ \Delta^{(k)},$$

and

$$\phi_2\,(x_1,\dots,x_k) : Y(\mathfrak{gl}_2) \to \Big(\mathbb{C}^2\Big)^{\otimes k} \otimes Y(\mathfrak{gl}_4),$$

$$\phi_2(x_1,\dots,x_k) = (\bar{\nu}^{\otimes k}\otimes \mathrm{id})\circ(\varpi(x_1)\otimes\cdots\otimes\varpi_2(x_k)\otimes\phi_2)\circ\widetilde{\Delta}^{(k)}.$$

For any element $g\in(\mathbb{C}^2)^{\otimes k}\otimes Y(\mathfrak{gl}_4)$, we define its components $g^{\boldsymbol{a}}$, $\boldsymbol{a}=(a_1,\dots,a_k)$, by the rule

$$g=\sum_{a_1,\dots,a_k=1}^{2}\mathbf{w}_{a_1}\otimes\cdots\otimes\mathbf{w}_{a_k}\otimes g^{\boldsymbol{a}}.$$

In the $\mathfrak{gl}_4$ case, we have $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$, and formula (2.2.10) takes the form

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})=\Big(\overset{[1]}{\mathbb{T}}(\boldsymbol{t}^1)\overset{[2]}{\mathbb{T}}(\boldsymbol{t}^2)\overset{[3]}{\mathbb{T}}(\boldsymbol{t}^3)\overset{[32]}{\mathbb{R}}(\boldsymbol{t}^3,\boldsymbol{t}^2)\overset{[31]}{\mathbb{R}}(\boldsymbol{t}^3,\boldsymbol{t}^1)\overset{[21]}{\mathbb{R}}(\boldsymbol{t}^2,\boldsymbol{t}^1)\Big)^{\mathbf{1}^{\xi_1}\,\mathbf{2}^{\xi_2}\,\mathbf{3}^{\xi_3}}_{\mathbf{2}^{\xi_1}\,\mathbf{3}^{\xi_2}\,\mathbf{4}^{\xi_3}}.$$

Proposition 2.3.1. Let v be a $Y(\mathfrak{gl}_4)$-singular vector, ξ_1,ξ_2,ξ_3 be nonnegative integers, and $\boldsymbol{t}=(t^1_1,\dots,t^1_{\xi_1};t^2_1,\dots,t^2_{\xi_2};t^3_1,\dots,t^3_{\xi_3})$. Then one has

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v=\sum_{\boldsymbol{a},\boldsymbol{b}}\Big(\mathcal{T}(\boldsymbol{t}^2)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}\Big(\phi_2(\boldsymbol{t}^2)\Big(\mathbb{B}^{\langle 2\rangle}_{\boldsymbol{\xi_1}}(\boldsymbol{t}^1)\Big)\Big)^{\boldsymbol{a}}\Big(\psi_2(\boldsymbol{t}^2)\Big(\mathbb{B}^{\langle 2\rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3)\Big)\Big)^{\boldsymbol{b-2}}v, \tag{2.3.3}$$

where the sum is taken over all sequences $\boldsymbol{a}=(a_1,a_2,\dots,a_{\xi_2})$, $\boldsymbol{b}=(b_1,b_2,\dots,b_{\xi_2})$, such that $a_i\in\{1,2\}$, $b_i\in\{3,4\}$ for all $i=1,\dots,\xi_2$, $\boldsymbol{b-2}=(b_1-2,b_2-2,\dots,b_{\xi_2}-2)$, and

$$\Big(\mathcal{T}(\boldsymbol{t}^j)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}=T(t^j_1)^{a_1}_{b_1}T(t^j_2)^{a_2}_{b_2}\dots T(t^j_{\xi_j})^{a_{\xi_2}}_{b_{\xi_2}}.$$

Proof. Formula (2.3.3) follows from the definition of the maps $\psi_2(\boldsymbol{t}^2)$ and $\phi_2(\boldsymbol{t}^2)$ and Lemma 2.3.2 below. □

Lemma 2.3.2. One has

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v=\sum_{\boldsymbol{a},\boldsymbol{b}}\Big(\mathcal{T}(\boldsymbol{t}^2)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}\Big(\overset{[21]}{\mathbb{R}}(\boldsymbol{t}^2,\boldsymbol{t}^1)\overset{[1]}{\mathbb{T}}(\boldsymbol{t}^1)\Big)^{\mathbf{1}^{\xi_1},\,\mathbf{2}^{\xi_2},\,\mathbf{3}^{\xi_3}}_{\mathbf{2}^{\xi_1},\,\boldsymbol{a},\,\mathbf{3}^{\xi_3}}\Big(\overset{[3]}{\mathbb{T}}(\boldsymbol{t}^3)\overset{[32]}{\mathbb{R}}(\boldsymbol{t}^3,\boldsymbol{t}^2)\Big)^{\mathbf{1}^{\xi_1},\,\boldsymbol{b},\,\mathbf{3}^{\xi_3}}_{\mathbf{1}^{\xi_1},\,\mathbf{3}^{\xi_2},\,\mathbf{4}^{\xi_3}}v,$$

where the sum over $\boldsymbol{a}$, $\boldsymbol{b}$ is the same as in formula (2.3.3).

Proof. Using Yang-Baxter equation (2.2.3), we can write $\mathbb{B}(\boldsymbol{t})$ in the following form,

$$\mathbb{B}_{\xi}(\boldsymbol{t})v=\Big(\overset{[21]}{\mathbb{R}}(\boldsymbol{t^2},\boldsymbol{t^1})\overset{[2]}{\mathbb{T}}(\boldsymbol{t^2})\overset{[1]}{\mathbb{T}}(\boldsymbol{t^1})\overset{[3]}{\mathbb{T}}(\boldsymbol{t^3})\overset{[31]}{\mathbb{R}}(\boldsymbol{t^3},\boldsymbol{t^1})\overset{[32]}{\mathbb{R}}(\boldsymbol{t^3},\boldsymbol{t^2})\Big)^{\mathbf{1}^{\xi_1},\,\mathbf{2}^{\xi_2},\,\mathbf{3}^{\xi_3}}_{\mathbf{2}^{\xi_1},\,\mathbf{3}^{\xi_2},\,\mathbf{4}^{\xi_3}}v.$$

Therefore,

$$\mathbb{B}_\xi(\boldsymbol{t})v = \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}} \begin{pmatrix} \overset{[21]}{\mathbb{R}} \overset{[2]}{\mathbb{T}} \overset{[1]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{3}^{\xi_3}} \begin{pmatrix} \overset{[3]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} \begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} \begin{pmatrix} \overset{[32]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_2},\boldsymbol{4}^{\xi_3}} v, \qquad (2.3.4)$$

where $\boldsymbol{p} = (p_1,\dots,p_{\xi_1})$, $\boldsymbol{q} = (q_1,\dots,q_{\xi_2})$, $\boldsymbol{r} = (r_1,\dots,r_{\xi_3})$, $\boldsymbol{s} = (s_1,\dots,s_{\xi_3})$. In (2.3.4), we omitted the arguments $\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3$ since they can be restored from the context.

We say that $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$ if $r_i \geq 3$ for all $i = 1,\dots,\xi_3$. Observe that by the definition of a singular vector and the commutation relations

$$T^3_b(w)T^3_d(u) = \frac{w-u-1}{w-u}\, T^3_d(u)T^3_b(w) - \frac{1}{w-u} T^3_d(w)T^3_b(u),$$

we have $\overset{[3]}{\mathbb{T}}(\boldsymbol{t^3})^{\boldsymbol{3}^{\xi_3}}_{\boldsymbol{r}} v = 0$ unless $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$.

Furthermore, for $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$, we have by induction on ξ_3 that $\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\boldsymbol{r},\boldsymbol{s}}$. Indeed, for $\xi_3 = 0$, the statement is true. Assume that $\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\boldsymbol{r},\boldsymbol{s}}$ for $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$ if $\xi_3 = n-1$, and consider the case $\xi_3 = n$. Let $\boldsymbol{r} = (r_1,\dots,r_n)$, $\boldsymbol{s} = (s_1,\dots,s_n)$, $\tilde{\boldsymbol{r}} = (r_1,\dots,r_{n-1})$, $\tilde{\boldsymbol{s}} = (s_1,\dots,s_{n-1})$, then we have

$$\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \sum_{\boldsymbol{x}} \left(\prod^{\rightarrow}_{1\leq i\leq n-1} \left(\prod^{\leftarrow}_{1\leq j\leq \xi_1} R^{(\xi^2+i,j)} \right)\right)^{\boldsymbol{p},\boldsymbol{q},\tilde{\boldsymbol{r}},r_n}_{\boldsymbol{x},\boldsymbol{q},\tilde{\boldsymbol{s}},r_n} \left(\prod^{\leftarrow}_{1\leq k\leq \xi_1} R^{(\xi^3,k)} \right)^{\boldsymbol{x},\boldsymbol{q},\tilde{\boldsymbol{s}},r_n}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\tilde{\boldsymbol{s}},s_n}. \qquad (2.3.5)$$

Observe that the R-matrix entry R^{jl}_{ik} with $i \neq l$ is not zero if and only if $i = j$ and $k = l$, and $R^{ik}_{ik} = 1$. Because of that and since $r_n \geq 3$, the last factor $\left(\prod R^{(3_n,1_k)}\right)^{\boldsymbol{x},\boldsymbol{q},\tilde{\boldsymbol{s}},r_n}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\tilde{\boldsymbol{s}},s_n}$ in (2.3.5) equals $\delta_{\boldsymbol{x},\boldsymbol{2}^{\xi_1}}\, \delta_{r_n,s_n}$, and we get

$$\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \left(\prod^{\rightarrow}_{1\leq i\leq n-1} \left(\prod^{\leftarrow}_{1\leq j\leq \xi_1} R^{(\xi^2+i,j)} \right)\right)^{\boldsymbol{p},\boldsymbol{q},\tilde{\boldsymbol{r}},r_n}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\tilde{\boldsymbol{s}},r_n} \delta_{r_n,s_n} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\tilde{\boldsymbol{r}},\tilde{\boldsymbol{s}}}\, \delta_{r_n,s_n} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\boldsymbol{r},\boldsymbol{s}},$$

by the induction assumption.

Since $\begin{pmatrix} \overset{[3]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} v = 0$ unless $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$ and $\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\boldsymbol{r},\boldsymbol{s}}$ for $\boldsymbol{r} \geq 3$, formula (2.3.4) becomes

$$\mathbb{B}_\xi(\boldsymbol{t})v = \sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s}} \begin{pmatrix} \overset{[21]}{\mathbb{R}} \overset{[2]}{\mathbb{T}} \overset{[1]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{3}^{\xi_3}} \begin{pmatrix} \overset{[3]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}} \begin{pmatrix} \overset{[32]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_2},\boldsymbol{4}^{\xi_3}} v,$$

and can be further transformed as

$$\mathbb{B}_\xi(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{r}} \begin{pmatrix} \overset{[2]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{c},\boldsymbol{a},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{c},\boldsymbol{b},\boldsymbol{3}^{\xi_3}} \begin{pmatrix} \overset{[21]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{3}^{\xi_3}} \begin{pmatrix} \overset{[1]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{c},\boldsymbol{b},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{b},\boldsymbol{3}^{\xi_1}} \begin{pmatrix} \overset{[3]}{\mathbb{T}} \end{pmatrix}^{\boldsymbol{2}^{\xi_1},\boldsymbol{b},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{b},\boldsymbol{r}} \begin{pmatrix} \overset{[32]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{2}^{\xi_1},\boldsymbol{b},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_3},\boldsymbol{4}^{\xi_3}} v, \tag{2.3.6}$$

where the sum is over all sequences $\boldsymbol{a} = (a_1. \dots, a_{\xi_2})$, $\boldsymbol{b} = (b_1. \dots, b_{\xi_2})$, $\boldsymbol{c} = (c_1, \dots, c_{\xi_1})$, $\boldsymbol{r} = (r_1, \dots, r_{\xi_3})$ such that $a_i, b_i, c_i, r_i \in \{1,2,3,4\}$. Since $\begin{pmatrix} \overset{[21]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{1}^{\xi_1},\boldsymbol{2}^{\xi_2},\boldsymbol{3}^{\xi_3}}_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{3}^{\xi_3}} = 0$ if $a_i \geq 3$ for some i, and $\begin{pmatrix} \overset{[32]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{2}^{\xi_1},\boldsymbol{b},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{3}^{\xi_3},\boldsymbol{4}^{\xi_3}} = 0$ if $b_i \leq 2$ for some i, terms in the sum in the right-hand side of (2.3.6) equal zero unless $a_i \in \{1,2\}$ and $b_i \in \{3,4\}$ for all i. Taking the sum over $\boldsymbol{c}$ and $\boldsymbol{r}$ in formula (2.3.6) we get the statement of Lemma 2.3.2. □

Example. Here we illustrate the proof of the relation $\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^{\xi_1},\boldsymbol{q},\boldsymbol{s}} = \delta_{\boldsymbol{p},\boldsymbol{2}^{\xi_1}}\, \delta_{\boldsymbol{r},\boldsymbol{s}}$ if $\boldsymbol{r} \geq \boldsymbol{3}^{\xi_3}$ for $\xi_1 = \xi_3 = 2$. In this case, $\boldsymbol{p} = (p_1, p_2)$, $\boldsymbol{r} = (r_1, r_2)$, $\boldsymbol{s} = (s_1, s_2)$, and

$$\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^2,\boldsymbol{q},\boldsymbol{s}} = \sum_{a,b,c,d} R^{p_2 r_1}_{ab}(t^3_1 - t^1_2)\, R^{p_1 b}_{cs_1}(t^3_1 - t^1_1)\, R^{ar_2}_{2d}(t^3_2 - t^1_2)\, R^{cd}_{2s_2}(t^3_2 - t^1_1).$$

For $r_1 \geq 3$, $r_2 \geq 3$, we have $R^{ar_2}_{2d}(t^3_2 - t^1_2) = \delta_{a,2}\, \delta_{r_2,d}$, thus

$$\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^2,\boldsymbol{q},\boldsymbol{s}} = \sum_{b,c} R^{p_2 r_1}_{2b}(t^3_1 - t^1_2)\, R^{p_1 b}_{cs_1}(t^3_1 - t^1_1)\, R^{cr_2}_{2s_2}(t^3_2 - t^1_1).$$

Then $R^{cr_2}_{2s_2}(t^3_2 - t^1_1) = \delta_{c,2}\, \delta_{r_2,s_2}$ and $R^{p_2 r_1}_{2b}(t^3_1 - t^1_2) = \delta_{p_2,2}\, \delta_{r_1,b}$, so that

$$\begin{pmatrix} \overset{[31]}{\mathbb{R}} \end{pmatrix}^{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}_{\boldsymbol{2}^2,\boldsymbol{q},\boldsymbol{s}} = R^{p_1 r_1}_{2s_1}(t^3_1 - t^1_1)\, \delta_{p_2,2}\, \delta_{r_2,s_2} = \delta_{p_2,2}\, \delta_{p_1,2}\, \delta_{r_1,s_1}\, \delta_{r_2,s_2} = \delta_{\boldsymbol{p},\boldsymbol{2}^2}\, \delta_{\boldsymbol{r},\boldsymbol{s}}.$$

2.3.2 Main theorem for the $\mathfrak{gl}_4$ case

The main result of this section is Theorem 2.3.9 formulated at the end of this section. We will approach it in several steps.

For a nonnegative integer k, set

$$Q_m\left(t_1,\ldots,t_m\right)=\prod_{1\leqslant i<j\leqslant m}\frac{t_i-t_j-1}{t_i-t_j}. \tag{2.3.7}$$

For an expression $f\left(t_1,\ldots,t_m\right)$, define

$$\operatorname{Sym}_{\boldsymbol{t}}\ f\left(t_1,\ldots,t_m\right)=\sum_{\sigma\in S_m} f\left(t_{\sigma_1},\ldots,t_{\sigma_m}\right),$$

and

$$\overline{\operatorname{Sym}_{\boldsymbol{t}}}\ f(t_1,\ldots,t_m)=\operatorname{Sym}_{\boldsymbol{t}}\left(f(t_1,\ldots,t_m)Q_m\left(t_1,\ldots,t_m\right)\right). \tag{2.3.8}$$

Proposition 2.3.3. Let ξ be a nonnegative integer and $\boldsymbol{t}=(t_1,\ldots,t_\xi)$. Then

$$\Delta\left(\mathbb{B}_\xi^{\langle 2\rangle}(\boldsymbol{t})\right)=$$
$$=\sum_{\eta=0}^{\xi}\frac{1}{(\xi-\eta)!\eta!}\overline{\operatorname{Sym}_{\boldsymbol{t}}}\Big[\Big(\mathbb{B}_\eta^{\langle 2\rangle}\left(t_1,\ldots,t_\eta\right)\otimes\mathbb{B}_{\xi-\eta}^{\langle 2\rangle}\left(t_{\eta+1},\ldots,t_\xi\right)\Big)\Big(\prod_{i=\eta+1}^{\xi}T_{22}^{\langle 2\rangle}(t_i)\otimes\prod_{j=1}^{\eta}T_{11}^{\langle 2\rangle}(t_j)\Big)\Big].$$

This proposition goes back to [55], [92]. For convenience, we give its proof in Section 2.3.5.

Given a subset I of $\{1,2\ldots,k\}$ denote by I^* the complement of I in $\{1,2,\ldots,k\}$. Define a vector $\mathbf{w}^I\in\left(\mathbb{C}^2\right)^{\otimes k}$ by the rule

$$\mathbf{w}^I=\mathbf{w}_{a_1}\otimes\mathbf{w}_{a_2}\otimes\ldots\otimes\mathbf{w}_{a_k},$$

where $a_i=2$ if $i\in I$, and $a_i=1$ if $i\notin I$.

Fix a $Y(\mathfrak{gl}_2)$-module V and a weight singular vector $v\in V$ with respect to the $Y(\mathfrak{gl}_2)$-action,

$$T_{21}^{\langle 2\rangle}(u)v=0,\qquad T_{11}^{\langle 2\rangle}(u)v=\langle T_{11}^{\langle 2\rangle}(u)v\rangle v,\quad T_{22}^{\langle 2\rangle}(u)v=\langle T_{22}^{\langle 2\rangle}(u)v\rangle v.$$

Here $\langle T_{11}^{\langle 2\rangle}(u)v\rangle$ and $\langle T_{22}^{\langle 2\rangle}(u)v\rangle$ are the corresponding eigenvalues. Given complex numbers $z_1,\ldots,z_k$, consider the $Y(\mathfrak{gl}_2)$-module $L(z_1)\otimes\cdots\otimes L(z_k)\otimes V$. Observe that $\mathbf{w}_1^{\otimes k}\otimes v$ is a weight singular vector with respect to the action of $Y(\mathfrak{gl}_2)$ in this module.

Proposition 2.3.4. For the $Y(\mathfrak{gl}_2)$-module $L(z_1)\otimes\cdots\otimes L(z_k)\otimes V$, we have

$$\begin{aligned}&\mathbb{B}^{\langle 2\rangle}_{\xi}(\boldsymbol{t})\left(\mathbf{w}_1^{\otimes k}\otimes v\right)=\\&=\sum_I\frac{1}{(\xi-|I|)!}\overline{\mathrm{Sym}_{\boldsymbol{t}}}\left[F_I(\boldsymbol{t},\boldsymbol{z})\prod_{a=1}^{|I|}\langle T^{\langle 2\rangle}_{11}(t_a)\,v\rangle\left(\mathbf{w}^I\otimes\mathbb{B}^{\langle 2\rangle}_{\xi-|I|}(t_{|I|+1},\dots,t_\xi)v\right)\right],\end{aligned}\tag{2.3.9}$$

where the sum is over all subsets $I\subset\{1,\dots,k\}$ such that $|I|\le\xi$, and for a given $I=\{i_1<i_2<\ldots<i_{|I|}\}$,

$$F_I(\boldsymbol{t},\boldsymbol{z})=\prod_{a=1}^{|I|}\left(\frac{1}{t_a-z_{i_a}}\prod_{m=i_a+1}^{k}\frac{t_a-z_m+1}{t_a-z_m}\right).\tag{2.3.10}$$

Proof. Observe that for each $Y(\mathfrak{gl}_2)$-module $L(z_i)$, $i=1,\dots,k$, the corresponding vector $\mathbf{w}_1\in L(z_i)$ is a weight singular vector:

$$T^{\langle 2\rangle}_{11}(u)\mathbf{w}_1=\left(1+(u-z_i)^{-1}\right)\mathbf{w}_1,\quad T^{\langle 2\rangle}_{22}(u)\mathbf{w}_1=\mathbf{w}_1,\quad T^{\langle 2\rangle}_{21}\mathbf{w}_1=0.$$

Moreover, $\mathbb{B}^{\langle 2\rangle}_1(u)\mathbf{w}_1=T^{\langle 2\rangle}_{12}(u)\mathbf{w}_1=(u-z_i)^{-1}\mathbf{w}_2$ and $\mathbb{B}^{\langle 2\rangle}_{\zeta}(u_1,\dots,u_\zeta)\mathbf{w}_1=0$ for $\zeta\ge 2$. Then formula (2.3.9) follows from Proposition 2.3.3 by induction on k. □

Given complex numbers $z_1,\dots,z_k$, consider the $Y(\mathfrak{gl}_2)$-module $V\otimes\bar L(z_k)\otimes\cdots\otimes\bar L(z_1)$. Observe that $v\otimes\mathbf{w}_2^{\otimes k}$ is a weight singular vector with respect to the action of $Y(\mathfrak{gl}_2)$ in this module.

Proposition 2.3.5. For the $Y(\mathfrak{gl}_2)$ module $V\otimes\bar L(z_k)\otimes\cdots\otimes\bar L(z_1)$, we have

$$\begin{aligned}&\mathbb{B}^{\langle 2\rangle}_{\xi}(\boldsymbol{t})\left(v\otimes\mathbf{w}_2^{\otimes k}\right)=\\&\sum_I\frac{1}{(\xi-|I|)!}\overline{\mathrm{Sym}_{\boldsymbol{t}}}\left[\tilde F_I(\boldsymbol{t},\boldsymbol{z})\prod_{i=1}^{|I|}\langle T^{\langle 2\rangle}_{22}\left(t_{\xi-|I|+i}\right)v\rangle\left(\mathbb{B}^{\langle 2\rangle}_{\xi-|I|}\left(t_1,\dots,t_{\xi-|I|}\right)v\otimes\mathbf{w}^{I^*}\right)\right],\end{aligned}\tag{2.3.11}$$

where the sum is over all subsets $I\subset\{1,\dots,k\}$ such that $|I|\le\xi$, and for a given $I=\{i_1<i_2<\ldots<i_{|I|}\}$,

$$\widetilde F_I(\boldsymbol{t},\boldsymbol{z})=\prod_{a=1}^{|I|}\left(\frac{1}{z_{i_a}-t_{\xi-a+1}}\prod_{m=i_a+1}^{k}\frac{z_m-t_{\xi-a+1}+1}{z_m-t_{\xi-a+1}}\right).\tag{2.3.12}$$

Proof. Observe that for each $Y(\mathfrak{gl}_2)$-module $\bar{L}(z_i)$, $i = 1, \dots, k$, the corresponding vector $\mathbf{w}_2 \in \bar{L}(z_i)$ is a weight singular vector:

$$T_{11}^{\langle 2\rangle}(u)\mathbf{w}_2 = \mathbf{w}_2, \quad T_{22}^{\langle 2\rangle}(u)\mathbf{w}_2 = \Big(1 + (z_i - u)^{-1}\Big)\,\mathbf{w}_2, \quad T_{21}^{\langle 2\rangle}(u)\mathbf{w}_2 = 0.$$

Moreover, $\mathbb{B}_1^{\langle 2\rangle}(u)\mathbf{w}_2 = T_{12}^{\langle 2\rangle}(u)\mathbf{w}_2 = (z_i - u)^{-1}\mathbf{w}_2$, and $\mathbb{B}_\zeta^{\langle 2\rangle}(u_1, \dots, u_\zeta)\mathbf{w}_2 = 0$ for $\zeta \geq 2$. Then formula (2.3.11) follows from Proposition 2.3.3 by induction on k. □

For $\boldsymbol{t} = (t_1, \dots, t_\xi)$, $\boldsymbol{z} = (z_1, \dots .z_k)$, $y \in \mathbb{C}$, and a subset $I = \{i_1 < i_2 < \dots < i_{|I|}\} \subset \{1, \dots, k\}$, define the functions

$$V_I(\boldsymbol{t}, \boldsymbol{z}, y) = \frac{1}{(\xi - |I|)!}\overline{\mathrm{Sym}}_{\boldsymbol{t}}\Big(F_I(\boldsymbol{t}, \boldsymbol{z})\prod_{a=1}^{|I|}(t_a - y)\Big) \tag{2.3.13}$$

and

$$\widetilde{V}_I(\boldsymbol{t}, \boldsymbol{z}, y) = \frac{1}{(\xi - |I|)!}\overline{\mathrm{Sym}}_{\boldsymbol{t}}\Big(\widetilde{F}_I(\boldsymbol{t}, \boldsymbol{z})\prod_{a=1}^{|I|}(t_{\xi-a+1} - y)\Big). \tag{2.3.14}$$

Consider the collection $\mathcal{S}_{p,q,r,k}$ of pairs of subsets of $\{1, \dots, k\}$ with given cardinalities of the subsets and their intersection. Namely,

$$\mathcal{S}_{p,q,r,k} = \{(I, J) \ \mid \ I, J \subset \{1, \dots, k\},\ |I| = p,\ |J| = q,\ |I \cap J| = r\}\,.$$

For $I \subset \{1, \dots, k\}$, set $\check{I} = \{k - i + 1,\ i \in I\}$.

Theorem 2.3.6. *Let V be a $\mathfrak{gl}_4$-module and let $v \in V$ be a $\mathfrak{gl}_4$-singular vector of weight $(\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4)$. Let ξ_1, ξ_2, ξ_3 be nonnegative integers, $\boldsymbol{t}^1 = (t^1_1, \dots, t^1_{\xi_1})$, $\boldsymbol{t}^2 = (t^2_1, \dots, t^2_{\xi_2})$, $\boldsymbol{t}^3 = (t^3_1, \dots, t^3_{\xi_3})$, and $\boldsymbol{t} = (\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3)$. For every triple (p, q, r), $p = 0, \dots, \min(\xi_2, \xi_3)$, $q = 0, \dots, \min(\xi_2, \xi_1)$, $r = \max(0, p + q - \xi_2), \dots, \min(p, q)$, fix a pair $(I_{p,q,r}, J_{p,q,r}) \in \mathcal{S}_{p,q,r,\xi_2}$. Then,*

1. *In the evaluation $Y(\mathfrak{gl}_4)$-module $V(x)$, one has*

$$\mathbb{B}_\xi(\boldsymbol{t})v = \prod_{a=1}^{3}\prod_{i=1}^{\xi_a}\frac{1}{t_i^a - x}\times$$
$$\times \sum_{p=0}^{\min(\xi_2,\xi_3)}\sum_{q=0}^{\min(\xi_2,\xi_1)}\sum_{r=\max(0,p+q-\xi_2)}^{\min(p,q)} \overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\Big(V_{\check{I}_{p,q,r}}(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3)\,\widetilde{V}_{J_{p,q,r}}(\boldsymbol{t}^1,\check{\boldsymbol{t}}^2,x-\Lambda^2)\Big)\times$$
$$\times\frac{e_{32}^{\xi_2-p-q+r}e_{31}^{q-r}e_{42}^{p-r}e_{41}^{r}e_{21}^{\xi_1-q}e_{43}^{\xi_3-p}v}{(p-r)!(q-r)!\,r!(\xi_2-p-q-r)!},\tag{2.3.15}$$

where $\check{\boldsymbol{t}}^2 = (t^2_{\xi_2},\dots,t^2_1)$.

2. *The function* $\overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\Big(V_{\check{I}_{p,q,r}}(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3)\widetilde{V}_{J_{p,q,r}}(\boldsymbol{t}^1,\check{\boldsymbol{t}}^2,x-\Lambda^2)\Big)$ *in* (2.3.15) *does not depend on the choice of the pair* $(I_{p,q,r},J_{p,q,r})$.

Proof. First statement follows from Propositions 2.3.7 and 2.3.8 given below. The second statement is an immediate corollary of Proposition 2.3.8.

Propositions 2.3.7 and 2.3.8 are proved in Sections 2.3.3 and 2.3.4, respectively. □

Proposition 2.3.7. In the notation of Theorem 2.3.6, we have

$$\mathbb{B}_\xi(\boldsymbol{t})v = \prod_{a=1}^{3}\prod_{i=1}^{\xi_a}\frac{1}{t_i^a - x}\times$$
$$\times \sum_{p=0}^{\min(\xi_2,\xi_3)}\sum_{q=0}^{\min(\xi_2,\xi_1)}\sum_{r=\max(0,p+q-\xi_2)}^{\min(p,q)}\Bigg(\sum_{(I,J)\in\mathcal{S}_{p,q,r,\xi_2}}\widetilde{V}_J(\boldsymbol{t}^1,\boldsymbol{t}^2,x-\Lambda^2)V_I(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3)\times$$
$$\times e_{32}^{\xi_2-p-q+r}e_{31}^{q-r}e_{42}^{p-r}e_{41}^{r}e_{21}^{\xi_1-q}e_{43}^{\xi_3-p}v\Bigg).\tag{2.3.16}$$

Proposition 2.3.8. In the notation of Theorem 2.3.6, we have

$$\sum_{(I,J)\in\mathcal{S}_{p,q,r,\xi_2}}\widetilde{V}_J(\boldsymbol{t}^1,\boldsymbol{t}^2,x-\Lambda^2)V_I(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3) =$$
$$= \frac{\overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\Big(V_{\check{I}_0}(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3)\,\widetilde{V}_{J_0}(\boldsymbol{t}^1,\check{\boldsymbol{t}}^2,x-\Lambda^2)\Big)}{(p-r)!(q-r)!\,r!(\xi_2-p-q-r)!},\tag{2.3.17}$$

where (I_0,J_0) is any pair from $\mathcal{S}_{p,q,r,\xi_2}$.

Below we reformulate Theorem 2.3.6 in a more closed form.

Theorem 2.3.9. *Let V be a $\mathfrak{gl}_4$-module and let $v \in V$ be a $\mathfrak{gl}_4$-singular vector of weight $(\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4)$. Let ξ_1, ξ_2, ξ_3 be nonnegative integers, $\boldsymbol{t}^1 = (t^1_1, \ldots, t^1_{\xi_1})$, $\boldsymbol{t}^2 = (t^2_1, \ldots, t^2_{\xi_2})$, $\boldsymbol{t}^3 = (t^3_1, \ldots, t^3_{\xi_3})$, and $\boldsymbol{t} = (\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3)$. For every triple (p, q, r), $p = 0, \ldots, \min(\xi_2, \xi_3)$, $q = 0, \ldots, \min(\xi_2, \xi_1)$, $r = \max(0, p+q-\xi_2), \ldots, \min(p, q)$, fix two sequences $\boldsymbol{i} = \{i_1 < \ldots < i_p\}$ and $\boldsymbol{j} = \{j_1 < \ldots < j_q\}$, such that $|\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\}| = r$. Then,*

1. *In the evaluation $Y(\mathfrak{gl}_4)$-module $V(x)$, one has*

$$\begin{aligned}\mathbb{B}_\xi(\boldsymbol{t})v = {} & \prod_{a=1}^{3} \prod_{i=1}^{\xi_a} \frac{1}{t^a_i - x} \times \\ & \times \sum_{p=0}^{\min(\xi_2,\xi_3)} \sum_{q=0}^{\min(\xi_2,\xi_1)} \sum_{r=\max(0,p+q-\xi_2)}^{\min(p,q)} \overline{\mathrm{Sym}}_{\boldsymbol{t}^1}\, \overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\, \overline{\mathrm{Sym}}_{\boldsymbol{t}^3}\; G_{\boldsymbol{i},\boldsymbol{j}}(\boldsymbol{t}) \times \\ & \times \frac{e_{32}^{\xi_2-p-q+r} e_{31}^{q-r} e_{42}^{p-r} e_{41}^{r} e_{21}^{\xi_1-q} e_{43}^{\xi_3-p} v}{(\xi_2-p-q+r)!(q-r)!(p-r)!\, r!(\xi_1-q)!(\xi_3-p)!},\end{aligned} \tag{2.3.18}$$

where

$$\begin{aligned}G_{\boldsymbol{i},\boldsymbol{j}}(\boldsymbol{t}) = {} & \prod_{a=1}^{p} \Bigg(\frac{t^3_a - x + \Lambda^3}{t^3_a - t^2_{i_a}} \prod_{m=i_a+1}^{\xi_2} \frac{t^3_a - t^2_m + 1}{t^3_a - t^2_m} \Bigg) \times \\ & \times \prod_{s=1}^{q} \Bigg(\frac{t^1_{\xi_1-q+s} - x + \Lambda^2}{t^2_{j_s} - t^1_{\xi_1-q+s}} \prod_{l=1}^{\xi_2-j_s} \frac{t^2_l - t^1_{\xi_1-q+s} + 1}{t^2_l - t^1_{\xi_1-q+s}} \Bigg).\end{aligned} \tag{2.3.19}$$

2. *The function $\overline{\mathrm{Sym}}_{\boldsymbol{t}^1}\, \overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\, \overline{\mathrm{Sym}}_{\boldsymbol{t}^3}\; G_{\boldsymbol{i},\boldsymbol{j}}(\boldsymbol{t})$ does not depend on the choice of the sequences $\boldsymbol{i}$, $\boldsymbol{j}$.*

Proof. Given the pair $(I_{p,q,r}, J_{p,q,r})$ from the formulation of Theorem 2.3.6, define the sequences $\boldsymbol{i} = \{i_1 < \ldots < i_p\}$ and $\boldsymbol{j} = \{j_1 < \ldots < j_q\}$ by the rule

$$I_{p,q,r} = \{\xi_2 - i_1 + 1, \xi_2 - i_2 + 1, \ldots, \xi_2 - i_p + 1\}, \quad J_{p,q,r} = \{\xi_2 - j_1 + 1, \xi_2 - j_2 + 1, \ldots, \xi_2 - j_q + 1\}.$$

Notice that $\{i_1, \ldots, i_p\} = \check{I}_{p,q,r}$, $\{j_1, \ldots, j_q\} = \check{J}_{p,q,r}$, and

$$|\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\}| = |I_{p,q,r} \cap J_{p,q,r}| = r.$$

Then combining formulas (2.3.10), (2.3.12),(2.3.13), (2.3.14), we obtain that

$$V_{\check I_{p,q,r}}(\boldsymbol{t}^3,\boldsymbol{t}^2,x-\Lambda^3)=\frac{1}{(\xi_3-p)!}\mathrm{Sym}_{\boldsymbol{t}^3}\prod_{a=1}^{p}\left(\frac{t^3_a-x+\Lambda^3}{t^3_a-t^2_{i_a}}\prod_{m=i_a+1}^{\xi_2}\frac{t^3_a-t^2_m+1}{t^3_a-t^2_m}\right) \tag{2.3.20}$$

and

$$\tilde V_{J_{p,q,r}}(\boldsymbol{t}^1,\check{\boldsymbol{t}}^2,x-\Lambda^2)=$$
$$=\frac{1}{(\xi_1-q)!}\mathrm{Sym}_{\boldsymbol{t}^1}\prod_{b=1}^{q}\left(\frac{t^1_{\xi_1-b+1}-x+\Lambda^2}{t^2_{j_{q-b+1}}-t^1_{\xi_1-b+1}}\prod_{l=1}^{\xi_2-j_{q-b+1}}\frac{t^2_l-t^1_{\xi_1-b+1}+1}{t^2_l-t^1_{\xi_1-b+1}}\right).$$

After substituting $b=q+1-s$, the last formula becomes

$$\tilde V_{J_{p,q,r}}(\boldsymbol{t}^1,\check{\boldsymbol{t}}^2,x-\Lambda^2)=$$
$$=\frac{1}{(\xi_1-q)!}\mathrm{Sym}_{\boldsymbol{t}^1}\prod_{s=1}^{q}\left(\frac{t^1_{\xi_1-q+s}-x+\Lambda^2}{t^2_{j_s}-t^1_{\xi_1-q+s}}\prod_{l=1}^{\xi_2-j_s}\frac{t^2_l-t^1_{\xi_1-q+s}+1}{t^2_l-t^1_{\xi_1-q+s}}\right). \tag{2.3.21}$$

Plugging (2.3.20), (2.3.21) into formula (2.3.15) we obtain formulas (2.3.18), (2.3.19).

Item 2) reformulates item 2) of Theorem 2.3.6. □

Example. Below we give two examples of natural choices of the sequences $\boldsymbol{i},\boldsymbol{j}$ in Theorem 2.3.9 and write down the corresponding expressions for the function $G_{\boldsymbol{i},\boldsymbol{j}}(\boldsymbol{t})$, see formula (2.3.19).

1. $\boldsymbol{i}=\boldsymbol{i}_1=\{1<\ldots<p\}$, $\boldsymbol{j}=\boldsymbol{j}_1=\{p+1-r<\ldots<p+q-r\}$. Then

$$G_{\boldsymbol{i_1},\boldsymbol{j_1}}(\boldsymbol{t})=\prod_{a=1}^{p}\left(\frac{t^3_a-\Lambda^3}{t^3_a-t^2_a}\prod_{m=a+1}^{\xi_2}\frac{t^3_a-t^2_m+1}{t^3_a-t^2_m}\right)\times$$
$$\times\prod_{c=1}^{q}\left(\frac{t^1_{\xi_1-q+c}-\Lambda^2}{t^2_{p-r+c}-t^1_{\xi_1-q+c}}\prod_{l=1}^{\xi_2-p+r-c}\frac{t^2_l-t^1_{\xi_1-q+c}+1}{t^2_l-t^1_{\xi_1-q+c}}\right).$$

2. $\boldsymbol{i}=\boldsymbol{i}_2=\{q+1-r<\ldots<q+p-r\}$, $\boldsymbol{j}=\boldsymbol{j}_2=\{1<\ldots<q\}$. Then

$$G_{\boldsymbol{i_2},\boldsymbol{j_2}}(\boldsymbol{t})=\prod_{a=1}^{p}\left(\frac{t^3_a-\Lambda^3}{t^3_a-t^2_{q-r+a}}\prod_{m=q-r+a+1}^{\xi_2}\frac{t^3_a-t^2_m+1}{t^3_a-t^2_m}\right)\times$$
$$\times\prod_{b=1}^{q}\left(\frac{t^1_{\xi_1-b+1}-\Lambda^2}{t^2_b-t^1_{\xi_1-b+1}}\prod_{l=1}^{\xi_2-b}\frac{t^2_l-t^1_{\xi_1-b+1}+1}{t^2_l-t^1_{\xi_1-b+1}}\right).$$

Notice that the equality

$$\overline{\mathrm{Sym}}_{\boldsymbol{t}^1}\ \overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\ \overline{\mathrm{Sym}}_{\boldsymbol{t}^3}\ G_{\boldsymbol{i}_1,\boldsymbol{j}_1}(\boldsymbol{t}) \ = \ \overline{\mathrm{Sym}}_{\boldsymbol{t}^1}\ \overline{\mathrm{Sym}}_{\boldsymbol{t}^2}\ \overline{\mathrm{Sym}}_{\boldsymbol{t}^3}\ G_{\boldsymbol{i}_2,\boldsymbol{j}_2}(\boldsymbol{t}),$$

stated in item 2) of Theorem 2.3.9, is not obvious.

2.3.3 Proof of Proposition 2.3.7

Let V be a $\mathfrak{gl}_4$-module and $v \in V$ a $\mathfrak{gl}_4$-singular vector of weight $(\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4)$. Let ξ_1, ξ_2, ξ_3 be nonnegative integers, $\boldsymbol{t}^1 = (t^1_1, \dots, t^1_{\xi_1})$, $\boldsymbol{t}^2 = (t^2_1, \dots, t^2_{\xi_2})$, $\boldsymbol{t}^3 = (t^3_1, \dots, t^3_{\xi_3})$ and $\boldsymbol{t} = (\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3)$. Recall that in the evaluation $Y(\mathfrak{gl}_4)$-module $V(x)$, we have $T^a_b(u) = \delta_{ab} + e_{ba}(u-x)^{-1}$, thus

$$T^a_a(u)v = \frac{u - x + \Lambda_a}{u - x} v.$$

By Proposition 2.3.1,

$$\mathbb{B}_\xi(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b}} \left(\overset{[2]}{\mathbb{T}}(\boldsymbol{t}^2) \right)^{\boldsymbol{a}}_{\boldsymbol{b}} \left(\phi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_1}}(\boldsymbol{t}^1) \right) \right)^{\boldsymbol{a}} \left(\psi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3) \right) \right)^{\boldsymbol{b-2}} v, \tag{2.3.22}$$

where the sum is taken over all sequences $\boldsymbol{a} = (a_1, a_2, \dots, a_{\xi_2})$, $\boldsymbol{b} = (b_1, b_2, \dots, b_{\xi_2})$, such that $a_i \in \{1, 2\}$, and $b_i \in \{3, 4\}$ for all $i = 1, \dots, \xi_2$.

Let ${}^{\psi}V(x)$ be the $Y(\mathfrak{gl}_2)$-module obtained by pulling back the module $V(x)$ through the embedding ψ_2. In order to compute $\left(\psi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3) \right) \right)^{\boldsymbol{b}} v$, we take the weight function $\mathbb{B}^2_{\xi_3}(\boldsymbol{t^3}) \left(\mathbf{w}_1^{\otimes \xi_2} \otimes v \right)$ in the $Y(\mathfrak{gl}_2)$-module $L(t^2_1) \otimes \dots \otimes L(t^2_{\xi_2}) \otimes {}^{\psi}V(x)$ and apply Proposition 2.3.4 for $k = \xi_2$. Then we obtain

$$\left(\psi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3) \right) \right)^{\boldsymbol{b-2}} v =$$
$$= \frac{1}{(\xi_3 - |I|)!} \overline{\mathrm{Sym}}_{\boldsymbol{t}^3} \left[F_I(\boldsymbol{t}^3, \boldsymbol{t}^2) \prod_{m=1}^{|I|} \frac{t^3_m - x + \Lambda^3}{t^3_m - x} \prod_{r=|I|+1}^{\xi_3} \frac{1}{t^3_r - x} \right] e_{43}^{\xi_3 - |I|} v,$$

where the subset $I \subset \{1, \dots, \xi_2\}$ and the sequence $\boldsymbol{b} = (b_1, \dots, b_{\xi_2})$ are related as follows: $b_j = 3$ if $j \notin I$ and $b_j = 4$ if $j \in I$. Therefore, by formula (2.3.14) we have

$$\left(\psi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3) \right) \right)^{\boldsymbol{b-2}} v = \prod_{r=1}^{\xi_3} \frac{1}{t^3_r - x} V_I(\boldsymbol{t}^3, \boldsymbol{t}^2, x - \Lambda^3)\, e_{43}^{\xi_3 - |I|} v.$$

The next step is to compute $\left(\phi_2(\boldsymbol{t}^2) \left(\mathbb{B}^{\langle 2 \rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3) \right) \right)^{\boldsymbol{a}} e_{43}^{\xi_3 - |I|} v$. Notice that for any nonnegative integer m, we have

$$T^2_1(u) e^m_{43} v = 0, \quad T^2_2(u) e^m_{43} v = \frac{u - x + \Lambda^2}{u - x} e^m_{43} v, \quad T^1_1(u) e^m_{43} v = \frac{u - x + \Lambda_1}{u - x} e^m_{43} v.$$

Let ${}^{\phi}V(x)$ the $Y(\mathfrak{gl}_2)$-module obtained by pulling back $V(x)$ through the embedding ϕ_2. To compute the $\left(\phi_2(\boldsymbol{t}^2)\left(\mathbb{B}^{\langle 2\rangle}_{\boldsymbol{\xi_3}}(\boldsymbol{t}^3)\right)\right)^{\boldsymbol{a}} e_{43}^{\xi_3-|I|}v$, we take the weight function $\mathbb{B}^2_{\xi_1}(\boldsymbol{t^1})(e_{43}^{\xi_3-|I|}v\otimes\mathbf{w}_2^{\otimes\xi_2})$ in the $Y(\mathfrak{gl}_2)$-module ${}^{\phi}V(x)\otimes L(t^2_{\xi_2})\otimes\cdots\otimes L(t^2_1)$ and apply Proposition 2.3.5 for $k=\xi_2$. Then we obtain

$$\left(\phi_2(\boldsymbol{t}^2)\left(\mathbb{B}^{\langle 2\rangle}_{\xi_1}(\boldsymbol{t}^1)\right)\right)^{\boldsymbol{a}} e_{43}^{\xi_3-|I|}v =$$

$$=\frac{1}{(\xi_1-|J|)!}\overline{\mathrm{Sym}}_{\boldsymbol{t}^1}\left[\tilde{F}_J(\boldsymbol{t}^1,\boldsymbol{t}^2)\prod_{m=1}^{|J|}\frac{t^1_m-x+\Lambda^2}{t^1_m-x}\prod_{r=|J|}^{\xi_1}\frac{1}{t^1_r-x}\right]e_{21}^{\xi_1-|J|}e_{43}^{\xi_3-|I|}v,$$

where the subset $J\subset\{1,\dots,\xi_2\}$ and the sequence $\boldsymbol{a}=(a_1,\dots,a_{\xi^2})$ are related as follows: $a_j=1$ if $j\in J$ and $a_j=2$ if $j\notin J$. Therefore, by formula (2.3.13) we have

$$\left(\phi_2(\boldsymbol{t}^2)\left(\mathbb{B}^{\langle 2\rangle}_{\xi_1}(\boldsymbol{t}^1)\right)\right)^{\boldsymbol{a}} e_{43}^{\xi_3-|I|}v=\prod_{r=1}^{\xi_1}\frac{1}{t^1_r-x}\widetilde{V}_J(\boldsymbol{t}^1,\boldsymbol{t}^2,x-\Lambda^2)\,e_{21}^{\xi_1-|J|}e_{43}^{\xi_3-|I|}v.$$

Finally, for the sequences $\boldsymbol{a},\boldsymbol{b}$ that are related to the sets I,J as above, we have

$$\left(\overset{[2]}{\mathbb{T}}(\boldsymbol{t}^2)\right)^{\boldsymbol{a}}_{\boldsymbol{b}}=\prod_{r=1}^{\xi_2}\frac{1}{t^2_r-x}e_{32}^{\xi_2-p-q+r}e_{31}^{q-r}e_{42}^{p-r}e_{41}^{r},\tag{2.3.23}$$

where $p=|I|,q=|J|,r=|I\cap J|$.

Now formula (2.3.16) follows from formulas (2.3.22)–(2.3.23).

2.3.4 Proof of Proposition 2.3.8

Consider the algebra $\mathcal{A}$ generated by two commuting copies of the symmetric group S_k and rational functions of $z_1, \dots, z_k$ subject to relations (2.3.24) below. We denote the copies of S_k in $\mathcal{A}$ by $\dot{S}_k$ and $\ddot{S}_k$, and mark elements of $\dot{S}_k$ and $\ddot{S}_k$ by the corresponding dots, keeping the notation S_k without dots for the abstract symmetric group.

Let $\boldsymbol{z} = (z_1, \dots, z_k)$ and $\boldsymbol{z}^\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(k)})$. The additional relations in $\mathcal{A}$ are

$$\dot{\sigma} f(\boldsymbol{z}) = f(\boldsymbol{z}^\sigma)\, \dot{\sigma}, \qquad \ddot{\tau} f(\boldsymbol{z}) = f(\boldsymbol{z})\, \ddot{\tau}. \tag{2.3.24}$$

For $a = 1, \dots, k-1$, let $s_a \in S_k$ be the transposition of a and $a+1$. Consider the elements $\hat{s}_1, \dots, \hat{s}_k$ of $\mathcal{A}$,

$$\hat{s}_a = \left(\frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} \ddot{s}_a - \frac{1}{z_a - z_{a+1} - 1} \right) \dot{s}_a. \tag{2.3.25}$$

It is straightforward to check that they satisfy the following relations,

$$\hat{s}_a \hat{s}_{a+1} \hat{s}_a = \hat{s}_{a+1} \hat{s}_a \hat{s}_{a+1}, \qquad \hat{s}_a^2 = 1.$$

Therefore, the assignment $s_a \mapsto \hat{s}_a$ defines an algebra homomorphism $\mathbb{C}S_k \to \mathcal{A}$. For any $\sigma \in S_k$, we denote by $\hat{\sigma}$ the corresponding element of $\mathcal{A}$. Every element $\hat{\sigma}$ can be written in the following form

$$\hat{\sigma} = \sum_{\tau \in S_k} X_{\sigma,\tau}(\boldsymbol{z})\, \ddot{\tau} \dot{\sigma}, \tag{2.3.26}$$

where $X_{\sigma,\tau}(\boldsymbol{z})$ are functions of $z_1, \dots, z_k$.

Let $|\sigma|$ denote the length of $\sigma \in S_k$.

Lemma 2.3.10. The functions $X_{\sigma,\tau}(\boldsymbol{z})$ have the following properties ,

$$X_{\sigma,\tau}(\boldsymbol{z}) = 0 \quad \text{if} \quad |\tau| > |\sigma|, \tag{2.3.27}$$

$$X_{\sigma,\tau}(\boldsymbol{z}) = \delta_{\sigma,\tau} X_{\sigma,\sigma}(\boldsymbol{z}) \quad \text{if} \quad |\tau| = |\sigma|, \tag{2.3.28}$$

$$X_{\sigma,\sigma}(\boldsymbol{z}) = \prod_{a<b,\sigma^{-1}(a)>\sigma^{-1}(b)} \frac{z_a - z_b}{z_a - z_b - 1}. \tag{2.3.29}$$

Proof. Formulas (2.3.27),(2.3.28) follow from formula (2.3.25) by inspection. Formula (2.3.29) can be shown by induction on $|\sigma|$. □

Denote by σ_0 the longest element of S_k, $\sigma_0(i) = k - i + 1$, $i = 1, \dots, k$. Let

$$\Phi(\boldsymbol{z}) = \prod_{a<b} \frac{z_a - z_b - 1}{z_a - z_b}\,.$$

Notice that

$$\Phi(\boldsymbol{z}) = \frac{1}{X_{\sigma_0,\sigma_0}(\boldsymbol{z})}\,. \tag{2.3.30}$$

Lemma 2.3.11. One has

$$\sum_{\lambda\in S_k} X_{\lambda,\rho}(\boldsymbol{z})\Phi(\boldsymbol{z}^{\lambda\sigma_0})X_{\sigma_0\lambda^{-1},\sigma_0\tau^{-1}}(\boldsymbol{z}^{\lambda\sigma_0}) = \delta_{\rho,\tau}\,. \tag{2.3.31}$$

Proof. Since $\widehat{\sigma\tau} = \hat{\sigma}\hat{\tau}$, by formula (2.3.26) we have

$$X_{\sigma\tau,\rho}(\boldsymbol{z}) = \sum_{\pi} X_{\sigma,\pi}(\boldsymbol{z})X_{\tau,\pi^{-1}\rho}(\boldsymbol{z}^{\sigma}). \tag{2.3.32}$$

Taking here $\rho = \sigma_0$, and using Lemma 2.3.10 and formula (2.3.30), we get

$$\sum_{\pi} X_{\sigma,\pi}(\boldsymbol{z})X_{\tau,\pi^{-1}\sigma_0}(\boldsymbol{z}^{\sigma}) = \delta_{\sigma\tau,\sigma_0}\,\frac{1}{\Phi(\boldsymbol{z})}. \tag{2.3.33}$$

Replacing now $\boldsymbol{z}$ by $\boldsymbol{z}^{\sigma^{-1}}$ in formula (2.3.33) and taking there $\tau = \mu^{-1}\sigma_0$, we get

$$\sum_{\pi} X_{\sigma,\pi}(\boldsymbol{z}^{\sigma^{-1}})X_{\mu^{-1}\sigma_0,\pi^{-1}\sigma_0}(\boldsymbol{z}) = \delta_{\sigma,\mu}\,\frac{1}{\Phi(\boldsymbol{z}^{\sigma^{-1}})}. \tag{2.3.34}$$

Formula (2.3.34) can be understood as the matrix equality $AB = C$ for $k! \times k!$ matrices A, B, C with entries labeled by permutations:

$$A_{\sigma,\pi} = X_{\sigma,\pi}(\boldsymbol{z}^{\sigma^{-1}}), \qquad B_{\pi,\mu} = X_{\mu^{-1}\sigma_0,\pi^{-1}\sigma_0}(\boldsymbol{z}), \quad C_{\sigma,\mu} = \delta_{\sigma,\mu}\,\frac{1}{\Phi(\boldsymbol{z}^{\sigma^{-1}})}.$$

Therefore the product $BC^{-1}A$ equals the identity matrix, which can be written as follows:

$$\sum_{\mu} X_{\mu^{-1}\sigma_0,\pi^{-1}\sigma_0}(\boldsymbol{z})\Phi(\boldsymbol{z}^{\mu^{-1}})X_{\mu,\sigma}(\boldsymbol{z}^{\mu^{-1}}) = \delta_{\pi,\sigma}\,.$$

After the substitution $\lambda = \mu^{-1}\sigma_0$, $\rho = \pi^{-1}\sigma_0$, $\tau = \sigma^{-1}\sigma_0$, we get formula (2.3.31). □

Lemma 2.3.12. One has

$$X_{\mu,\sigma}(\boldsymbol{z}^{s_a}) = \frac{z_a - z_{a+1}}{z_a - z_{a+1} + 1} X_{s_a\mu,s_a\sigma}(\boldsymbol{z}) + \frac{1}{z_a - z_{a+1} + 1} X_{s_a\mu,\sigma}(\boldsymbol{z}). \tag{2.3.35}$$

Proof. By formulas (2.3.25), (2.3.26), we have

$$X_{s_a,s_a}(\boldsymbol{z}) = \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1}, \quad X_{s_a,\mathrm{id}}(\boldsymbol{z}) = \frac{-1}{z_a - z_{a+1} - 1},$$

and $X_{s_a,\tau}(\boldsymbol{z}) = 0$, otherwise. Therefore, by formula (2.3.32) we obtain

$$X_{s_a\pi,\sigma}(\boldsymbol{z}) = \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} X_{\pi,s_a\sigma}(\boldsymbol{z}^{s_a}) - \frac{1}{z_a - z_{a+1} - 1} X_{\pi,\sigma}(\boldsymbol{z}^{s_a}).$$

Replacing here $\boldsymbol{z}$ by $\boldsymbol{z}^{s_a}$ and making the substitution $\pi = s_a\mu$, we get formula (2.3.35). □

Lemma 2.3.13. One has

$$X_{\mu,\sigma}(\boldsymbol{z}^{s_a\mu^{-1}}) = \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} X_{\mu s_a,\sigma s_a}(\boldsymbol{z}^{s_a\mu^{-1}}) - \frac{1}{z_a - z_{a+1} - 1} X_{\mu s_a,\sigma}(\boldsymbol{z}^{s_a\mu^{-1}}). \tag{2.3.36}$$

Proof. By formula (2.3.32), we have

$$X_{\mu s_a,\sigma}(\boldsymbol{z}) = \sum_{\pi} X_{\mu,\pi}(\boldsymbol{z}) X_{s_a,\pi^{-1}\sigma}(\boldsymbol{z}^{\mu}).$$

Thus

$$X_{\mu s_a,\sigma}(\boldsymbol{z}) = \sum_{\pi} X_{\mu,\pi}(\boldsymbol{z}) X_{s_a,\pi^{-1}\sigma}(\boldsymbol{z}^{\mu}) = X_{\mu,\sigma}(\boldsymbol{z}) X_{s_a,id}(\boldsymbol{z}^{\mu}) + X_{\mu,\sigma s_a}(\boldsymbol{z}) X_{s_a,s_a}(\boldsymbol{z}^{\mu}).$$

Replacing here $\boldsymbol{z}$ by $\boldsymbol{z}^{\mu^{-1}}$, we get

$$X_{\mu s_a,\sigma}(\boldsymbol{z}^{\mu^{-1}}) = X_{\mu,\sigma}(\boldsymbol{z}^{\mu^{-1}})X_{s_a,id}(\boldsymbol{z}) + X_{\mu,\sigma s_a}(\boldsymbol{z}^{\mu^{-1}})X_{s_a,s_a}(\boldsymbol{z}).$$

Substituting now μ with μs_a, we obtain (2.3.36). □

For $\sigma \in S_n$ and a subset $I = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, denote

$$\sigma(I) = \{\sigma(i_1), \dots, \sigma(i_m)\}.$$

Recall the functions $V_I(\boldsymbol{t}, \boldsymbol{z}, y)$, $\tilde{V}_J(\boldsymbol{t}, \boldsymbol{z}, y)$, see formulas (2.3.10),(2.3.13) and (2.3.12),(2.3.14).

Lemma 2.3.14. For each $a = 1, \dots, k-1$, we have

$$V_I(\boldsymbol{t}; \boldsymbol{z}^{s_a}, y) = \frac{z_{a+1} - z_a}{z_{a+1} - z_a - 1} V_{s_a(I)}(\boldsymbol{t}, \boldsymbol{z}, y) - \frac{1}{z_{a+1} - z_a - 1} V_I(\boldsymbol{t}, \boldsymbol{z}, y), \tag{2.3.37}$$

$$\tilde{V}_I(\boldsymbol{t}; \boldsymbol{z}^{s_a}, y) = \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} \tilde{V}_{s_a(I)}(\boldsymbol{t}, \boldsymbol{z}, y) - \frac{1}{z_a - z_{a+1} - 1} \tilde{V}_I(\boldsymbol{t}, \boldsymbol{z}, y). \tag{2.3.38}$$

Proof. By the structure of formulas (2.3.10),(2.3.13) for the function $V_I(\boldsymbol{t}, \boldsymbol{z}, y)$, it is enough to prove formula (2.3.37) for $k = 2$. In this case, the statement follows from the identities:

$$1 = \frac{z' - z}{z' - z - 1} - \frac{1}{z' - z - 1},$$

$$\frac{1}{t - z'} \cdot \frac{t - z + 1}{t - z} = \frac{z' - z}{z' - z - 1} \cdot \frac{1}{t - z'} - \frac{1}{z' - z - 1} \cdot \frac{1}{t - z} \cdot \frac{t - z' + 1}{t - z'},$$

$$(t' - z)(t - z' + 1)(t - t' - 1) - (t - z)(t' - z' + 1)(t' - t - 1) =$$
$$= (t' - z')(t - z + 1)(t - t' - 1) - (t - z')(t' - z + 1)(t' - t - 1).$$

The proof of formula (2.3.38) is similar by using formulas (2.3.12),(2.3.14) for functions $\tilde{V}_J(\boldsymbol{t}, \boldsymbol{z}, y)$. □

The statement of Proposition 2.3.8 is given by formula (2.3.17). It can be written as follows

$$\sum_{(I,J)\in\mathcal{S}_{p,q,r,k}} V_I(\boldsymbol{t^3},\boldsymbol{z},x-\Lambda^3)\,\widetilde{V}_J(\boldsymbol{t}^1,\boldsymbol{z},x-\Lambda^2) = \\ = \frac{1}{C_{p,q,r,k}}\sum_{\sigma\in S_k} V_{\sigma_0(I_0)}(\boldsymbol{t}^3,\boldsymbol{z}^\sigma,x-\Lambda^3)\,\widetilde{V}_{J_0}(\boldsymbol{t}^1,\boldsymbol{z}^{\sigma\sigma_0},x-\Lambda^2)\,\Phi(\boldsymbol{z}^\sigma), \tag{2.3.39}$$

where $\mathcal{S}_{p,q,r,k}$ in the left-hand side is the set of all pairs of subsets I,J of $\{1,\dots,k\}$, such that $|I|=p$, $|J|=q$, $|I\cap J|=r$, and we use $k=\xi^2$, $\boldsymbol{z}=\boldsymbol{t}^2$. In the right-hand side, $C_{p,q,r,k}=(p-r)!(q-r)!\;r!(k-p-q-r)!$ and (I_0,J_0) is any fixed pair from $\mathcal{S}_{p,q,r,k}$. We also expanded $\overline{\mathrm{Sym}}_{\boldsymbol{t}^2}$ according to formulas (2.3.7),((2.3.8), and observed that $\check{I}_0=\sigma_0(I_0)$, $\check{\boldsymbol{t}}^2=\boldsymbol{z}^{\sigma_0}$.

In the rest of the proof, we will suppress the arguments $\boldsymbol{t}^1,\boldsymbol{t}^3,x-\Lambda^2,x-\Lambda^3$ because they are the same in both sides of formula (2.3.39) and will never be changed in the reasoning.

Notice that every pair $(I,J)\in\mathcal{S}_{p,q,r,k}$ can be obtained from an arbitrary fixed pair $(I_0,J_0)\in\mathcal{S}_{p,q,r,k}$ by the action of the symmetric group S_k. Therefore, the left-hand side of the formula (2.3.39) can be written in the following way

$$\sum_{(I,J)\in\mathcal{S}_{p,q,r,k}}\tilde{V}_J(\boldsymbol{z})V_I(\boldsymbol{z}) = \frac{1}{C_{p,q,r,k}}\sum_{\sigma\in S_k}\tilde{V}_{\sigma(J_0)}(\boldsymbol{z})V_{\sigma(I_0)}(\boldsymbol{z}). \tag{2.3.40}$$

Using Lemma 2.3.11, we get

$$\sum_{\sigma\in S_k}\tilde{V}_{\sigma(J_0)}(\boldsymbol{z})V_{\sigma(I_0)}(\boldsymbol{z}) = \sum_{\sigma,\pi,\tau\in S_k}\tilde{V}_{\sigma(J_0)}(\boldsymbol{z})X_{\pi,\sigma}(\boldsymbol{z})\Phi(\boldsymbol{z}^{\pi\sigma_0})X_{\sigma_0\pi^{-1},\sigma_0\tau^{-1}}(\boldsymbol{z}^{\pi\sigma_0})V_{\tau(I_0)}(\boldsymbol{z}), \tag{2.3.41}$$

since from formula (2.3.31)

$$\sum_{\pi\in S_k}X_{\pi,\sigma}(\boldsymbol{z})\Phi(\boldsymbol{z}^{\pi\sigma_0})X_{\sigma_0\pi^{-1},\sigma_0\tau^{-1}}(\boldsymbol{z}^{\pi\sigma_0}) = \delta_{\sigma,\tau}.$$

Lemma 2.3.15. We have

$$\sum_{\sigma\in S_k}\tilde{V}_{\sigma(J_0)}(\boldsymbol{z})X_{\pi,\sigma}(\boldsymbol{z}) = \tilde{V}_{J_0}(\boldsymbol{z}^\pi). \tag{2.3.42}$$

Proof. We will use induction on the length of the permutation π. For $\pi = \mathrm{id}$, formula (2.3.42) is clear, and for $\pi = s_a$ with some $a = 1, \ldots, k-1$, formula (2.3.42) coincides with formula (2.3.38). For the induction step, we find a such that $|s_a\pi| = |\pi| - 1$, and denote $\rho = s_a\pi$. Then by the induction assumption

$$\sum_\sigma \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}) X_{\rho,\sigma}(\boldsymbol{z}) = \tilde{V}_{J_0}(\boldsymbol{z}^\rho).$$

Replacing here $\boldsymbol{z}$ by $\boldsymbol{z}^{s_a}$, we get

$$\sum_\sigma \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}^{s_a}) X_{\rho,\sigma}(\boldsymbol{z}^{s_a}) = \tilde{V}_{J_0}(\boldsymbol{z}^{s_a\rho}) = \tilde{V}_{J_0}(\boldsymbol{z}^\pi). \tag{2.3.43}$$

Using formulas (2.3.35)), (2.3.38), the left-hand side of (2.3.43)) becomes

$$\sum_\sigma \frac{(z_a - z_{a+1})^2}{(z_a - z_{a+1})^2 - 1} \tilde{V}_{s_a\sigma(J_0)}(\boldsymbol{z}) X_{s_a\rho, s_a\sigma}(\boldsymbol{z}) + \sum_\sigma \frac{z_a - z_{a+1}}{(z_a - z_{a+1})^2 - 1} \tilde{V}_{s_a\sigma(J_0)}(\boldsymbol{z}) X_{s_a\rho,\sigma}(\boldsymbol{z}) -$$
$$- \sum_\sigma \frac{z_a - z_{a+1}}{(z_a - z_{a+1})^2 - 1} \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}) X_{s_a\rho, s_a\sigma}(\boldsymbol{z}) - \sum_\sigma \frac{1}{(z_a - z_{a+1})^2 - 1} \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}) X_{s_a\rho,\sigma}(\boldsymbol{z}).$$

Changing the summation index in the first and second sums from σ to $s_a\sigma$, we observe that the second and third sums cancel each other, while the first and forth sums combine together and simplify to the expression

$$\sum_\sigma \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}) X_{s_a\rho,\sigma}(\boldsymbol{z}) = \sum_\sigma \tilde{V}_{\sigma(J_0)}(\boldsymbol{z}) X_{\pi,\sigma}(\boldsymbol{z}),$$

which appears in the left-hand side of formula (2.3.42). □

Lemma 2.3.16. We have

$$\sum_\tau V_{\tau(I_0)}(\boldsymbol{z}) X_{\sigma_0\pi^{-1}, \sigma_0\tau^{-1}}(\boldsymbol{z}^{\pi\sigma_0}) = V_{\sigma_0(I_0)}(\boldsymbol{z}^{\pi\sigma_0}). \tag{2.3.44}$$

Proof. Recall the notation $\sigma_0(I_0) = \check{I}_0$. Transform formula (2.3.44) by making the substitutions $\mu = \sigma_0\pi^{-1}$, $\sigma = \sigma_0\tau^{-1}$,

$$\sum_\sigma V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\mu,\sigma}(\boldsymbol{z}^{\mu^{-1}}) = V_{\check{I}_0}(\boldsymbol{z}^{\mu^{-1}}). \tag{2.3.45}$$

The rest of the proof is analogous to that of Lemma 2.3.15.

To prove formula (2.3.45), we will use induction on the length of μ. For $\mu = \mathrm{id}$, formula (2.3.45) is clear, and for $\mu = s_a$ with some $a = 1, \dots, k-1$, formula (2.3.45) coincides with formula (2.3.37). For the induction step, we find a such that $|\mu s_a| = |\mu| - 1$, and denote $\rho = \mu s_a$. Then by the induction assumption,

$$\sum_\sigma V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho,\sigma}(\boldsymbol{z}^{\rho^{-1}}) = V_{\check{I}_0}(\boldsymbol{z}^{\rho^{-1}}),$$

and replacing here $\boldsymbol{z}$ by $\boldsymbol{z}^{s_a}$, we get

$$\sum_\sigma V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z}^{s_a})X_{\rho,\sigma}(\boldsymbol{z}^{s_a\rho^{-1}}) = V_{\check{I}_0}(\boldsymbol{z}^{\mu^{-1}}) = V_{\check{I}_0}(\boldsymbol{z}^{s_a\rho^{-1}}). \tag{2.3.46}$$

Using formulas (2.3.36), (2.3.37), the left-hand of (2.3.46) side becomes

$$\begin{aligned}
&\sum_\sigma \frac{(z_{a+1}-z_a)^2}{(z_{a+1}-z_a)^2-1} V_{s_a\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho s_a,\sigma s_a}(\boldsymbol{z}^{s_a\rho^{-1}}) - \\
&\quad - \sum_\sigma \frac{z_a - z_{a+1}}{(z_a - z_{a+1})^2-1} V_{s_a\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho s_a,\sigma}(\boldsymbol{z}^{s_a\rho^{-1}}) + \\
&+ \sum_\sigma \frac{z_a - z_{a+1}}{(z_a - z_{a+1})^2-1} V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho s_a,\sigma s_a}(\boldsymbol{z}^{s_a\rho^{-1}}) - \\
&\quad - \sum_\sigma \frac{1}{(z_{a+1}-z_a)^2-1} V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho s_a,\sigma}(\boldsymbol{z}^{s_a\rho^{-1}}).
\end{aligned}$$

Changing the summation index in the first and second sums from σ to σs_a, we observe that the second and third sums cancel each other, while the first and the forth sums combine together and simplify to the expression

$$\sum_\sigma V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\rho s_a,\sigma}(\boldsymbol{z}^{s_a\rho^{-1}}) = \sum_\sigma V_{\sigma^{-1}(\check{I}_0)}(\boldsymbol{z})X_{\mu,\sigma}(\boldsymbol{z}^{\mu^{-1}}),$$

which appears in the left-hand side of formula (2.3.44). □

Using Lemmas 2.3.15 and 2.3.16, we evaluate the sums over σ and τ in the right-hand side of the formula (2.3.41) and get the equality

$$\sum_{\sigma\in S_k} \tilde{V}_{\sigma(J_0)}(\boldsymbol{z})V_{\sigma(I_0)}(\boldsymbol{z}) = \sum_{\pi} V_{\sigma_0(I_0)}(\boldsymbol{z}^{\pi\sigma_0})\tilde{V}_{J_0}(\boldsymbol{z}^{\pi})\Phi(\boldsymbol{z}^{\pi\sigma_0}). \tag{2.3.47}$$

Using formula (2.3.40) in the left-hand side and making the substitution $\pi = \sigma\sigma_0$ in the right-hand side we obtain that (2.3.47) can be written as

$$\sum_{(I,J)\in\mathcal{S}_{p,q,r,k}} \tilde{V}_J(\boldsymbol{z})V_I(\boldsymbol{z}) = \frac{1}{C_{p,q,r,k}}\sum_{\sigma} V_{\sigma_0(I_0)}(\boldsymbol{z}^{\sigma})\tilde{V}_{J_0}(\boldsymbol{z}^{\sigma\sigma_0})\Phi(\boldsymbol{z}^{\sigma}),$$

which is formula (2.3.39). Proposition 2.3.8 is proved.

2.3.5 Proof of the coproduct formula

In this section we will consider only the algebra $Y(\mathfrak{gl}_2)$ and, for convenience, we will not write the superscript $\langle 2\rangle$. We will use the commutation relations

$$T_{11}(u)T_{11}(t) = T_{11}(t)T_{11}(u), \quad T_{12}(u)T_{12}(t) = T_{12}(t)T_{12}(u), \quad T_{22}(u)T_{22}(u) = T_{22}(t)T_{22}(u), \tag{2.3.48}$$

$$T_{11}(u)T_{12}(t) = \frac{u-t-1}{u-t} T_{12}(t)T_{11}(u) + \frac{1}{u-t} T_{12}(u)T_{11}(t), \tag{2.3.49}$$

$$T_{22}(u)T_{12}(t) = \frac{u-t+1}{u-t} T_{12}(t)T_{22}(u) - \frac{1}{u-t} T_{12}(u)T_{22}(t), \tag{2.3.50}$$

following from the defining relations in $Y(\mathfrak{gl}_2)$, see (2.2.5). We will also use the next statement.

Proposition 2.3.17. One has

$$\begin{aligned} T_{11}(u)T_{12}(t_1)\dots T_{12}(t_k) &= \prod_{i=1}^{k} \frac{u-t_i-1}{u-t_i} T_{12}(t_1)\dots T_{12}(t_k)T_{11}(u)+ \\ &+ \sum_{l=1}^{k} \frac{1}{u-t_l} \prod_{\substack{m=1\\ m\neq l}}^{k} \frac{t_l-t_m-1}{t_l-t_m} T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_k)T_{12}(u)T_{11}(t_l), \end{aligned} \tag{2.3.51}$$

and

$$\begin{aligned} T_{22}(u)T_{12}(t_1)\dots T_{12}(t_k) &= \prod_{i=1}^{k} \frac{u-t_i+1}{u-t_i} T_{12}(t_1)\dots T_{12}(t_k)T_{22}(u)- \\ &- \sum_{l=1}^{k} \frac{1}{u-t_l} \prod_{\substack{m=1\\ m\neq l}}^{k} \frac{t_l-t_m+1}{t_l-t_m} T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_k)T_{12}(u)T_{22}(t_l). \end{aligned} \tag{2.3.52}$$

Proof. The statement goes back to [31]. We will prove it by induction on k. Consider formula (2.3.51). The statement for $k = 1$ is given by formula (2.3.49). We use the induction assumption to move $T_{11}(u)$ through the product $T_{12}(t_1)\dots T_{12}(t_{k-1})$:

$$T_{11}(u)T_{12}(t_1)\dots T_{12}(t_{k-1})T_{12}(t_k) = \prod_{i=1}^{k-1}\frac{u-t_i-1}{u-t_i}T_{12}(t_1)\dots T_{12}(t_{k-1})T_{11}(u)T_{12}(t_k)+$$
$$+\sum_{l=1}^{k-1}\frac{1}{u-t_l}\prod_{\substack{m=1\\m\neq l}}^{k-1}\frac{t_l-t_m-1}{t_l-t_m}T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_{k-1})T_{12}(u)T_{11}(t_l)T_{12}(t_k). \tag{2.3.53}$$

Then we apply (2.3.49) to the product $T_{11}(u)T_{12}(t_k)$ and $T_{11}(t_l)T_{12}(t_k)$ and the right-hand side of (2.3.53) becomes

$$\prod_{i=1}^{k}\frac{u-t_i-1}{u-t_i}T_{12}(t_1)\dots T_{12}(t_{k-1})T_{12}(t_k)T_{11}(u)+$$
$$+\frac{1}{u-t_k}\prod_{i=1}^{k-1}\frac{u-t_i-1}{u-t_i}T_{12}(t_1)\dots T_{12}(t_{k-1})T_{12}(u)T_{11}(t_k)$$
$$+\sum_{l=1}^{k-1}\frac{1}{u-t_l}\prod_{\substack{m=1\\m\neq l}}^{k}\frac{t_l-t_m-1}{t_l-t_m}T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_{k-1})T_{12}(u)T_{12}(t_k)T_{11}(t_l)$$
$$+\sum_{l=1}^{k-1}\frac{1}{u-t_l}\frac{1}{t_l-t_k}\prod_{\substack{m=1\\m\neq l}}^{k-1}\frac{t_l-t_m-1}{t_l-t_m}\Big(T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_{k-1})\times$$
$$\times T_{12}(u)T_{12}(t_l)T_{11}(t_k)\Big)$$

The first term here coincide with the first term in the right-hand side of formula (2.3.51). The third term here is the second term of (2.3.51) without $l = k$ summand. We also used that $T_{12}(u)$ and $T_{12}(t_k)$ commute, see (2.3.48). The second and forth summands combine into the product

$$\frac{1}{u-t_k}\prod_{m=1}^{k-1}\frac{t_k-t_m-1}{t_k-t_m}T_{12}(t_1)\dots T_{12}(t_{l-1})T_{12}(t_{l+1})\dots T_{12}(t_{k-1})T_{12}(u)T_{11}(t_l), \tag{2.3.54}$$

using the following identity

$$\frac{1}{u-t_k}\prod_{i=1}^{k-1}\frac{u-t_i-1}{u-t_i}+\sum_{l=1}^{k-1}\frac{1}{(u-t_l)(t_l-t_k)}\prod_{\substack{m=1\\ m\neq l}}^{k-1}\frac{t_l-t_m-1}{t_l-t_m}=\frac{1}{u-t_k}\prod_{m=1}^{k-1}\frac{t_k-t_m-1}{t_k-t_m}.$$

The product (2.3.54) is exactly the summand with $l=k$ of the second term in (2.3.51). Formula (2.3.51) is proved.

The proof of formula (2.3.52) is similar to that of formula (2.3.51) with relation (2.3.50) used instead of (2.3.49). □

Recall that for the $\mathfrak{gl}_2$ case we have

$$\mathbb{B}_\xi(\boldsymbol{t})=T_{12}(t_1)\dots T_{12}(t_\xi),$$

and thus Proposition 2.3.3 can be rewritten as follows.

Proposition 2.3.18. Let ξ be a nonnegative integer and $\boldsymbol{t}=(t_1,\dots,t_\xi)$. Then

$$\begin{aligned}&\Delta\left(T_{12}(t_1)\dots T_{12}(t_\xi)\right)=\\&=\sum_{\eta=0}^{\xi}\frac{1}{(\xi-\eta)!\eta!}\overline{\mathrm{Sym}}_{\boldsymbol{t}}\Big[\Big(\prod_{i=1}^{\eta}T_{12}(t_i)\otimes\prod_{j=\eta+1}^{\xi}T_{12}(t_j)\Big)\Big(\prod_{k=\eta+1}^{\xi}T_{22}(t_k)\otimes\prod_{l=1}^{\eta}T_{11}(t_l)\Big)\Big].\end{aligned}\tag{2.3.55}$$

Remark. Notice that according to (2.3.48), the factors in each of the large products commute among themselves, so the order of the factors is irrelevant. (2.3.55).

Proof. Consider the summand from the right-hand side of (2.3.55) with a given η,

$$F_{\eta,\xi-\eta}(\boldsymbol{t})=\overline{\mathrm{Sym}}_{\boldsymbol{t}}\Big[\Big(\prod_{i=1}^{\eta}T_{12}(t_i)\otimes\prod_{j=\eta+1}^{\xi}T_{12}(t_j)\Big)\Big(\prod_{k=\eta+1}^{\xi}T_{22}(t_k)\otimes\prod_{l=1}^{\eta}T_{11}(t_l)\Big)\Big].\tag{2.3.56}$$

Let

$$P_{\eta,\xi-\eta}(\boldsymbol{t}) = \prod_{1\le i\le\eta<j\le\xi} \frac{t_i - t_j - 1}{t_i - t_j} \Big(\prod_{i=1}^{\eta} T_{12}(t_i) \otimes \prod_{j=\eta+1}^{\xi} T_{12}(t_j)\Big)\Big(\prod_{k=\eta+1}^{\xi} T_{22}(t_k) \otimes \prod_{l=1}^{\eta} T_{11}(t_l)\Big),$$

$$U_{\eta,\xi-\eta}(\boldsymbol{t}) = \prod_{1\le i<j\le\eta} \frac{t_i - t_j - 1}{t_i - t_j} \prod_{\eta+1\le i<j\le\xi} \frac{t_i - t_j - 1}{t_i - t_j},$$

$$\boldsymbol{t}^\sigma = (t_{\sigma(1)}, \dots, t_{\sigma(\xi)}).$$

Using this notation, formula (2.3.56) can be written as

$$F_{\eta,\xi-\eta}(\boldsymbol{t}) = \sum_{\sigma\in S_\xi} U_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma).$$

Observe that $F_{\eta,\xi-\eta}(\boldsymbol{t})$ is symmetric in $t_1, \dots, t_\xi$. Denote by $S_\eta \times S_{\xi-\eta}$ the subgroup of S_ξ stabilizing the subsets $\{1, \dots, \eta\}$ and $\{\eta+1, \dots, \xi\}$. We have

$$F_{\eta,\xi-\eta}(\boldsymbol{t}) = \frac{1}{\eta!(\xi-\eta)!} \sum_{\tau\in S_\eta\times S_{\xi-\eta}} F_{\eta,\xi-\eta}(\boldsymbol{t}^\tau) = \frac{1}{\eta!(\xi-\eta)!} \sum_{\tau\in S_\eta\times S_{\xi-\eta}} \sum_{\sigma\in S_\xi} U_{\eta,\xi-\eta}(\boldsymbol{t}^{\tau\sigma}) P_{\eta,\xi-\eta}(\boldsymbol{t}^{\tau\sigma}).$$

Changing the summation variable in the inner sum, $\sigma = \tau^{-1}\rho\tau$, and using the fact that $P_{\eta,\xi-\eta}(\boldsymbol{t}^{\rho\tau}) = P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho)$ for all $\tau \in S_\eta \times S_{\xi-\eta}$, we get

$$\begin{aligned} F_{\eta,\xi-\eta}(\boldsymbol{t}) &= \frac{1}{\eta!(\xi-\eta)!} \sum_{\tau\in S_\eta\times S_{\xi-\eta}} \sum_{\rho\in S_\xi} U_{\eta,\xi-\eta}(\boldsymbol{t}^{\rho\tau}) P_{\eta,\xi-\eta}(\boldsymbol{t}^{\rho\tau}) = \\ &= \frac{1}{\eta!(\xi-\eta)!} \sum_{\rho\in S_\xi} P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho) \sum_{\tau\in S_\eta\times S_{\xi-\eta}} U_{\eta,\xi-\eta}(\boldsymbol{t}^{\rho\tau}). \end{aligned}$$

Furthermore, using the identity

$$\sum_{\tau\in S_n} \prod_{1\le i<j\le n} \frac{x_{\tau(i)} - x_{\tau(j)} - 1}{x_{\tau(i)} - x_{\tau(j)}} = n!,$$

we obtain that $\sum_{\tau\in S_\eta\times S_{\xi-\eta}} U_{\eta,\xi-\eta}(\boldsymbol{t}^{\rho\tau}) = \eta!(\xi-\eta)!$ and

$$F_{\eta,\xi-\eta}(\boldsymbol{t}) = \sum_{\rho\in S_\xi} P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho). \tag{2.3.57}$$

Using formula (2.3.57), the statement of Proposition 2.3.18 can be formulated as follows:

$$\Delta\left(T_{12}(t_1)\dots T_{12}(t_\xi)\right)=\sum_{\eta=0}^{\xi}\frac{1}{(\xi-\eta)!\eta!}\sum_{\rho\in S_\xi}P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho). \tag{2.3.58}$$

We will prove this formula using the induction on ξ. The base of induction at $\xi=1$ is given by formula (2.2.6):

$$\Delta\Big(T_{12}(t_1)\Big)=T_{12}(t_1)\otimes T_{11}(t_1)+T_{22}(t_1)\otimes T_{12}(t_1). \tag{2.3.59}$$

To make the induction step, we use that

$$\Delta\left(\mathbb{B}_\xi(\boldsymbol{t})\right)=\Delta\Big(T_{12}(t_1)\Big)\,\Delta\Big(T_{12}(t_2)\dots T_{12}(t_\xi)\Big), \tag{2.3.60}$$

expand the first factor according to (2.3.59), and apply the induction assumption to expand the second factor. Denote by $S'_{\xi-1}\subset S_\xi$ the subgroup of permutations ρ, such that $\rho(1)=1$. Then the right-hand side of formula (2.3.60) becomes

$$\begin{aligned}&T_{12}(t_1)\otimes T_{11}(t_1)\sum_{\eta=1}^{\xi}\frac{1}{(\xi-\eta)!(\eta-1)!}\sum_{\tau\in S'_{\xi-1}}P_{\eta-1,\xi-\eta}(t_{\tau(2)},\dots,t_{\tau(\xi)})+\\&+T_{22}(t_1)\otimes T_{12}(t_1)\sum_{\eta=0}^{\xi-1}\frac{1}{(\xi-1-\eta)!\eta!}\sum_{\rho\in S'_{\xi-1}}P_{\eta,\xi-\eta-1}(t_{\rho(2)},\dots,t_{\rho(\xi)}),\end{aligned} \tag{2.3.61}$$

where in the first term we shifted the summation variable of the exterior sum. Using the definition of $P_{\eta,\xi-1-\eta}(\boldsymbol{t})$ and $P_{\eta-1,\xi-\eta}(\boldsymbol{t})$, we further expand expression (2.3.61):

$$\sum_{\eta=1}^{\xi}\frac{1}{(\xi-\eta)!(\eta-1)!}\sum_{\tau\in S'_{\xi-1}}\prod_{1<i\leq\eta<j\leq\xi}\frac{t_{\tau(i)}-t_{\tau(j)}-1}{t_{\tau(i)}-t_{\tau(j)}}\times$$
$$\times\Big(T_{12}(t_1)\prod_{i=2}^{\eta}T_{12}(t_{\tau(i)})\otimes T_{11}(t_1)\prod_{j=\eta+1}^{\xi}T_{12}(t_{\tau(j)})\Big)\Big(\prod_{k=\eta+1}^{\xi}T_{22}(t_{\tau(k)})\otimes\prod_{l=2}^{\eta}T_{11}(t_{\tau(l)})\Big)+$$
$$+\sum_{\eta=0}^{\xi-1}\frac{1}{(\xi-\eta-1)!\eta!}\sum_{\rho\in S'_{\xi-1}}\prod_{1<i\leq\eta+1<j\leq\xi}\frac{t_{\rho(i)}-t_{\rho(j)}-1}{t_{\rho(i)}-t_{\rho(j)}}\times$$
$$\times\Big(T_{22}(t_1)\prod_{i=2}^{\eta+1}T_{12}(t_{\rho(i)})\otimes T_{12}(t_1)\prod_{j=\eta+2}^{\xi}T_{12}(t_{\rho(j)})\Big)\Big(\prod_{k=\eta+2}^{\xi}T_{22}(t_{\rho(k)})\otimes\prod_{l=2}^{\eta+1}T_{11}(t_{\rho(l)})\Big).$$

In the first term we move $T_{11}(t_1)$ through the product $\prod_{j=\eta+1}^{\xi}T_{12}(t_{\tau(j)})$ using formula (2.3.51):

$$T_{11}(t_1)\prod_{j=\eta+1}^{\xi}T_{12}(t_{\tau(j)})=\Big(\prod_{j=\eta+1}^{\xi}\frac{t_1-t_{\tau(j)}-1}{t_1-t_j}T_{12}(t_{\tau(j)})\Big)T_{11}(t_1)+$$
$$+\sum_{p=\eta+1}^{\xi}\frac{1}{t_1-t_{\tau(p)}}\Big(\prod_{\substack{j=\eta+1\\ j\neq p}}^{\xi}\frac{t_{\tau(p)}-t_{\tau(j)}-1}{t_{\tau(p)}-t_{\tau(j)}}T_{12}(t_{\tau(j)})\Big)T_{12}(t_1)T_{11}(t_{\tau(p)}).$$

Similarly, in the second term we move $T_{22}(t_1)$ through the product $\prod_{m=2}^{\eta}T_{12}(t_{\rho(m)})$ using formula (2.3.52):

$$T_{22}(t_1)\prod_{i=2}^{\eta+1}T_{12}(t_{\rho(i)})=\Big(\prod_{i=2}^{\eta+1}\frac{t_1-t_{\rho(i)}+1}{t_1-t_{\rho(i)}}T_{12}(t_{\rho(m)})\Big)T_{22}(t_1)-$$
$$-\sum_{s=2}^{\eta+1}\frac{1}{t_1-t_s}\Big(\prod_{\substack{i=1\\ i\neq s}}^{\eta+1}\frac{t_{\rho(s)}-t_{\rho(i)}+1}{t_{\rho(s)}-t_{\rho(i)}}T_{12}(t_{\rho(i)})\Big)T_{12}(t_1)T_{22}(t_{\rho(s)}),$$

After all, the right-hand side of (2.3.60) becomes a sum of four terms:

$$\Delta\left(\mathbb{B}_{\xi}(\boldsymbol{t})\right)=Y_1(\boldsymbol{t})+Y_2(\boldsymbol{t})+Y_3(\boldsymbol{t})+Y_4(\boldsymbol{t}),$$

where

$$Y_1(\boldsymbol{t}) = \sum_{\eta=1}^{\xi} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{\tau\in S'_{\xi-1}} \Bigg[\prod_{l=\eta+1}^{\xi} \frac{t_1 - t_{\tau(l)} - 1}{t_1 - t_{\tau(l)}} \prod_{1<i\le\eta<j\le\xi} \frac{t_{\tau(i)} - t_{\tau(j)} - 1}{t_{\tau(i)} - t_{\tau(j)}} \times$$
$$\times \Big(T_{12}(t_1) \prod_{i=2}^{\eta} T_{12}(t_{\tau(i)}) \otimes \prod_{j=\eta+1}^{\xi} T_{12}(t_{\tau(j)}) \Big) \Big(\prod_{k=\eta+1}^{\xi} T_{22}(t_{\tau(k)}) \otimes T_{11}(t_1) \prod_{l=2}^{\eta} T_{11}(t_{\tau(l)}) \Big) \Bigg],$$

$$Y_2(\boldsymbol{t}) = \sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{\tau\in S'_{\xi-1}} \sum_{l=\eta+1}^{\xi} \Bigg[\frac{1}{t_1 - t_{\tau(l)}} \prod_{1<i\le\eta<j\le\xi} \frac{t_{\tau(i)} - t_{\tau(j)} - 1}{t_{\tau(i)} - t_{\tau(j)}} \prod_{\substack{m=\eta+1\\ m\neq l}}^{\xi} \frac{t_{\tau(l)} - t_{\tau(m)} - 1}{t_{\tau(l)} - t_{\tau(m)}}$$
$$\times \Big(T_{12}(t_1) \prod_{k=2}^{\eta} T_{12}(t_{\tau(k)}) \otimes T_{12}(t_1) \prod_{\substack{m=\eta+1\\ m\neq l}}^{\xi} T_{12}(t_{\tau(m)}) \Big) \Big(\prod_{i=\eta+1}^{\xi} T_{22}(t_{\tau(i)}) \otimes T_{11}(t_{\tau(l)}) \prod_{j=2}^{\eta} T_{11}(t_{\tau(j)}) \Big) \Bigg].$$

$$Y_3(\boldsymbol{t}) = \sum_{\eta=0}^{\xi-1} \frac{1}{(\xi-\eta-1)!\eta!} \sum_{\rho\in S'_{\xi-1}} \Bigg[\prod_{m=2}^{\eta+1} \frac{t_{\rho(m)} - t_1 - 1}{t_{\rho(m)} - t_1} \prod_{1<i\le\eta+1<j\le\xi} \frac{t_{\rho(i)} - t_{\rho(j)} - 1}{t_{\rho(i)} - t_{\rho(j)}} \times$$
$$\times \Big(\prod_{m=2}^{\eta+1} T_{12}(t_{\rho(m)}) \otimes T_{12}(t_1) \prod_{l=\eta+2}^{\xi} T_{12}(t_{\rho(l)}) \Big) \Big(T_{22}(t_1) \prod_{i=\eta+2}^{\xi} T_{22}(t_{\rho(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\rho(j)}) \Big) \Bigg].$$

$$Y_4(\boldsymbol{t}) = -\sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta-1)!\eta!} \sum_{\rho\in S'_{\xi-1}} \Bigg[\sum_{k=2}^{\eta+1} \frac{1}{t_1 - t_{\rho(k)}} \prod_{1<i\le\eta+1<j\le\xi} \frac{t_{\rho(i)} - t_{\rho(j)} - 1}{t_{\rho(i)} - t_{\rho(j)}} \prod_{\substack{m=2\\ m\neq k}}^{\eta+1} \frac{t_{\rho(k)} - t_{\rho(m)} + 1}{t_{\rho(k)} - t_{\rho(m)}}$$
$$\times \Big(T_{12}(t_1) \prod_{\substack{m=2\\ m\neq k}}^{\eta+1} T_{12}(t_{\rho(m)}) \otimes T_{12}(t_1) \prod_{l=\eta+2}^{\xi} T_{12}(t_{\rho(l)}) \Big) \Big(T_{22}(t_{\rho(k)}) \prod_{i=\eta+2}^{\xi} T_{22}(t_{\rho(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\rho(j)}) \Big) \Bigg].$$

To complete the proof we will show that

$$Y_1(\boldsymbol{t}) + Y_3(\boldsymbol{t}) = \sum_{\eta=0}^{\xi} \frac{1}{(\xi-\eta)!\eta!} \sum_{\rho\in S_\xi} P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho) \qquad \text{and} \qquad Y_2(\boldsymbol{t}) + Y_4(\boldsymbol{t}) = 0. \tag{2.3.62}$$

We will start with the first equality in (2.3.62).

Observe that

$$Y_1(\boldsymbol{t}) = \sum_{\eta=1}^{\xi} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{\substack{\sigma\in S_\xi\\ \sigma(1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma),$$

and

$$Y_3(\boldsymbol{t}) = \sum_{\eta=0}^{\xi-1} \frac{1}{(\xi-\eta-1)!\eta!} \sum_{\substack{\sigma\in S_\xi \\ \sigma(\eta+1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma).$$

On the other hand, we have

$$\sum_{\eta=0}^{\xi} \frac{1}{(\xi-\eta)!\eta!} \sum_{\sigma\in S_\xi} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) =$$
$$\sum_{\eta=1}^{\xi} \frac{1}{(\xi-\eta)!\eta!} \sum_{\substack{\sigma\in S_\xi \\ \sigma^{-1}(1)\in\{1,\dots,\eta\}}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) + \sum_{\eta=0}^{\xi-1} \frac{1}{(\xi-\eta)!\eta!} \sum_{\substack{\sigma\in S_\xi \\ \sigma^{-1}(1)\in\{\eta+1,\dots,\xi\}}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma).$$

Denote by $s_{a,b} \in S_\xi$ the transposition of a and b. Then we have

$$\sum_{\substack{\sigma\in S_\xi \\ \sigma^{-1}(1)\in\{1,\dots,\eta\}}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) = \sum_{l=1}^{\eta} \sum_{\substack{\tau\in S_\xi, \\ \tau(1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^{\tau s_{1,l}}) = \eta \sum_{\substack{\tau\in S_\xi, \\ \tau(1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\tau).$$

For the first step we used $l = \sigma^{-1}(1)$ and $\tau = \sigma s_{1,\sigma^{-1}(1)}$, so that $\tau(1) = 1$. For the second step we used the equality $P_{\eta,\xi-\eta}(\boldsymbol{t}^{\tau s_{1,l}}) = P_{\eta,\xi-\eta}(\boldsymbol{t}^\tau)$.

Similarly,

$$\sum_{\substack{\sigma\in S_\xi \\ \sigma^{-1}(1)\in\{\eta+1,\dots,\xi\}}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) = (\xi-\eta) \sum_{\substack{\rho\in S_\xi, \\ \rho(\eta+1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\rho).$$

Therefore,

$$\sum_{\eta=0}^{\xi} \frac{1}{(\xi-\eta)!\eta!} \sum_{\sigma\in S_\xi} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) =$$
$$= \sum_{\eta=1}^{\xi} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{\substack{\sigma\in S_\xi \\ \sigma(1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) + \sum_{\eta=0}^{\xi-1} \frac{1}{(\xi-1-\eta)!\eta!} \sum_{\substack{\sigma\in S_\xi \\ \sigma(\eta+1)=1}} P_{\eta,\xi-\eta}(\boldsymbol{t}^\sigma) =$$
$$= Y_1(\boldsymbol{t}) + Y_3(\boldsymbol{t}).$$

Finally, we show that $Y_2(\boldsymbol{t}) + Y_4(\boldsymbol{t}) = 0$. Observe that $Y_2(\boldsymbol{t})$ can be written as

$$Y_2(\boldsymbol{t}) = \sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{l=\eta+1}^{\xi} \sum_{\tau\in S'_{\xi-1}} \Bigg[\frac{1}{t_1 - t_{\tau(l)}} \prod_{1<i\le\eta<j\le\xi} \frac{t_{\tau(i)} - t_{\tau(j)} - 1}{t_{\tau(i)} - t_{\tau(j)}} \prod_{\substack{m=\eta+1\\ m\neq l}}^{\xi} \frac{t_{\tau(l)} - t_{\tau(m)} - 1}{t_{\tau(l)} - t_{\tau(m)}}$$
$$\times \Big(T_{12}(t_1) \prod_{k=2}^{\eta} T_{12}(t_{\tau(k)}) \otimes T_{12}(t_1) \prod_{\substack{m=\eta+1\\ m\neq l}}^{\xi} T_{12}(t_{\tau(m)}) \Big) \Big(\prod_{i=\eta+1}^{\xi} T_{22}(t_{\tau(i)}) \otimes T_{11}(t_{\tau(l)}) \prod_{j=2}^{\eta} T_{11}(t_{\tau(j)}) \Big) \Bigg].$$

Changing the summation variable in the inner sum, $\tau = \sigma s_{l,\eta+1}$, we obtain that

$$Y_2(\boldsymbol{t}) = \sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta)!(\eta-1)!} \sum_{l=\eta+1}^{\xi} \sum_{\sigma\in S'_{\xi-1}} \Bigg[\frac{1}{t_1 - t_{\sigma(\eta+1)}} \times$$
$$\times \prod_{m=\eta+2}^{\xi} \frac{t_{\sigma(\eta+1)} - t_{\sigma(m)} - 1}{t_{\sigma(\eta+1)} - t_{\sigma(m)}} \prod_{1<i\le\eta<j\le\xi} \frac{t_{\sigma(i)} - t_{\sigma(j)} - 1}{t_{\sigma(i)} - t_{\sigma(j)}} \times$$
$$\times \Big(T_{12}(t_1) \prod_{k=2}^{\eta} T_{12}(t_{\sigma(k)}) \otimes T_{12}(t_1) \prod_{m=\eta+1}^{\xi} T_{12}(t_{\sigma(m)}) \Big) \Big(\prod_{i=\eta+2}^{\xi} T_{22}(t_{\sigma(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\sigma(j)}) \Big) \Bigg].$$

The expression under the inner sum over σ does not depend on l, so after a certain redistribution of factors, we get

$$Y_2(\boldsymbol{t}) = \sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta-1)!(\eta-1)!} \sum_{\sigma\in S'_{\xi-1}} \Bigg[\frac{1}{t_1 - t_{\sigma(\eta+1)}} \times$$
$$\times \prod_{m=\eta+2}^{\xi} \frac{t_{\sigma(\eta+1)} - t_{\sigma(m)} - 1}{t_{\sigma(\eta+1)} - t_{\sigma(m)}} \prod_{i=2}^{\eta} \frac{t_{\sigma(i)} - t_{\sigma(\eta+1)} - 1}{t_{\sigma(i)} - t_{\sigma(\eta+1)}} \prod_{1<i<\eta+1<j\le\xi} \frac{t_{\sigma(i)} - t_{\sigma(j)} - 1}{t_{\sigma(i)} - t_{\sigma(j)}} \times$$
$$\times \Big(T_{12}(t_1) \prod_{k=2}^{\eta} T_{12}(t_{\sigma(k)}) \otimes T_{12}(t_1) \prod_{m=\eta+1}^{\xi} T_{12}(t_{\sigma(m)}) \Big) \Big(\prod_{i=\eta+2}^{\xi} T_{22}(t_{\sigma(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\sigma(j)}) \Big) \Bigg]. \tag{2.3.63}$$

In a similar way $Y_4(\boldsymbol{t})$ can be written as

$$Y_4(\boldsymbol{t}) = -\sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-1-\eta)!\eta!} \sum_{k=2}^{\eta+1} \sum_{\rho\in S'_{\xi-1}} \Bigg[\frac{1}{t_1 - t_{\rho(k)}} \prod_{1<i\le\eta+1<j\le\xi} \frac{t_{\rho(i)} - t_{\rho(j)} - 1}{t_{\rho(i)} - t_{\rho(j)}} \prod_{\substack{m=2\\ m\neq k}}^{\eta+1} \frac{t_{\rho(k)} - t_{\rho(m)} + 1}{t_{\rho(k)} - t_{\rho(m)}}$$
$$\times \Big(T_{12}(t_1) \prod_{\substack{m=2\\ m\neq k}}^{\eta+1} T_{12}(t_{\rho(m)}) \otimes T_{12}(t_1) \prod_{l=\eta+2}^{\xi} T_{12}(t_{\rho(l)}) \Big) \Big(T_{22}(t_{\rho(k)}) \prod_{i=\eta+2}^{\xi} T_{22}(t_{\rho(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\rho(j)}) \Big) \Bigg],$$

and changing the summation variable in the inner sum, $\rho = \sigma s_{k,\eta+1}$, we get

$$Y_4(\boldsymbol{t}) = -\sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-1-\eta)!\eta!} \sum_{k=2}^{\eta+1} \sum_{\sigma \in S'_{\xi-1}} \Big[\frac{1}{t_1 - t_{\sigma(\eta+1)}} \times$$
$$\times \prod_{m=2}^{\eta} \frac{t_{\sigma(\eta+1)} - t_{\sigma(m)} + 1}{t_{\sigma(\eta+1)} - t_{\sigma(m)}} \prod_{1<i\le\eta+1<j\le\xi} \frac{t_{\sigma(i)} - t_{\sigma(j)} - 1}{t_{\sigma(i)} - t_{\sigma(j)}} \times$$
$$\times \Big(T_{12}(t_1) \prod_{m=2}^{\eta} T_{12}(t_{\sigma(m)}) \otimes T_{12}(t_1) \prod_{l=\eta+2}^{\xi} T_{12}(t_{\sigma(l)}) \Big) \Big(\prod_{i=\eta+1}^{\xi} T_{22}(t_{\sigma(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\sigma(j)}) \Big) \Big].$$

The expression under the inner sum over σ does not depend on k, so after a certain redistribution of factors we get

$$Y_4(\boldsymbol{t}) = -\sum_{\eta=1}^{\xi-1} \frac{1}{(\xi-\eta-1)!(\eta-1)!} \sum_{\sigma \in S'_{\xi-1}} \Big[\frac{1}{t_1 - t_{\sigma(\eta+1)}} \times$$
$$\times \prod_{m=2}^{\eta} \frac{t_{\sigma(\eta+1)} - t_{\sigma(m)} + 1}{t_{\sigma(\eta+1)} - t_{\sigma(m)}} \prod_{q=\eta+2}^{\xi} \frac{t_{\sigma(\eta+1)} - t_{\sigma(q)} - 1}{t_{\sigma(\eta+1)} - t_{\sigma(q)}} \prod_{1<i<\eta+1<j\le\xi} \frac{t_{\sigma(i)} - t_{\sigma(j)} - 1}{t_{\sigma(i)} - t_{\sigma(j)}} \times$$
$$\times \Big(T_{12}(t_1) \prod_{m=2}^{\eta} T_{12}(t_{\sigma(m)}) \otimes T_{12}(t_1) \prod_{l=\eta+2}^{\xi} T_{12}(t_{\sigma(l)}) \Big) \Big(\prod_{i=\eta+1}^{\xi} T_{22}(t_{\sigma(i)}) \otimes \prod_{j=2}^{\eta+1} T_{11}(t_{\sigma(j)}) \Big) \Big]. \tag{2.3.64}$$

Formulas (2.3.63) and (2.3.64) show that $Y_2(\boldsymbol{t}) + Y_4(\boldsymbol{t}) = 0$. This complete the proof of formula (2.3.58). Proposition 2.3.18 is proved. □

2.4 Combinatorial formulae for the $\mathfrak{gl}_n$ case

In this section we will generalize the result of the previous section and obtain combinatorial formulas for the weight functions in the $\mathfrak{gl}_n$ case. We start with the generalization of Proposition 2.3.1.

2.4.1 Splitting property

Let $T_{ab}^{\langle r\rangle}(u)$ be series (2.2.4) for the algebra $Y(\mathfrak{gl}_r)$, and $R^{\langle r\rangle}(u)$ be the corresponding rational R-matrix, see (2.2.1). For the rest of the section we fix integers m and n, $1 \le m < n$.

Consider $Y(\mathfrak{gl}_{n-m})$-module structure on the vector space $\mathbb{C}^{n-m}$ given by the rule

$$\pi(x) : T^{\langle n-m\rangle}(u) \mapsto (u-x)^{-1} R^{\langle n-m\rangle}(u-x). \tag{2.4.1}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-m}$ be the standard basis of the space $\mathbb{C}^{n-m}$. The $Y(\mathfrak{gl}_{n-m})$-module defined by (2.4.1) is a highest weight evaluation module with $\mathfrak{gl}_{n-m}$ highest weight $(1, \ldots, 0, 0)$ and highest weight vector $\mathbf{v}_1$.

Consider $Y(\mathfrak{gl}_m)$-module structure on the vector space $\mathbb{C}^m$ given by the rule

$$\varpi(x) : T^{\langle m\rangle}(u) \mapsto (x-u)^{-1} \Big(\Big(R^{\langle m\rangle}(x-u) \Big)^{(21)} \Big)^{t_2}, \tag{2.4.2}$$

where the superscript t_2 stands for the matrix transposition in the second tensor factor.

We denote the standard basis of the space $\mathbb{C}^m$ by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$. The $Y(\mathfrak{gl}_m)$-module defined by (2.4.2) is a highest weight evaluation module with $\mathfrak{gl}_m$ highest weight $(0, \ldots, 0, -1)$ and highest weight vector $\mathbf{w}_m$.

For any $Z \in \mathrm{End}(\mathbb{C}^{n-m})$, set $\nu(Z) = Z\mathbf{v}_1$, and for any $X \in \mathrm{End}(\mathbb{C}^m)$, set $\bar\nu(X) = X\mathbf{w}_m$. Recall the coproducts Δ and $\widetilde{\Delta}$, see (2.2.6) and (2.2.7), and the embeddings $\psi_{n-m} : Y(\mathfrak{gl}_{n-m}) \to Y(\mathfrak{gl}_n)$, $\phi_m : Y(\mathfrak{gl}_m) \to Y(\mathfrak{gl}_n)$ given by (2.2.11). For any r, denote by $(\Delta^{\langle r\rangle})^{(k)} : Y(\mathfrak{gl}_r) \to (Y(\mathfrak{gl}_r))^{\otimes(k+1)}$ and $(\widetilde{\Delta}^{\langle r\rangle})^{(k)} : Y(\mathfrak{gl}_r) \to (Y(\mathfrak{gl}_r))^{\otimes(k+1)}$ the corresponding iterated coproduct and opposite coproduct. Consider the maps

$$\psi_{n-m}(x_1, \ldots, x_k) : Y(\mathfrak{gl}_{n-m}) \to \Big(\mathbb{C}^{n-m}\Big)^{\otimes k} \otimes Y(\mathfrak{gl}_n),$$

$$\psi_{n-m}(x_1,\dots,x_k) = (\nu^{\otimes k}\otimes \mathrm{id})\circ(\pi(x_1)\otimes\dots\otimes\pi(x_k)\otimes\psi_{n-m})\circ(\Delta^{\langle n-m\rangle})^{(k)},$$

and

$$\phi_m(x_1,\dots,x_k) : Y(\mathfrak{gl}_m)\to(\mathbb{C}^m)^{\otimes k}\otimes Y(\mathfrak{gl}_n),$$

$$\phi_m(x_1,\dots,x_k) = (\bar\nu^{\otimes k}\otimes \mathrm{id})\circ(\varpi(x_1)\otimes\dots\otimes\varpi(x_k)\otimes\phi_m)\circ(\widetilde{\Delta}^{\langle m\rangle})^{(k)}.$$

For any element $g\in(\mathbb{C}^{n-m})^{\otimes k}\otimes Y(\mathfrak{gl}_n)$, we define its components $g^{\boldsymbol{b}}$, $\boldsymbol{b}=(b_1,\dots,b_k)$, by the rule

$$g=\sum_{b_1,\dots,b_k=m+1}^{n}\mathbf{v}_{b_1-m}\otimes\dots\otimes\mathbf{v}_{b_k-m}\otimes g^{\boldsymbol{b}}.$$

For any element $h\in(\mathbb{C}^m)^{\otimes k}\otimes Y(\mathfrak{gl}_n)$, we define its components $h^{\boldsymbol{a}}$, $\boldsymbol{a}=(a_1,\dots,a_k)$, by the rule

$$h=\sum_{a_1,\dots,a_k=1}^{m}\mathbf{w}_{a_1}\otimes\dots\otimes\mathbf{w}_{a_k}\otimes h^{\boldsymbol{a}}.$$

Given nonnegative integers $\xi_1,\dots,\xi_{n-1}$, and the variables $\boldsymbol{t}=(t^1_1,\dots,t^{n-1}_{\xi_{n-1}})$, see (2.2.8), introduce

$$\begin{aligned}
\boldsymbol{\xi} &= (\xi_1,\dots,\xi_{n-1}),\\
\dot{\boldsymbol{\xi}} &= (\xi_1,\dots,\xi_{m-1}),\\
\ddot{\boldsymbol{\xi}} &= (\xi_{m+1},\dots,\xi_{n-1}),\\
\boldsymbol{t} &= (t^1_1,\dots,t^1_{\xi_1};t^2_1,\dots,t^2_{\xi_2};t^{n-1}_1,\dots,t^{n-1}_{\xi_{n-1}}),\\
\dot{\boldsymbol{t}} &= (t^1_1,\dots,t^1_{\xi_1};t^2_1,\dots,t^2_{\xi_2};t^{m-1}_1,\dots,t^{m-1}_{\xi_{m-1}}),\\
\ddot{\boldsymbol{t}} &= (t^{m+1}_1,\dots,t^{m+1}_{\xi_{m+1}};t^{m+2}_1,\dots,t^{m+2}_{\xi_{m+2}};t^{n-1}_1,\dots,t^{n-1}_{\xi_{n-1}}).
\end{aligned}\tag{2.4.3}$$

Recall that $\xi^a=\xi_1+\xi_2+\ldots+\xi_a$, $a=1,\dots,n-1$.

Let V be a $\mathfrak{gl}_n$-module, $x\in\mathbb{C}$, and $V(x)$ be the evaluation $Y(\mathfrak{gl}_n)$-module. Consider a weight singular vector $v\in V(x)$ of $\mathfrak{gl}_n$-weight $\boldsymbol{\Lambda}=(\Lambda_1,\dots,\Lambda_n)$. That is, we have

$$T^a_b(u)v=0,\ 1\le b<a\le n,\quad T^a_a v=\frac{u-x+\Lambda_a}{u-x}v,\ a=1,\dots,n.$$

We are interested in finding a formula for the vector $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v$, where $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})$ is given by (2.2.10).

Proposition 2.4.1. Let $v \in V(x)$ be a weight singular vector and $\xi_1, \dots, \xi_{n-1}$ be nonnegative integers. Then one has

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b}} \mathcal{T}(\boldsymbol{t}^m)^{\boldsymbol{a}}_{\boldsymbol{b}} \left(\phi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle m\rangle}_{\dot{\xi}}(\dot{\boldsymbol{t}})\right)\right)^{\boldsymbol{a}} \left(\psi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\xi}}(\ddot{\boldsymbol{t}})\right)\right)^{\boldsymbol{b}} v, \tag{2.4.4}$$

where the sum is taken over all sequences $\boldsymbol{a} = (a_1, a_2, \dots, a_{\xi_m})$, $\boldsymbol{b} = (b_1, b_2, \dots, b_{\xi_m})$, such that $a_i \in \{1, 2, \dots, m\}$, $b_i \in \{m+1, m+2, \dots, n\}$ for all $i = 1, \dots, \xi_m$, and

$$\mathcal{T}(\boldsymbol{t}^m)^{\boldsymbol{a}}_{\boldsymbol{b}} = T(t^m_1)^{a_1}_{b_1} T(t^m_2)^{a_2}_{b_2} \dots T(t^m_{\xi_m})^{a_{\xi_m}}_{b_{\xi_m}}\,. \tag{2.4.5}$$

Proof. Using the definition of the maps $\psi_m(\boldsymbol{t}^m)$ and $\phi_m(\boldsymbol{t}^m)$, formula (2.4.4) can be written as

$$\begin{aligned} \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b}} \mathcal{T}(\boldsymbol{t}^m)^{\boldsymbol{a}}_{\boldsymbol{b}} & \left(\left(\prod^{\rightarrow}_{1<j\le m} \prod^{\rightarrow}_{1\le i<j} \overset{[j\,i]}{\mathbb{R}}(\boldsymbol{t}^j, \boldsymbol{t}^i)\right) \overset{[m-1]}{\mathbb{T}}(\boldsymbol{t}^{m-1}) \dots \overset{[1]}{\mathbb{T}}(\boldsymbol{t}^1)\right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_2(\boldsymbol{a})} \times \\ \times \left(\overset{[m+1]}{\mathbb{T}}(\boldsymbol{t}^{m+1}) \dots \overset{[n-1]}{\mathbb{T}}(\boldsymbol{t}^{n-1}) \left(\prod^{\rightarrow}_{m+1\le j\le n-1} \prod^{\rightarrow}_{m\le i<j} \overset{[j\,i]}{\mathbb{R}}(\boldsymbol{t}^j, \boldsymbol{t}^i)\right)\right)^{\boldsymbol{\ell}_2(\boldsymbol{b})}_{\boldsymbol{\ell}_3} & v, \end{aligned} \tag{2.4.6}$$

where the sequences $\boldsymbol{a}, \boldsymbol{b}$ are as in (2.4.4) and

$$\begin{aligned} \boldsymbol{\ell}_1 &= (\mathbf{1}^{\xi_1}, \dots, (\boldsymbol{m}-\mathbf{1})^{\xi_{m-1}}, \boldsymbol{m}^{\xi_m}, (\boldsymbol{m}+\mathbf{1})^{\xi_m+1}, \dots, (\boldsymbol{n}-\mathbf{1})^{\xi_n}), \\ \boldsymbol{\ell}_2(\boldsymbol{a}) &= (\mathbf{2}^{\xi_1}, \dots, \boldsymbol{m}^{\xi_{m-1}}, \boldsymbol{a}, (\boldsymbol{m}+\mathbf{1})^{\xi_m+1}, \dots, (\boldsymbol{n}-\mathbf{1})^{\xi_n}), \\ \boldsymbol{\ell}_3 &= (\mathbf{2}^{\xi_1}, \dots, \boldsymbol{m}^{\xi_{m-1}}, (\boldsymbol{m}+\mathbf{1})^{\xi_m}, (\boldsymbol{m}+\mathbf{2})^{\xi_m+1}, \dots, \boldsymbol{n}^{\xi_n}). \end{aligned}$$

To prove formula (2.4.6), we take the definition of $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v$ following from formulas (2.2.9), (2.2.10), and observe that using the Yang-Baxter equation, one can express $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v$ as follows:

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \left(\left(\prod^{\rightarrow}_{1\le i<j\le m} \overset{[j\,i]}{\mathbb{R}}\right) \overset{[m]}{\mathbb{T}}\,\overset{[m-1]}{\mathbb{T}} \dots \overset{[1]}{\mathbb{T}}\,\overset{[m+1]}{\mathbb{T}} \dots \overset{[n-1]}{\mathbb{T}} \left(\prod^{\rightarrow}_{m+1\le j\le n-1} \prod^{\rightarrow}_{1\le i<j} \overset{[j\,i]}{\mathbb{R}}\right)\right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_3} v.$$

In detail, that gives

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \mathcal{T}(\boldsymbol{t}^m)^{\boldsymbol{a}}_{\boldsymbol{b}} \left(\prod^{\rightarrow}_{1<j\leq m} \prod^{\rightarrow}_{1\leq i<j} \overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i) \right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_4(\boldsymbol{y},\boldsymbol{a})} \mathcal{T}(\boldsymbol{t}^{m-1})^{\boldsymbol{y}^{(m-1)}}_{\boldsymbol{x}^{(m-1)}} \dots \mathcal{T}(\boldsymbol{t}^1)^{\boldsymbol{y}^{(1)}}_{\boldsymbol{x}^{(1)}} \times$$
$$\times \mathcal{T}(\boldsymbol{t}^{m+1})^{\bullet}_{\boldsymbol{z}^{(m+1)}} \dots \mathcal{T}(\boldsymbol{t}^{n-1})^{\bullet}_{\boldsymbol{z}^{(n-1)}} \left(\prod^{\rightarrow}_{m+1\leq j\leq n-1} \prod^{\rightarrow}_{1\leq i<j} \overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i) \right)^{\boldsymbol{\ell}_5(\boldsymbol{x},\boldsymbol{b},\boldsymbol{z})}_{\boldsymbol{\ell}_3} v, \tag{2.4.7}$$

where

$$\boldsymbol{x} = (\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m-1)}), \qquad \boldsymbol{y} = (\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(m-1)}), \qquad \boldsymbol{z} = (\boldsymbol{z}^{(m+1)},\dots,\boldsymbol{z}^{(n-1)}),$$
$$\boldsymbol{x}^{(l)} = (x^l_1,\dots,x^l_{\xi_l}), \qquad \boldsymbol{y}^{(l)} = (y^l_1,\dots,y^l_{\xi_l}), \qquad \boldsymbol{z}^{(l)} = (z^l_1,\dots,z^l_{\xi_l}),$$

$$\mathcal{T}(\boldsymbol{t}^s)^{\boldsymbol{y}^{(s)}}_{\boldsymbol{x}^{(s)}} = T(t^s_1)^{y^s_1}_{x^s_1} T(t^s_2)^{y^s_2}_{x^s_2} \dots T(t^s_{\xi_s})^{y^s_{\xi_s}}_{x^s_{\xi_s}}, \qquad \mathcal{T}(\boldsymbol{t}^s)^{\bullet}_{\boldsymbol{z}^{(s)}} = T(t^s_1)^{s+1}_{z^s_1} T(t^s_2)^{s+1}_{z^s_2} \dots T(t^s_{\xi_s})^{s+1}_{z^s_{\xi_s}},$$

$$\boldsymbol{\ell}_4(\boldsymbol{y},\boldsymbol{a}) = \left(\boldsymbol{y}^{(1)},\dots,\boldsymbol{y}^{(m-1)},\boldsymbol{a},(\boldsymbol{m+1})^{\xi_{m+1}},\dots,(\boldsymbol{n-1})^{\xi_n}\right),$$
$$\boldsymbol{\ell}_5(\boldsymbol{x},\boldsymbol{b},\boldsymbol{z}) = \left(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m-1)},\boldsymbol{b},\boldsymbol{z}^{(m+1)},\dots,\boldsymbol{z}^{(n-1)}\right),$$

and the sum is taken over sequences $\boldsymbol{a},\boldsymbol{b},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$ with entries belonging to $\{1,\dots,n\}$.

The next step is to show that for every $j = m+1,\dots,n-1$, the product $\prod^{\rightarrow}_{1\leq i<j} \overset{[j\ i]}{\mathbb{R}}$ in the second line of formula (2.4.7) can be truncated to $\prod^{\rightarrow}_{m\leq i<j} \overset{[j\ i]}{\mathbb{R}}$ and the sum over $\boldsymbol{x} = (\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m-1)})$ reduces to a single term with $\boldsymbol{x} = (\boldsymbol{2}^{\xi_1},\dots,\boldsymbol{m}^{\xi_{m-1}})$. This can be done by induction on j. The key idea is to combine two observations. First, since v is a weight singular vector, we have $z^s_k \geq s$ for all $s = m+1,\dots,n-1$, and $k = 1,\dots,\xi_s$. And second, the entries of R-matrix (2.2.1) have the property $R^{a\,b}_{c\,d} = \delta_{ac}\delta_{bd}$ if $b > c$, see formula (2.2.2). We have worked out this reasoning in detail for the $\mathfrak{gl}_4$-case in previous section.

After the last step, formula (2.4.7) becomes

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}} \mathcal{T}(\boldsymbol{t}^m)^{\boldsymbol{a}}_{\boldsymbol{b}} \left(\prod^{\rightarrow}_{1<j\leq m} \prod^{\rightarrow}_{1\leq i<j} \overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i) \right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_4(\boldsymbol{y},\boldsymbol{a})} \mathcal{T}(\boldsymbol{t}^{m-1})^{\boldsymbol{y}^{(m-1)}}_{\boldsymbol{2}^{\xi_{m-1}}} \dots \mathcal{T}(\boldsymbol{t}^1)^{\boldsymbol{y}^{(1)}}_{\boldsymbol{2}^{\xi_1}} \times$$
$$\times \mathcal{T}(\boldsymbol{t}^{m+1})^{\bullet}_{\boldsymbol{z}^{(m+1)}} \dots \mathcal{T}(\boldsymbol{t}^{n-1})^{\bullet}_{\boldsymbol{z}^{(n-1)}} \left(\prod^{\rightarrow}_{m+1\leq j\leq n-1} \prod^{\rightarrow}_{1\leq i<j} \overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i) \right)^{\boldsymbol{\ell}_6(\boldsymbol{b},\boldsymbol{z})}_{\boldsymbol{\ell}_3} v, \tag{2.4.8}$$

where

$$\boldsymbol{\ell}_6(\boldsymbol{b},\boldsymbol{z}) = \left(\mathbf{2}^{\xi_1},\ldots,\boldsymbol{m}^{\xi_{m-1}},\boldsymbol{b},\boldsymbol{z}^{(m+1)},\ldots,\boldsymbol{z}^{(n-1)}\right).$$

Formula (2.2.2) for entries of the R-matrix implies that the entry $\Big(\prod\limits^{\rightarrow}_{1<j\le m}\prod\limits^{\rightarrow}_{1\le i<j}\overset{[j\ i]}{\mathbb{R}}\Big)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_4(\boldsymbol{y},\boldsymbol{a})}$ equals zero unless $a_i\in\{1,2,\ldots,m\}$ for all $i=1,\ldots,\xi_m$. Similarly, we have that the entry $\Big(\prod\limits^{\rightarrow}_{m+1\le j\le n-1}\prod\limits^{\rightarrow}_{1\le i<j}\overset{[j\ i]}{\mathbb{R}}\Big)^{\boldsymbol{\ell}_6(\boldsymbol{b},\boldsymbol{z})}_{\boldsymbol{\ell}_3}$ equals zero unless $b_i\in\{m+1,m+2,\ldots,n\}$ for all $i=1,\ldots,\xi_m$. Therefore, the sum over $\boldsymbol{a},\boldsymbol{b}$ in formula (2.4.8) reduces to the same range of summation variables as in formula (2.4.6). Furthermore, the sums over $\boldsymbol{y}$ and $\boldsymbol{z}$ in (2.4.8) can be evaluated,

$$\sum_{\boldsymbol{y}}\left(\prod^{\rightarrow}_{1<j\le m}\prod^{\rightarrow}_{1\le i<j}\overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i)\right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_4(\boldsymbol{y},\boldsymbol{a})}\mathcal{T}(\boldsymbol{t}^{m-1})^{\boldsymbol{y}^{(m-1)}}_{\boldsymbol{m}^{\xi_{m-1}}}\ldots\mathcal{T}(\boldsymbol{t}^1)^{\boldsymbol{y}^{(1)}}_{\mathbf{2}^{\xi_1}} =$$
$$\left(\left(\prod^{\rightarrow}_{1<j\le m}\prod^{\rightarrow}_{1\le i<j}\overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i)\right)\overset{[m-1]}{\mathbb{T}}(\boldsymbol{t}^{m-1})\ldots\overset{[1]}{\mathbb{T}}(\boldsymbol{t}^1)\right)^{\boldsymbol{\ell}_1}_{\boldsymbol{\ell}_2(\boldsymbol{a})}$$

and

$$\sum_{\boldsymbol{z}}\mathcal{T}(\boldsymbol{t}^{m+1})^{\bullet}_{\boldsymbol{z}^{(m+1)}}\ldots\mathcal{T}(\boldsymbol{t}^{n-1})^{\bullet}_{\boldsymbol{z}^{(n-1)}}\left(\prod^{\rightarrow}_{m+1\le j\le n-1}\prod^{\rightarrow}_{1\le i<j}\overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i)\right)^{\boldsymbol{\ell}_6(\boldsymbol{b},\boldsymbol{z})}_{\boldsymbol{\ell}_3} =$$
$$\left(\overset{[m+1]}{\mathbb{T}}(\boldsymbol{t}^{m+1})\ldots\overset{[n-1]}{\mathbb{T}}(\boldsymbol{t}^{n-1})\left(\prod^{\rightarrow}_{m+1\le j\le n-1}\prod^{\rightarrow}_{m\le i<j}\overset{[j\ i]}{\mathbb{R}}(\boldsymbol{t}^j,\boldsymbol{t}^i)\right)\right)^{\boldsymbol{\ell}_2(\boldsymbol{b})}_{\boldsymbol{\ell}_3}.$$

Thus we transformed formula (2.4.8) to formula (2.4.6). Proposition 2.4.1 is proved. □

Remark. The statement of Proposition 2.4.1 holds for any singular vector v in any $Y(\mathfrak{gl}_n)$-module.

Remark. For $\xi_m=0$, Proposition 2.4.1 takes the form

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \phi_m\left(\mathbb{B}^{\langle m\rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right)\ \psi_m\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\right)v. \tag{2.4.9}$$

Notice that the transformation of formula (2.4.8) to formula (2.4.6) in the case $\xi_m=0$ remains nontrivial.

2.4.2 Weight functions

Fix a positive integer M. Consider a collection of nonnegative integers $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ such that $\sum_{k=1}^m \lambda_k = M$. Set $\lambda^s = \sum_{k=1}^s \lambda_k, \quad s = 1, \dots, m$. Introduce the variables $u_1^s, \dots, u_{\lambda^s}^s,\ s = 1, \dots, m$. Denote $\boldsymbol{u}^s = (u_1^s, \dots, u_{\lambda^s}^s)$ and $\boldsymbol{u} = (\boldsymbol{u}^1, \dots, \boldsymbol{u}^{m-1})$. Let $\boldsymbol{J} = (J_1, \dots, J_m)$ be a partition of $\{1, \dots, M\}$ into disjoint subsets $J_1, \dots, J_m$ such that $|J_k| = \lambda_k$, $k = 1, \dots, m$.

Similarly, consider a collection of nonnegative integers $\boldsymbol{\mu} = (\mu_{m+1}, \dots, \mu_n)$, such that $\sum_{l=m+1}^n \mu_l = M$. Set $\mu^r = \sum_{k=r+1}^n \mu_k, \quad r = m, \dots, n-1$. Introduce the variables $v_1^r, \dots, v_{\mu^r}^r,\ r = m, \dots, n-1$. Denote $\boldsymbol{v}^r = \left(v_1^r, \dots, v_{\mu^r}^r\right)$ and $\boldsymbol{v} = (\boldsymbol{v}^{m+1}, \dots, \boldsymbol{v}^{n-1})$. Let $\boldsymbol{I} = (I_{m+1}, \dots, I_n)$ be a partition of $\{1, \dots, M\}$ into disjoint subsets $I_{m+1}, \dots, I_n$ such that $|I_j| = \mu_j$, $j = m+1, \dots, n$.

Given partitions $\boldsymbol{I}, \boldsymbol{J}$ as above, we define rational functions $U_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{v}^m)$ and $\widetilde{U}_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{u}^m)$ that will be extensively used later in this paper. Set

$$U_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{v}^m) = \\ = \prod_{l=m+1}^{n-1} \prod_{a=1}^{\mu^l} \left(\prod_{\substack{c=1 \\ i_c^{(l-1)} = i_a^{(l)}}}^{\mu^{l-1}} \left(\frac{1}{v_a^l - v_c^{l-1}} \right) \prod_{\substack{d=1 \\ i_d^{(l-1)} > i_a^{(l)}}}^{\mu^{l-1}} \left(\frac{v_a^l - v_d^{l-1} + 1}{v_a^l - v_d^{l-1}} \right) \prod_{b=a+1}^{\mu^l} \frac{v_b^l - v_a^l + 1}{v_b^l - v_a^l} \right), \tag{2.4.10}$$

where the numbers $i_c^{(l)}$ are defined as follows

$$\bigcup_{k=l+1}^{n} I_k = \{i_1^{(l)} < \ldots < i_{\mu^l}^{(l)}\}.$$

Similarly, set

$$\widetilde{U}_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{u}^m) = \\ = \prod_{l=2}^{m} \prod_{a=1}^{\lambda^l} \left(\prod_{\substack{c=1 \\ j_c^{(l-1)} < j_a^{(l)}}}^{\lambda^{l-1}} \left(\frac{u_a^l - u_c^{l-1} + 1}{u_a^l - u_c^{l-1}} \right) \prod_{\substack{d=1 \\ j_d^{(l-1)} = j_a^{(l)}}}^{\lambda^{l-1}} \left(\frac{1}{u_a^l - u_d^{l-1}} \right) \prod_{b=a+1}^{\lambda^l} \frac{u_a^l - u_b^l + 1}{u_a^l - u_b^l} \right), \tag{2.4.11}$$

where the numbers $j_c^{(l)}$ are defined as follows

$$\bigcup_{k=1}^{l} J_k = \{j_1^{(l)} < \ldots < j_{\lambda^l}^{(l)}\}.$$

For a function $f(x_1, \ldots, x_k)$ of some variables, denote

$$\mathrm{Sym}_{x_1,\ldots,x_k} f(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} f\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right).$$

Introduce the weight functions $W_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{v}^m)$ and $\widetilde{W}_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{u}^m)$

$$W_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{v}^m) = \mathrm{Sym}_{v_1^{m+1},\ldots,v_{\mu^{m+1}}^{m+1}} \ldots \mathrm{Sym}_{v_1^{n-1},\ldots,v_{\mu^{n-1}}^{n-1}} U_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{v}^m), \tag{2.4.12}$$

$$\widetilde{W}_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{u}^m) = \mathrm{Sym}_{u_1^1,\ldots,u_{\lambda^1}^1} \ldots \mathrm{Sym}_{u_1^{m-1},\ldots,u_{\lambda^{m-1}}^{m-1}} U_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{u}^m), \tag{2.4.13}$$

For $\sigma \in S_M$ and $\boldsymbol{J} = (J_1, \ldots, J_m)$, set $\sigma(\boldsymbol{J}) = (\sigma(J_1), \ldots, \sigma(J_m))$. Similarly, for $\boldsymbol{I} = (I_m, \ldots, I_n)$, $\sigma(\boldsymbol{I}) = (\sigma(I_m), \ldots, \sigma(I_n))$. For $a, b = 1, \ldots, n$, let $s_{a,b}$ be the transposition of a, b.

Here is the main property of the weight functions, see for instance [80].

Lemma 2.4.2. Let $\boldsymbol{z} = (z_1, \ldots, z_{\xi_m})$. Then one has

$$W_{\boldsymbol{I}}(\boldsymbol{v}; z_1, \ldots, z_{a+1}, z_a, \ldots, z_{\xi_m}) = \\ = \frac{z_{a+1} - z_a}{z_{a+1} - z_a - 1} W_{s_{a,a+1}(\boldsymbol{I})}(\boldsymbol{v}; \boldsymbol{z}) - \frac{1}{z_{a+1} - z_a - 1} W_{\boldsymbol{I}}(\boldsymbol{v}; \boldsymbol{z}),$$

$$\widetilde{W}_{\boldsymbol{J}}(\boldsymbol{u}; z_1, \ldots, z_{a+1}, z_a, \ldots, z_{\xi_m}) = \\ = \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} \widetilde{W}_{s_{a,a+1}(\boldsymbol{J})}(\boldsymbol{u}; \boldsymbol{z}) - \frac{1}{z_a - z_{a+1} - 1} \widetilde{W}_{\boldsymbol{J}}(\boldsymbol{u}; \boldsymbol{z}).$$

This property provides us with the tool for the proof of the main result of this paper, Theorem 2.4.7.

2.4.3 Main theorem for the $\mathfrak{gl}_n$ case

The main result of this section is Theorem 2.4.7 formulated at the end of this section. It will be approached in several steps. We use the notation given in (2.4.3).

Definition 2.4.1. For a collection $\boldsymbol{a} = (a_1, \dots, a_{\xi_m})$ such that $a_i \in \{1, 2, \dots, m\}$ for all $i = 1, \dots, \xi_m$, we define a partition $\boldsymbol{J}(\boldsymbol{a}) = (J_1, \dots, J_m)$ of $\{1, \dots, \xi_m\}$ by the rule $J_l = \{j | a_j = l\}$. Denote $| \cup_{r=1}^{s} J_r| = \zeta_s$, so that we have $\xi_m = \zeta_m \geq \zeta_{m-1} \geq \dots \geq \zeta_1 \geq 0$. Denote $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{m-1})$.

For a collection $\boldsymbol{b} = (b_1, \dots, b_{\xi_m})$ such that $b_i \in \{m+1, m+2, \dots, n\}$ for all $i = 1, \dots, \xi_m$, we define a partition $\boldsymbol{I}(\boldsymbol{b}) = (I_{m+1}, \dots, I_n)$ of $\{1, \dots, \xi_m\}$ by the rule $I_l = \{i | b_i = l\}$. Denote $| \cup_{r=s+1}^{n} I_r| = \eta_s$, so that we have $\xi_m = \eta_m \geq \eta_{m+1} \geq \dots \geq \eta_{n-1} \geq 0$. Denote $\boldsymbol{\eta} = (\eta_{m+1}, \dots, \eta_{n-1})$.

Lemma 2.4.3. The correspondence $\boldsymbol{a} \mapsto \boldsymbol{J}(\boldsymbol{a})$ and $\boldsymbol{b} \mapsto \boldsymbol{I}(\boldsymbol{b})$ are bijections.

Proof. By inspection. □

Consider $\boldsymbol{\eta} \in \mathbb{Z}_{\geq 0}^{n-m-1}$, $\boldsymbol{\zeta} \in \mathbb{Z}_{\geq 0}^{m-1}$, such that $\eta_i \leq \xi_i$ and $\zeta_j \leq \xi_j$ for all relevant i, j. Set

$$
\begin{aligned}
\boldsymbol{t}^m &= (t_1^m, \dots, t_{\xi_m}^m) \\
\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]} &= \left(t_1^{m+1}, \dots, t_{\eta_{m+1}}^{m+1}; \dots; t_1^{n-1}, \dots, t_{\eta_{n-1}}^{n-1}\right), \\
\ddot{\boldsymbol{t}}_{(\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}]} &= \left(t_{\eta_{m+1}+1}^{m+1}, \dots, t_{\xi_{m+1}}^{m+1}; \dots; t_{\eta_{n-1}+1}^{n-1}, \dots, t_{\xi_{n-1}}^{n-1}\right), \\
\dot{\boldsymbol{t}}_{[\dot{\boldsymbol{\xi}} - \boldsymbol{\zeta}]} &= \left(t_1^1, \dots, t_{\xi_1 - \zeta_1}^1; \dots; t_1^{m-1}, \dots, t_{\xi_{m-1} - \zeta_{m-1}}^{m-1}\right), \\
\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}} - \boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}]} &= \left(t_{\xi_1 - \zeta_1 + 1}^1, \dots, t_{\xi_1}^1; \dots; t_{\xi_{m-1} - \zeta_{m-1} + 1}^{m-1}, \dots, t_{\xi_{m-1}}^{m-1}\right).
\end{aligned}
$$

Lemma 2.4.4. For a given sequence $\boldsymbol{b} = (b_1, \dots, b_{\xi_m})$, where $b_i \in \{m+1, m+2, \dots, n\}$, consider the corresponding partition $\boldsymbol{I}(\boldsymbol{b})$ and the decreasing sequence $\eta_m = \xi_m \geq \eta_{m+1} \geq \dots \geq \eta_{n-1}$ as described in Definition 2.4.1. Denote $\boldsymbol{\eta} = (\eta_{m+1}, \dots, \eta_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-m-1}$. Let $\tilde{v}$ be a vector, such that $T_c^a(u)\tilde{v} = 0$, for $m \leq c < a \leq n$, and $T_c^c(u)\tilde{v} = (1 + \Lambda^c(u-x)^{-1})\tilde{v}$, $c = m, \dots, n$. Then for $\ddot{\boldsymbol{\xi}} - \boldsymbol{\eta} \in \mathbb{Z}_{\geq 0}^{n-m-1}$, we have

$$\left(\psi_{n-m}(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\right)\right)^{\boldsymbol{b}}\tilde{v} = \prod_{b=m+1}^{n-1}\frac{1}{(\xi_b-\eta_b)!}\mathrm{Sym}_{\ddot{\boldsymbol{t}}}\left[U_{\boldsymbol{I}(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\right)L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}},\boldsymbol{t}^m)\right]\tilde{v},$$

where $U_{\boldsymbol{I}(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\right)$ is given by formula (2.4.10) and

$$L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}) = \prod_{a=m+1}^{n-2}\prod_{i=1}^{\eta_{a+1}}\prod_{j=\eta_a+1}^{\xi_a}\frac{t_i^{a+1}-t_j^a+1}{t_i^{a+1}-t_j^a}\prod_{l=m+1}^{n-1}\prod_{i=1}^{\eta_l}\frac{t_i^l-x+\Lambda^l}{t_i^l-x}\psi_{n-m}\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}-\boldsymbol{\eta}}\left(\ddot{\boldsymbol{t}}_{(\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}]}\right)\right). \tag{2.4.14}$$

If $\ddot{\boldsymbol{\xi}}-\boldsymbol{\eta}\notin\mathbb{Z}^{n-m-1}_{\geq 0}$, then $\left(\psi_{n-m}(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\right)\right)^{\boldsymbol{b}}\tilde{v}=0$.

Proof. The statement coincide with Lemma 4.3 from [89] up to an adjustment of notation. □

Lemma 2.4.5. For a given sequence $\boldsymbol{a}=(a_1,\dots,a_{\xi_m})$, where $a_i\in\{1,2,\dots,m\}$, consider the corresponding partition $\boldsymbol{J}(\boldsymbol{a})$ and the increasing sequence $\zeta_1\leq\zeta_2\leq\dots\leq\zeta_{m-1}\leq\zeta_m=\xi_m$ as described Definition 2.4.1. Denote $\boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_{m-1})\in\mathbb{Z}^{m-1}_{\geq 0}$. Let $\hat{v}$ be a vector, such that $T^c_b(u)\hat{v}=0$, for $1\leq b<c\leq m$, and $T^b_b(u)\hat{v}=(1+\Lambda^b(u-x)^{-1})\hat{v}$, $b=1,\dots,m$. Then for $\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta}\in\mathbb{Z}^{m-1}_{\geq 0}$, we have

$$\left(\phi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle m\rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right)\right)^{\boldsymbol{a}}\hat{v} = \prod_{l=1}^{m}\frac{1}{(\xi_l-\zeta_l)!}\mathrm{Sym}_{\dot{\boldsymbol{t}}}\left[\widetilde{U}_{\boldsymbol{J}(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m)\widetilde{L}_{\zeta,\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right]\hat{v},$$

where $\widetilde{U}_{\boldsymbol{J}(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m)$ is given by formula (2.4.11) and

$$\begin{aligned}\widetilde{L}_{\zeta,\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}) = \prod_{a=1}^{m-2}\prod_{i=1}^{\xi_{a+1}-\eta_{a+1}}\prod_{j=\xi_a-\eta_a+1}^{\xi_a}\frac{t_i^{a+1}-t_j^a+1}{t_i^{a+1}-t_j^a}\times\\ \times\prod_{l=1}^{m-1}\prod_{i=0}^{\eta^l-1}\frac{t^l_{\xi_l-i}-x+\Lambda^{l+1}}{t^l_{\xi_l-i}-x}\phi_m\left(\mathbb{B}^{\langle m\rangle}_{\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta}}\left(\dot{\boldsymbol{t}}_{[\dot{\boldsymbol{\xi}}-\zeta]}\right)\right).\end{aligned} \tag{2.4.15}$$

If $\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta}\notin\mathbb{Z}^{m-1}_{\geq 0}$, then $\left(\phi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle m\rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right)\right)^{\boldsymbol{a}}\hat{v}=0$.

Proof. The proof is analogous to the proof of Lemma 4.3 in [89] with Corollary 3.4 there replaced by Corollary 3.2 *op.cit.* □

Consider collections $\boldsymbol{q} = (q_{sp} : \; s = m+1, \dots, n, \; p = 1, \dots, m)$ of nonnegative integers, and introduce

$$\begin{aligned} \eta_k(\boldsymbol{q}) &= \sum_{s=k+1}^{n} \sum_{p=1}^{m} q_{sp}, \quad k = m+1, \dots, n-1, \\ \zeta_l(\boldsymbol{q}) &= \sum_{s=m+1}^{n} \sum_{p=1}^{l} q_{sp}, \quad l = 1, \dots, m-1. \end{aligned} \tag{2.4.16}$$

Given $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1})$, define a set

$$\mathcal{Q}_{m,n} = \left\{ \boldsymbol{q} = (q_{sp}) : \; \sum_{s=m+1}^{n} \sum_{p=1}^{m} q_{sp} = \xi_m, \quad \eta_k(\boldsymbol{q}) \le \xi_k, \quad \zeta_l(\boldsymbol{q}) \le \xi_l, \; \text{ for all } k, l \right\}. \tag{2.4.17}$$

In addition, for any $\boldsymbol{q} \in \mathcal{Q}_{m,n}$, we consider the set $\mathcal{S}_{\boldsymbol{q}}$ of all pairs $(\boldsymbol{I}, \boldsymbol{J})$ of partitions $\boldsymbol{I} = (I_{m+1}, \dots, I_n)$, $\boldsymbol{J} = (J_1, \dots, J_m)$ of $\{1, \dots, \xi_m\}$ with given cardinalities of intersections,

$$\mathcal{S}_{\boldsymbol{q}} = \Big\{ \; (\boldsymbol{I}, \boldsymbol{J}) : \; \; |I_s \cap J_p| = q_{sp}, \; \; s = m+1, \dots, n, \; p = 1, \dots, m \; \Big\}. \tag{2.4.18}$$

Notice that,

$$\left| \cup_{r=k+1}^{n} I_r \right| = \eta_k(\boldsymbol{q}), \quad \left| \cup_{r=1}^{l} J_r \right| = \zeta_l(\boldsymbol{q}).$$

Proposition 2.4.6. Let $v \in V(x)$ be a weight singular vector of $\mathfrak{gl}_n$-weight $(\Lambda_1, \dots \Lambda_n)$, then one has

$$\begin{aligned} \mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{q} \in \mathcal{Q}_{m,n}} \Bigg(\prod_{m+1 \le i \le n} \; \prod_{1 \le j \le m} e_{ij}^{q_{ij}} \Bigg) \prod_{a=1}^{m-1} \frac{1}{(\xi_a - \zeta_a)!} \prod_{b=m+1}^{n-1} \frac{1}{(\xi_b - \eta_b)!} \prod_{l=1}^{\xi_m} \frac{1}{t_l^m - x} \times \\ \times \operatorname{Sym}_{\ddot{\boldsymbol{t}}} \operatorname{Sym}_{\dot{\boldsymbol{t}}} \Bigg[L_{\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}) \widetilde{L}_{\boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}) \sum_{(\boldsymbol{I}, \boldsymbol{J}) \in \mathcal{S}_{\boldsymbol{q}}} \widetilde{U}_{\boldsymbol{J}}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}} - \zeta, \dot{\boldsymbol{\xi}}]}, \boldsymbol{t}^m) U_{\boldsymbol{I}} \left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^m \right) \Bigg] v, \end{aligned} \tag{2.4.19}$$

where the sequences $\boldsymbol{\eta} = (\eta_{m+1}, \dots, \eta_{n-1})$ and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{m-1})$, are given by the rule

$$\eta_k = \eta_k(\boldsymbol{q}), \quad \zeta_l = \zeta_l(\boldsymbol{q}), \quad k = m+1, \dots, n-1, \quad l = 1, \dots, m-1.$$

Proof. Taking formula (2.4.4) for $\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v$, we apply Lemma 2.4.4 to the expression $\Big(\psi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\right)\Big)^{\boldsymbol{b}}v$. Next we observe that the vector $\psi_{n-m}\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}-\boldsymbol{\eta}}\left(\ddot{\boldsymbol{t}}_{(\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}]}\right)\right)v$ in the right-hand side of (2.4.14) satisfies the conditions for the vector $\hat{v}$ in Lemma 2.4.5. Applying Lemma 2.4.5 to the expression $\Big(\phi_m(\boldsymbol{t}^m)\left(\mathbb{B}^{\langle m\rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right)\Big)^{\boldsymbol{a}}\psi_{n-m}\left(\mathbb{B}^{\langle n-m\rangle}_{\ddot{\boldsymbol{\xi}}-\boldsymbol{\eta}}\left(\ddot{\boldsymbol{t}}_{(\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}]}\right)\right)v$, we obtain

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v=\sum_{\boldsymbol{a},\boldsymbol{b}}\Big(\mathcal{T}(\boldsymbol{t}^m)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}\operatorname{Sym}_{\ddot{\boldsymbol{t}}}\operatorname{Sym}_{\dot{\boldsymbol{t}}}\left[\widetilde{U}_{\boldsymbol{J}(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m)U_{\boldsymbol{I}(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\right)L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\widetilde{L}_{\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right]v,$$

where $\Big(\mathcal{T}(\boldsymbol{t}^m)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}$ is given by (2.4.5), while the partitions $\boldsymbol{J}(\boldsymbol{a}),\boldsymbol{I}(\boldsymbol{b})$ and the sequences $\boldsymbol{\eta}=(\eta_{m+1},\dots,\eta_{n-1})$, $\boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_{m-1})$ are described in Definition 2.4.1.

On the next step, we use the bijection between the sequences $\boldsymbol{a},\boldsymbol{b}$ and the partitions $\boldsymbol{I},\boldsymbol{J}$, see Lemma 2.4.3, to rewrite the sum $\sum_{\boldsymbol{a},\boldsymbol{b}}$ as the sum $\sum_{\boldsymbol{I},\boldsymbol{J}}$ over pairs of partitions. The latter sum can be further written as the double sum $\sum_{\boldsymbol{q}\in\mathcal{Q}_{m,n}}\sum_{(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}}$ over collections $\boldsymbol{q}\in\mathcal{Q}_{m,n}$ and pairs of partitions $(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}$. Thus we obtain

$$\begin{aligned}\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v=\sum_{\boldsymbol{q}\in\mathcal{Q}_{m,n}}\Bigg(\sum_{(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}}&\Big(\mathcal{T}(\boldsymbol{t}^m)\Big)^{\boldsymbol{a}(\boldsymbol{J})}_{\boldsymbol{b}(\boldsymbol{I})}\times\\ &\times\operatorname{Sym}_{\ddot{\boldsymbol{t}}}\operatorname{Sym}_{\dot{\boldsymbol{t}}}\left[\widetilde{U}_{\boldsymbol{J}(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m)U_{\boldsymbol{I}(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\right)L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\widetilde{L}_{\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right]\Bigg)v,\end{aligned}$$

where $\boldsymbol{a}(\boldsymbol{J})$, $\boldsymbol{b}(\boldsymbol{I})$ are the sequences corresponding to the partitions $\boldsymbol{I}$, $\boldsymbol{J}$.

Finally, we observe that in the module $V(x)$ we have $\Big(\mathcal{T}(\boldsymbol{t}^m)\Big)^{\boldsymbol{a}}_{\boldsymbol{b}}=\prod_{i,j}e_{ji}^{q_{ij}}\prod_{l=1}^{\xi_m}\dfrac{1}{t^m_l-x}$. The last expression and the product $L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\widetilde{L}_{\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})$ depend only on the collection $\boldsymbol{q}$ and can be moved out of the sum $\sum_{(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}}$. Proposition 2.4.6 is proved. □

The next theorem is the main result of this paper. It will be proved in Section 2.4.4.

Theorem 2.4.7. *Let $v \in V(x)$ be a weight singular vector of $\mathfrak{gl}_n$-weight $(\Lambda_1, \dots \Lambda_n)$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1})$ be a collection of nonnegative numbers. Then one has*

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{q} \in \mathcal{Q}_{m,n}} \Bigg(\prod_{m+1 \le i \le n} \prod_{1 \le j \le m} \frac{e_{ij}^{q_{ij}}}{(q_{ij})!} \Bigg) \prod_{a=1}^{m-1} \frac{1}{(\xi_a - \zeta_a)!} \prod_{b=m+1}^{n-1} \frac{1}{(\xi_b - \eta_b)!} \prod_{l=1}^{\xi_m} \frac{1}{t_l^m - x} \times$$
$$\times \operatorname{Sym}_{\boldsymbol{t}^m} \operatorname{Sym}_{\dot{\boldsymbol{t}}} \operatorname{Sym}_{\ddot{\boldsymbol{t}}} \Big[L_{\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}) \widetilde{L}_{\boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}\,)\ \widetilde{U}_{\boldsymbol{J}_0}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}} - \boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}]}, \check{\boldsymbol{t}}^m) U_{\check{\boldsymbol{I}}_0}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^m\right) \Phi(\boldsymbol{t}^m) \Big] v. \tag{2.4.20}$$

Here the set $\mathcal{Q}_{m,n}$ is given by (2.4.17), $\boldsymbol{I}_{\boldsymbol{0}} = (I_{m+1}, \dots, I_n)$, $\boldsymbol{J}_{\boldsymbol{0}} = (J_1, \dots, J_m)$ is any pair of partitions of $\{1, \dots, \xi_m\}$ such that $(\boldsymbol{I}_0, \boldsymbol{J}_0) \in \mathcal{S}_{\boldsymbol{q}}$, see (2.4.18), $\check{\boldsymbol{I}}_0 = (\sigma_0(I_{m+1}), \dots, \sigma_0(I_n))$, where $\sigma_0 \in S_{\xi_m}$ is the longest permutation, $\sigma_0(i) = \xi_m - i + 1$, the sequences $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{m-1})$, $\boldsymbol{\eta} = (\eta_{m+1}, \dots, \eta_{n-1})$, are given by the rule $\eta_k = \eta_k(\boldsymbol{q})$, $\zeta_l = \zeta_l(\boldsymbol{q})$ for all k, l, see (2.4.16), the functions $L_{\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}, \boldsymbol{t}^m)$, $\widetilde{L}_{\boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}, \boldsymbol{t}^m)$ are given by (2.4.14),(2.4.15), the functions $U_{\check{\boldsymbol{I}}_0(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^m\right)$, $\widetilde{U}_{\boldsymbol{J}_0(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}} - \boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}]}, \check{\boldsymbol{t}}^m)$ are given by (2.4.10), (2.4.11), $\check{\boldsymbol{t}}^m = (t^m_{\xi_m}, \dots, t^m_1)$ and

$$\Phi(\boldsymbol{t}^m) = \prod_{1 \le a < b \le \xi_m} \frac{t_a^m - t_b^m - 1}{t_a^m - t_b^m}.$$

Remark. For $\xi_m = 0$, we have $\boldsymbol{\eta} = (0, \dots, 0)$, $\boldsymbol{\zeta} = (0, \dots, 0)$. Then $\widetilde{U}_{\boldsymbol{J}_0(\boldsymbol{a})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}} - \boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}]}, \check{\boldsymbol{t}}^m) = 1$, $U_{\check{\boldsymbol{I}}_0(\boldsymbol{b})}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^m\right) = 1$, $L_{\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}\,) = \psi_{n-m}\left(\mathbb{B}^{\langle n-m \rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}\,)\right)$, $\widetilde{L}_{\boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}\,) = \phi_m\left(\mathbb{B}^{\langle m \rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}\,)\right)$. Notice that $\psi_{n-m}\left(\mathbb{B}^{\langle n-m \rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}\,)\right)$ is a symmetric function of $t^s_1, \dots, t^s_{\xi_s}$ for each $s = m+1, \dots, n-1$, while $\phi_m\left(\mathbb{B}^{\langle m \rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}\,)\right)$ is a symmetric function of $t^p_1, \dots, t^p_{\xi_p}$ for each $p = 1, \dots, m-1$. Therefore, formula (2.4.20) reduces to formula (2.4.9):

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \phi_m\left(\mathbb{B}^{\langle m \rangle}_{\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\right)\ \psi_m\left(\mathbb{B}^{\langle n-m \rangle}_{\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\right) v.$$

Remark. In cases $m = 1$ Theorem 2.4.7 becomes Theorem 3.3 from [89],

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \prod_{l=1}^{\xi_1} \frac{1}{t_l^1 - x} \sum_{\boldsymbol{q} \in \mathcal{Q}_n} \Bigg(\prod_{2 \le i \le n} \frac{e_{i1}^{q_i}}{(q_i)!} \Bigg) \prod_{b=2}^{n-1} \frac{1}{(\xi_b - \eta_b)!} \times$$
$$\times \operatorname{Sym}_{\boldsymbol{t}^1} \operatorname{Sym}_{\ddot{\boldsymbol{t}}} \left[L_{\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}) U_{\boldsymbol{I}_0}\left(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^1\right)\ \Phi(\boldsymbol{t}^1) \psi_{n-1}\left(\mathbb{B}^{\langle n-1 \rangle}_{\ddot{\boldsymbol{\xi}} - \boldsymbol{\eta}}\left(\ddot{\boldsymbol{t}}_{(\boldsymbol{\eta}, \ddot{\boldsymbol{\xi}}]}\right)\right)\right] v.$$

In case $m = n-1$ it becomes Theorem 3.1 from [89]:

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \prod_{l=1}^{\xi_{n-1}} \frac{1}{t_l^{n-1} - x} \sum_{\boldsymbol{q} \in \mathcal{Q}_{m,n}} \Bigg(\prod_{j=1}^{n-1} \frac{e_{nj}^{q_j}}{(q_j)!} \Bigg) \prod_{a=1}^{n-2} \frac{1}{(\xi_a - \zeta_a)!} \times$$
$$\times \operatorname{Sym}_{\boldsymbol{t}^{n-1}} \operatorname{Sym}_{\dot{\boldsymbol{t}}} \Big[\widetilde{L}_{\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}\,)\, \widetilde{U}_{\boldsymbol{J}_0}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}]}, \boldsymbol{t}^{n-1}) \Phi(\boldsymbol{t}^{n-1}) \Big]\, v.$$

2.4.4 Proof of Theorem 2.4.7

Throughout this section, we use the notation from Theorem 2.4.7. We will begin with two auxiliary lemmas.

Lemma 2.4.8. Consider functions $F(x_1,\dots,x_k)$ and $G(x_1,\dots,x_l)$, $k \le l$, such that the function $G(x_1,\dots,x_l)$ is a symmetric function of $x_1,\dots,x_k$ and of $x_{k+1},\dots,x_l$. Then

$$\begin{aligned}&\operatorname{Sym}_{x_1,\dots,x_l}\Big[F(x_1,\dots,x_k)G(x_1,\dots,x_l)\Big] =\\ &\qquad = \frac{1}{k!}\operatorname{Sym}_{x_1,\dots,x_k}\Big[\Big(\operatorname{Sym}_{x_1,\dots,x_l}F(x_1,\dots,x_k)\Big)G(x_1,\dots,x_l)\Big].\end{aligned}$$

Proof. By inspection. □

Recall the weight functions $W_{\boldsymbol I}(\boldsymbol v;\boldsymbol z)$, $\widetilde W_{\boldsymbol J}(\boldsymbol u;\boldsymbol z)$, see (2.4.12),(2.4.13), and the functions $L_{\boldsymbol\eta,\ddot{\boldsymbol\xi}}(\ddot{\boldsymbol t},\boldsymbol t^m)$, $\widetilde L_{\boldsymbol\zeta,\dot{\boldsymbol\xi}}(\dot{\boldsymbol t},\boldsymbol t^m)$ given by (2.4.14),(2.4.15).

Lemma 2.4.9. One has

$$\operatorname{Sym}_{\ddot{\boldsymbol t}}\Big[U_{\boldsymbol I}\Big(\ddot{\boldsymbol t}_{[\boldsymbol\eta]},\boldsymbol t^m\Big)L_{\boldsymbol\eta,\ddot{\boldsymbol\xi}}(\ddot{\boldsymbol t})\Big] = \prod_{s=m+1}^{n-1}\frac{1}{\eta_s!}\operatorname{Sym}_{\ddot{\boldsymbol t}}\Big[W_{\boldsymbol I}\Big(\ddot{\boldsymbol t}_{[\boldsymbol\eta]},\boldsymbol t^m\Big)L_{\boldsymbol\eta,\ddot{\boldsymbol\xi}}(\ddot{\boldsymbol t})\Big],$$

$$\operatorname{Sym}_{\dot{\boldsymbol t}}\Big[\widetilde U_{\boldsymbol J}(\dot{\boldsymbol t}_{(\dot{\boldsymbol\xi}-\boldsymbol\zeta,\dot{\boldsymbol\xi}]},\check{\boldsymbol t}^m)\widetilde L_{\boldsymbol\zeta,\dot{\boldsymbol\xi}}(\dot{\boldsymbol t})\Big] = \prod_{s=1}^{m-1}\frac{1}{\zeta_s!}\operatorname{Sym}_{\dot{\boldsymbol t}}\Big[\widetilde W_{\boldsymbol J}(\dot{\boldsymbol t}_{(\dot{\boldsymbol\xi}-\boldsymbol\zeta,\dot{\boldsymbol\xi}]},\check{\boldsymbol t}^m)\widetilde L_{\boldsymbol\zeta,\dot{\boldsymbol\xi}}(\dot{\boldsymbol t})\Big].$$

Proof. Observe that for every $s = m+1,\dots,n-1$, the function $L_{\boldsymbol\eta,\ddot{\boldsymbol\xi}}(\ddot{\boldsymbol t})$ is symmetric in $t^s_1,\dots,t^s_{\eta_s}$ and in $t^s_{\eta_s+1},\dots,t^s_{\xi_s}$. Similarly, for every $s = 1,\dots,m-1$, the function $\widetilde L_{\boldsymbol\zeta,\dot{\boldsymbol\xi}}(\dot{\boldsymbol t})$ is a symmetric in $t^s_1,\dots,t^s_{\xi_s-\zeta_s}$ and in $t^s_{\xi_s-\zeta_s+1},\dots,t^s_{\xi_s}$. Then the statement follows from the formulas (2.4.12),(2.4.13) by applying Lemma 2.4.8 several times. □

Recall that for $\sigma \in S_{\xi_m}$ and a partition $\boldsymbol J = (J_1,\dots,J_m)$ of $\{1,\dots,\xi_m\}$, we have $\sigma(\boldsymbol J) = (\sigma(J_1),\dots,\sigma(J_m))$. Similarly, for $\boldsymbol I = (I_{m+1},\dots,I_n)$, we have $\sigma(\boldsymbol I) = (\sigma(I_{m+1}),\dots,\sigma(I_n))$.

The next proposition is the key point of the proof. For a collection $\boldsymbol z = (z_1,\dots,z_{\xi_m})$ set

$$\Phi(\boldsymbol z) = \prod_{1\le a<b\le\xi_m}\frac{z_a - z_b - 1}{z_a - z_b}.$$

Proposition 2.4.10. One has

$$\sum_{\sigma\in S_{\xi_m}} W_{\sigma(\boldsymbol{I})}\Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{z}\Big)\widetilde{W}_{\sigma(\boldsymbol{J})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{z}) = \\ = \mathrm{Sym}_{\boldsymbol{z}}\Big[W_{\sigma_0(\boldsymbol{I})}(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{z})\widetilde{W}_{\boldsymbol{J}}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{z}^{\sigma_0})\Phi(\boldsymbol{z})\Big], \tag{2.4.21}$$

where σ_0 is the longest permutation in S_{ξ_m}, $\sigma_0(i)=\xi_m-i+1$, and $\boldsymbol{z}^{\sigma_0}=(z_{\xi_m},\dots,z_1)$.

Proof. Formula (2.4.21) can be written as

$$\sum_{\sigma\in S_{\xi_m}} W_{\sigma(\boldsymbol{I})}\Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{z}\Big)\,\widetilde{W}_{\sigma(\boldsymbol{J})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{z}) = \sum_{\pi\in S_{\xi_m}} W_{\sigma_0(\boldsymbol{I})}(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{z}^{\pi})\widetilde{W}_{\boldsymbol{J}}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{z}^{\pi\sigma_0})\Phi(\boldsymbol{z}^{\pi}), \tag{2.4.22}$$

$\boldsymbol{z}^{\pi}=(z_{\pi(1)},\dots,z_{\pi(\xi_m)})$. Then the proof of formula (2.4.22) is literately the same as that of formula (2.3.47) with Corollary 2.4.2 substituting Lemma 2.3.14. □

Proof of Theorem 2.4.7: First we apply Lemma 2.4.9 to formula (2.4.19) and get

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v = \sum_{\boldsymbol{q}\in\mathcal{Q}_{m,n}}\Big(\prod_{m+1\le i\le n}\ \prod_{1\le j\le m} e_{ij}^{q_{ij}}\Big)\prod_{a=1}^{m-1}\frac{1}{(\xi_a-\zeta_a)!}\prod_{b=m+1}^{n-1}\frac{1}{(\xi_b-\eta_b)!}\prod_{l=1}^{\xi_m}\frac{1}{t_l^m-x}\times \\ \times\prod_{s=1}^{m-1}\frac{1}{\zeta_s!}\prod_{s=m+1}^{n-1}\frac{1}{\eta_s!}\mathrm{Sym}_{\ddot{\boldsymbol{t}}}\,\mathrm{Sym}_{\dot{\boldsymbol{t}}}\Bigg[L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}})\widetilde{L}_{\zeta,\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}})\sum_{(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}}\widetilde{W}_{\boldsymbol{J}}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m)W_{\boldsymbol{I}}\Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\Big)\Bigg]v. \tag{2.4.23}$$

Notice that every pair $(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}$ can be obtained from an arbitrary fixed pair $(\boldsymbol{I}_0,\boldsymbol{J}_0)\in\mathcal{S}_{\boldsymbol{q}}$ by the action of the symmetric group S_{ξ_m}. Therefore, the inner sum in the right-hand side of formula (2.4.23) can be written in the following way,

$$\sum_{(\boldsymbol{I},\boldsymbol{J})\in\mathcal{S}_{\boldsymbol{q}}} W_{\boldsymbol{I}}\Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\Big)\,\widetilde{W}_{\boldsymbol{J}}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m) = \\ = \Big(\prod_{m+1\le i\le n}\ \prod_{1\le j\le m}\frac{1}{(q_{ij})!}\Big)\sum_{\sigma\in S_{\xi_m}} W_{\sigma(\boldsymbol{I_0})}\Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]},\boldsymbol{t}^m\Big)\,\widetilde{W}_{\sigma(\boldsymbol{J_0})}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\zeta,\dot{\boldsymbol{\xi}}]},\boldsymbol{t}^m). \tag{2.4.24}$$

Here $\prod_{i,j}(q_{i,j})!$ is the cardinality of the isotropic subgroup of the pair $(\boldsymbol{I_0}, \boldsymbol{J_0})$. Applying Proposition 2.4.10 for $\boldsymbol{z} = \boldsymbol{t}^m$ to the sum over permutations $\sum_{\sigma \in S_{\xi_m}}$ in the right-hand side of (2.4.24), we get

$$\mathbb{B}_{\boldsymbol{\xi}}(\boldsymbol{t})v =$$

$$= \sum_{\boldsymbol{q} \in \mathcal{Q}_{m,n}} \Big(\prod_{m+1 \le i \le n} \prod_{1 \le j \le m} \frac{e_{ij}^{q_{ij}}}{(q_{ij})!} \Big) \prod_{a=1}^{m-1} \frac{1}{(\xi_a - \zeta_a)!} \prod_{b=m+1}^{n-1} \frac{1}{(\xi_b - \eta_b)!} \prod_{l=1}^{\xi_m} \frac{1}{t_l^m - x} \times$$

$$\times \prod_{s=m+1}^{n-1} \frac{1}{\eta_s!} \prod_{s=1}^{m-1} \frac{1}{\zeta_s!} \operatorname{Sym}_{\boldsymbol{t}^m} \operatorname{Sym}_{\dot{\boldsymbol{t}}} \operatorname{Sym}_{\ddot{\boldsymbol{t}}} \Big[L_{\boldsymbol{\eta},\ddot{\boldsymbol{\xi}}}(\ddot{\boldsymbol{t}}) \widetilde{L}_{\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}}(\dot{\boldsymbol{t}}) \, \widetilde{W}_{\boldsymbol{J}_0}(\dot{\boldsymbol{t}}_{(\dot{\boldsymbol{\xi}}-\boldsymbol{\zeta},\dot{\boldsymbol{\xi}}]}, \check{\boldsymbol{t}}^m) W_{\check{\boldsymbol{I}}_0} \Big(\ddot{\boldsymbol{t}}_{[\boldsymbol{\eta}]}, \boldsymbol{t}^m \Big) \Phi(\boldsymbol{t}^m) \Big] v$$

Finally, we use Lemma 2.4.9 once more to transform the symmetrization $\operatorname{Sym}_{\dot{\boldsymbol{t}}} \operatorname{Sym}_{\ddot{\boldsymbol{t}}}$ in the last formula to the form in formula (2.4.20). $\square$

3. ASYMPTOTICS OF tt^* EQUATION

The idea of topological field theories (TFT) as solvable models was proposed in [99]. In [99, 22, 23, 100] it was shown that topological correlators (at tree level) in a 2D TFT model are holomorphic functions on moduli of the TFT model obeying an overdetermined system of nonlinear PDE (the equations of associativity of primary operator algebra). Integrability of this equations was proved in [26].

The problem of calculation of the ground state metric of a family of TFT was studied in general situation (for both massless and massive theories) in [13]. In this paper a system of PDE for the ground state metric (being a Hermitian metric on the moduli space of TFT) was derived. The topological and "antitopological" (i.e. complex conjugate) correlators serve as coefficients of these PDE. This general construction of calculating of ground state metric was called in [6] a *topological-antitopological fusion* or *tt^* fusion.* The equation of the same form arise for the metric on moduli space of Calabi-Yau varieties [88, 10]. The Hermitian metric on the moduli space in this case is the same as the Zamolodchikov metric [103] of the underlying $N = 2$ superconformal field theories. In [13, 11, 12] a number of particular integrable reductions of the main equations was found. It was shown that under some symmetry assumption the equations of topological-antitopological fusion can be reduced to affine Toda equations (particularly, to Euclidean sinh-Gordon) and to some other integrable systems of the soliton theory. For massive perturbations of topological conformal field theory (TCFT) many particular reductions of the main equations can be solved via the Painlevé transcendents of the third kind.

In [27] the integrability of the equations of topological-antitopological fusion was proven in the general case. This integrability immediately follows from the zero-curvature representation of these equations depending on a spectral parameter.

3.1 Main result

The two-dimensional Toda lattice is an important integrable system. For instance it is an example of a nonabelian Chern-Simons theory in classical field theory (see [102]), and it can be interpreted in differential geometry as the equation for primitive harmonic maps taking values in a compact flag manifold ([8],[9]). Such maps are closely related to harmonic maps into symmetric spaces. These maps in turn have many geometrical interpretations, e.g. surfaces in $\mathbb{R}^3$ of constant mean curvature (see [25]), or special Lagrangian cones in $\mathbb{C}^3$([66]).

We shall be concerned with the so called "negative sign tt^*-Toda" equation (see [14, 15, 16]). This is the system

$$2\,(w_i)_{z\bar{z}} = e^{2(w_{i+1}-w_i)} - e^{2(w_i-w_{i-1})}, \qquad w_i : U \to \mathbb{R},\ i \in \mathbb{Z}, \tag{3.1.1}$$

where U is an open subset of $\mathbb{C} = \mathbb{R}^2$. We assume the following conditions on w_i:

- periodicity: $w_i = w_{i+n+1}$ for all $i \in \mathbb{Z}$, for some $n \in \mathbb{Z}_{\geq 0}$,
- $w_0 + \cdots + w_n = 0$,
- anti-symmetry conditions: $w_i + w_{n-i} = 0$, $i \in \mathbb{Z}$.

All these conditions are clearly consistent with equation (3.1.1)

The first manifestation of the integrability of this system is its zero curvature formulation:

$$d\alpha + \alpha \wedge \alpha = 0 \quad \text{ for all } \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\},$$

where

$$\alpha(\lambda) = \left(w_z + \frac{1}{\lambda} W^t\right) dz + \left(-w_{\bar{z}} + \lambda W\right) d\bar{z},$$

and

$$w = \operatorname{diag}\left(w_0, \ldots, w_n\right), \qquad W = \left(\begin{array}{c|c|c|c} & e^{w_1-w_0} & & \\ \hline & & \ddots & \\ \hline & & & e^{w_n-w_{n-1}} \\ \hline e^{w_0-w_n} & & & \end{array}\right).$$

In other words, (3.1.1) is the compatibility condition $2w_{z\bar{z}} = -\left[W^t, W\right]$ for the linear system

$$\begin{cases} \Psi_z = \left(w_z + \frac{1}{\lambda}W\right)\Psi \\ \Psi_{\bar{z}} = \left(-w_{\bar{z}} + \lambda W^t\right)\Psi. \end{cases} \tag{3.1.2}$$

A solution of (3.1.1) is called radial if every $w_i = w_i(z, \bar{z})$, $i = 1, \ldots, n$, depend only on the real variable $x = |z|$. For radial solution of (3.1.1), system (3.1.2) reduces to

$$\Psi_x = \frac{1}{x}\left(z\Psi_z + \bar{z}\Psi_{\bar{z}}\right) = \left(\frac{1}{\lambda}\frac{z}{x}W + \lambda\frac{\bar{z}}{x}W^t\right)\Psi.$$

This can be regarded as the equation for an isomonodromic deformation, which is another, perhaps more famous, manifestation of integrability, and a well known approach to studying equations of Painlevé type (see [35],[48] and also [41]). Namely, if we impose the (Euler-type) homogeneity condition

$$\lambda\Psi_\lambda + z\Psi_z - \bar{z}\Psi_{\bar{z}} = 0,$$

we obtain

$$\Psi_\lambda = \left(-\frac{1}{\lambda^2}zW - \frac{1}{\lambda}xw_x + \bar{z}W^t\right)\Psi,$$

which is a meromorphic o.d.e. in λ with poles of order two at 0 and ∞. Writing $\mu = \lambda x/z$, we obtain

$$\Psi_\mu = \left(-\frac{1}{\mu^2}xW - \frac{1}{\mu}xw_x + xW^t\right)\Psi. \tag{3.1.3}$$

The compatibility condition of the linear system

$$\begin{cases} \Psi_\mu = \left(-\frac{1}{\mu^2}xW - \frac{1}{\mu}xw_x + xW^t\right)\Psi \\ \Psi_x = \left(\frac{1}{\mu}W + \mu W^t\right)\Psi \end{cases} \tag{3.1.4}$$

is the radial version $(xw_x)_x = 2x\,[W^t, W]$ of (1.1).

For the case $n = 2$, constraints on w_i give $w_2 = -w_0$ and $w_1 = 0$. Thus, (3.1.1) becomes

$$2(w_0)_{t\bar{t}} = e^{-2w_0} - e^{4w_0}. \tag{3.1.5}$$

This equation was studied by Kitaev in [52] using the WKB method.

Remark. The equation (3.1.5) is also known as the Bullough-Dodd equation. By substituting

$$\begin{cases} x = \dfrac{3}{4} s^{\frac{2}{3}} \\ \tilde{w} = s^{\frac{1}{3}} e^{-2w_0}, \end{cases}$$

one can show that it becomes the following equation:

$$\tilde{w}_{ss} = \frac{\tilde{w}_s^2}{w} - \frac{\tilde{w}_s}{s} - \frac{\tilde{w}^2}{s} + \frac{1}{\tilde{w}}.$$

This is a special case of the Painlevé III equation [77, 44].

We are interested in asymptotics of the solutions of (3.1.5). The technique that we used to achieve this goal is the Riemann-Hilbert problem (see [17, 61, 19, 18]). Recall that to solve the Riemann-Hilbert problem on an oriented contour Γ with a jump matrix $J(z)$, $z \in \Gamma$, means to find matrix-valued function $\Psi(z)$ which satisfies the following properties

- $\Psi(z)$ is analytic outside of contour Γ;
- $\Psi(z)$ satisfies the jump condition $\Psi_+(z) = \Psi_-(z)J(z)$ on the contour Γ, where " + " means the boundary value from the left of the contour Γ and " $-$" means the boundary value from the right of the contour Γ;
- $\Psi(z)$ satisfies appropriate normalization conditions .

For the case $n = 2$, we showed that Ψ_μ from (3.1.3) satisfies the following Riemann-Hilbert problem: the contour Γ is the union of the unit circle and the imaginary axis, see the picture,

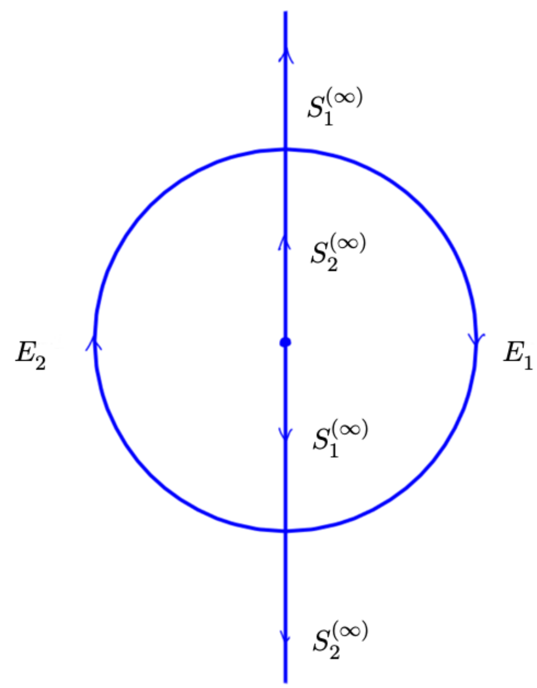

Figure 3.1. Riemann-Hilbert problem for tt^* equation.

and the jump matrices are

$$S_1^{(\infty)} = \begin{pmatrix} 1 & \omega^2 s & 0 \\ 0 & 1 & 0 \\ \omega^2 s & -\omega s & 1 \end{pmatrix}, \quad S_2^{(\infty)} = \begin{pmatrix} 1 & 0 & -\omega s \\ -\omega s & 1 & \omega^2(s+1) \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} A & B & \overline{B} \\ B & \omega s A - \omega^2 s B + \overline{B} & A \\ \overline{B} & A & \omega^2 s A + B - \omega s \overline{B} \end{pmatrix}$$

ă

$$E_2 = \frac{1}{9} d_3 E_1^{-1} d_3, \quad d_3 = \mathrm{diag}(1, \omega, \omega^2),$$

where $\omega = \exp\frac{2\pi i}{3}$, A, s are real numbers, B is a complex number. Triple A, B, s satisfy the following equations

$$(1+s)A + \omega^2 B + \omega^2 \bar{B} = \frac{1}{3},$$
$$A^2 - \frac{1}{3}A = |B|^2\,.$$

Our main result describes the connection between the monodromy data (A, B, s) of Ψ_μ and the asymptotics of the solutions w_0 of (3.1.5).

Theorem 3.1.1. *The asymptotic behavior of the radial solutions $w_0(x)$ as $x \to \infty$ is*

$$w_0(x) = \frac{\rho}{\sqrt{x}}\cos(2\sqrt{3}x + \nu\ln x + \gamma) + O\left(\frac{1}{x}\right),$$

where

$$\rho^2 = \frac{\sqrt{3}}{2}\nu, \quad \rho > 0,$$
$$\gamma = \nu\ln(24\sqrt{3}) + \frac{\pi}{12} + \arg(\omega^2 B) + \arg\Gamma(i\nu),$$
$$\nu = \frac{1}{2\pi}\ln 3A.$$

The proof of this theorem is given in the next sections. Let us outline the structure.

Our main tools is the RHP and its connection to the asymptotics of the Painlevé equation. We recall the main ideas of this method in Section 3.2. Then, following the procedure discussed in that section, we start with the symmetries of the equation (3.1.4) and describe the monodromy data (Stokes matrices S_k and the connection matrices E_k) in Sections 3.3 and 3.4 . In Section 3.5 we modify our RHP using the decomposition of $S_n = Q_n Q_{n+\frac{1}{3}} Q_{n+\frac{1}{3}}$, described in Subsection 3.3.5, and "open lenses" using decomposition $E_i = L_i D_i R_i$, described in Section 3.6. In Section 3.7 we solve the global parametrix problem and the local problems. We observe that the local problem at point $\xi = 1$ has a block structure and can be reduced to a standard RHP for the parabolic cylinder function. Other local problems can be obtained from $\xi = 1$ local problem using symmetries described in Section 3.3. Finally, in Section 3.8 we show that the error problem satisfies the conditions of the small norm theorem and in Subsection 3.8.2 we obtain the asymptotic behavior stated in Theorem 3.1.1.

3.2 Riemann-Hilbert problem and isomonodromy

The six classical Painlevé transcendents were introduced at the turn of the twentieth century by Paul Painlevé and his school, as the solution of a specific classification problem for second order ODEs of the type

$$u_{xx} = F\left(x, u, u_x\right),$$

where F is a function meromorphic in x and rational in u and u_x. The problem was to find all equations of this form, which have the property that their solutions are free from movable critical points, i.e., the locations of possible branch points ind essential singularities of the solution do not depend on the initial data. The motivation for posing this problem is quite clear: the absence of movable critical points means that every solution of the equation can be meromorphically extended to the entire universal covering of a punctured complex sphere, determined only by the equation. This implies that such equations share one of the fundamental properties of linear equations.

It was shown by Painlevé and Gambier (1900, 1910), that within a Möbius transformation,

$$u \mapsto \frac{\alpha(x)u + \beta(x)}{\gamma(x)u + \delta(x)}, \quad x \mapsto \varphi(x),$$
$$\alpha, \beta, \gamma, \delta, \varphi \text{ - are meromorphic in } x,$$

there exist only fifty such equations (see monograph [47]). Each of them either can be integrated in terms of known functions, or can be mapped to a set of six equations, which cannot be integrated in terms of known functions. These six equations are called Painlevé equations, and their general solutions are called Painlevé functions or Painlevé transcendents. The canonical forms for the Painlevé equations are:

1. $u_{xx} = 6u^2 + x$
2. $u_{xx} = xu + 2u^3 - \alpha$
3. $u_{xx} = \frac{1}{u}u_x^2 - \frac{u_x}{x} + \frac{1}{x}\left(\alpha u^2 + \beta\right) + \gamma u^3 + \frac{\delta}{u}$

4. $u_{xx} = \frac{1}{2u}u_x^2 + \frac{3}{2}u^3 + 4xu^2 + 2\left(x^2 - \alpha\right)u + \frac{\beta}{u}$

5. $u_{xx} = \frac{3u-1}{2u(u-1)}u_x^2 - \frac{1}{x}u_x + \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1}$

6.
$$u_{xx} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)u_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)u_x + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right)$$

Here, $\alpha, \beta, \gamma, \delta$ are complex parameters. During the rest of the twentieth century a great deal of facts about these equations were discovered: the structure of movable singularities, families of explicit solutions, their transformation properties, etc.

During the last thirty years, great progress in the theory of Painlevé equations themselves has been achieved. Just as for their linear counterparts, it is now possible to derive explicit connection formulae for the Painlevé transcendents relating the relevant asymptotics parameters at different critical points. This fact is based on the *Isomonodromy Method.* This method was introduced in 1980 by H. Flaschka and A.C. Newell [35] , and by M. Jimbo, T. Miwa and K. Ueno [48], and it is based on the intrinsic relation of the Painlevé functions to the monodromy theory of systems of linear ODE with rational coefficients.

Let us outline the isomonodromy formulation of the Painlevé equations. Consider the generic case of a linear system with rational coefficients, i.e., the Fuchsian system,

$$\frac{d\Psi}{d\lambda} = \sum_{j=1}^{n} \frac{A_j}{\lambda - a_j}\Psi, \quad \Psi, A_j - N \times N \text{ matrices}$$

The monodromy group of this system is defined as a representation of the fundamental group of the punctured Riemann sphere (more exactly, a conjugate class of representations),

$$\rho : \pi_1\left(\mathbb{CP}^1 \setminus \{a_1, \dots, a_n, \infty\}\right) \mapsto \mathbf{GL}(N, \mathbb{C})$$

generated by encircling the singular points $a_1, \dots, a_n, a_\infty = \infty$:

$$\Psi(\lambda)|_{(\lambda - a_j) \mapsto (\lambda - a_j)e^{2\pi i}} = \Psi(\lambda)M_j; \quad \lambda - a_\infty \equiv \lambda^{-1},$$
$$M_\infty M_n \dots M_1 = I.$$

The matrices $M_1, \ldots, M_n, M_\infty$ are called monodromy matrices, and the set

$$\mathfrak{m} = \{M_1, \ldots, M_n\},$$

monodromy data. This set completely defines (up to a conjugation) the monodromy group $\mathfrak{M}$ of the above Fuchsian equation. Following [48] we will call the set

$$\mathbb{A} = \{a_1, \ldots, a_n; A_1, \ldots, A_n\}$$

singular data of the Fuchsian system and the set

$$\mathbb{M} = \{a_1, \ldots, a_n; \mathbb{M}_1, \ldots, \mathbb{M}_n\} \equiv \{a_1, \ldots, a_n; \mathfrak{m}\}$$

its extended monodromy data. We will also use the notations

$$\mathcal{A} \equiv \{\mathbb{A}\}, \quad \mathcal{M} \equiv \{\mathfrak{m}\}, \quad \text{and} \quad \mathcal{M}_e \equiv \{\mathbb{M}\}$$

for the sets of singular, monodromy, and extended monodromy data, respectively. The associated Riemann-Hilbert problem consists of proving the existence of a Fuchsian system with given singular points and monodromy group, i.e., with the given set

$$\{a_1, \ldots, a_n; \mathfrak{M}\} \equiv \mathbb{M}.$$

More generally, one has to analyze the direct and inverse monodromy maps, i.e., the maps

$$\mathcal{A} \mapsto \mathcal{M}$$

and

$$\mathcal{M}_e \mapsto \mathcal{A}$$

respectively. This constitutes the central question of the global theory of Fuchsian systems.

Let us consider the situation when the the matrix size is $N = 2$, and we have four singular points, $n = 3$. In the generic situation, the system can be written as follows: (1)

$$\frac{d\Psi}{d\lambda} = \left(\frac{A}{\lambda} + \frac{B}{\lambda - 1} + \frac{C}{\lambda - x}\right)\Psi \equiv \mathrm{A}(\lambda)\Psi,$$

with

$$\operatorname{tr} A = \operatorname{tr} B = \operatorname{tr} C = 0,$$

$$A + B + C = \operatorname{diag}(A + B + C).$$

Thus

$$\dim \mathcal{A} = 8$$

In this case it can be shown that

$$\dim \mathcal{M} = 7 = \dim \mathcal{A} - 1.$$

Hence, one expects a one-parameter family of equations, i.e., a curve in the space $\mathcal{A}$, with a given monodromy group $\mathfrak{M}$. This is in fact the point where the Painlevé equations appear. Indeed, if one writes (cf. [48]) the $(1, 2)$-th element of matrix $A(\lambda)$ as

$$A_{12}(\lambda) = \frac{w(\lambda - u)}{\lambda(\lambda - 1)(\lambda - x)}$$

then,

$$\mathfrak{m} \equiv \text{ const } \iff u = u(x) \text{ satisfies the PVI equation.}$$

The Riemann-Hilbert (RH) problem has a long and illustrious history. A comprehensive solution was obtained only relatively recently by A.A. Bolibruch [7]. We will limit our discussions to how the monodromy problem can be used to study the Painlevé equations. In order to put the other Painlevé equations into a similar context, we consider more general linear systems, namely systems with irregular singularities.

3.2.1 The Stokes phenomenon.

Let A(λ) be an $N \times N (N > 1)$ matrix-valued rational function of the complex variable λ. Consider in the complex λ-plane the first order linear ODE,

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi \tag{3.2.1}$$

for an $N \times N$ matrix-valued function $\Psi(\lambda)$. The local analytic theory of systems of linear ordinary differential equations (ODEs) with rational coefficients deals with the behavior of the solutions of equation (1.1.1) near a given point $\lambda_0 \in \mathbb{C}P^1$.

Let $\lambda_0 \in \mathbb{C}P^1$ be a non-Fuchsian singular point of equation (3.2.1), i.e., let equation (3.2.1) have the following local form

$$\frac{d\Psi}{d\xi} = \left(\sum_{k=-r-1}^{\infty} A_{k+1}\xi^k \right) \Psi, \quad \lambda \in D_{\lambda_0} \setminus \{\lambda_0\}, \quad r > 0, \tag{3.2.2}$$

Here ξ and D_{λ_0} are the local variable and the disk centered at λ_0 respectively :

$$\xi = \begin{cases} \lambda - \lambda_0, & \text{if } \lambda_0 \neq \infty \\ 1/\lambda, & \text{if } \lambda_0 = \infty \end{cases}$$

$$D_{\lambda_0} = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \rho\}, & 0 < \rho \leq \infty, \quad \text{if} \quad \lambda_0 \neq \infty, \\ \{\lambda \in \mathbb{C} : |\lambda| > \rho\} \cup \{\infty\}, & 0 \leq \rho < \infty, \quad \text{if} \quad \lambda_0 = \infty, \end{cases}$$

We assume that the coefficient A_{-r} of the leading order singularity in the Laurent series in (3.2.2) has distinct eigenvalues. This means that we only discuss the generic case when the Jordan form Λ_{-r} of the matrix A_{-r} is a diagonal matrix with distinct entries,

$$\begin{aligned} P^{-1}A_{-r}P = \Lambda_{-r}, \quad \det P \neq 0, \\ \Lambda_{-r} = \operatorname{diag}(\alpha_1, \ldots, \alpha_N), \quad \alpha_i \neq \alpha_j, \quad i \neq j. \end{aligned} \tag{3.2.3}$$

Under this generic conditions it was proven in [48, 96] that the formal solution of (3.2.1) exists and it is unique.

Proposition. *Equation (3.2.1) has a unique formal fundamental solution*

$$\begin{aligned}\Psi_f(\lambda) &= P\left(\sum_{k=0}^{\infty}\Psi_k\xi^k\right)\xi^{\Lambda_0}\exp\left\{\frac{\Lambda_{-r}}{-r}\xi^{-r}+\cdots+\frac{\Lambda_{-1}}{-1}\xi^{-1}\right\}\\ &\equiv P\left(\sum_{k=0}^{\infty}\Psi_k\xi^k\right)e^{\Lambda(\xi)}, \quad \Psi_0 = I,\end{aligned} \tag{3.2.4}$$

where

$$\Lambda(\xi) = \sum_{k=-r}^{-1}\frac{\Lambda_k}{k}\xi^k + \Lambda_0\ln\xi, \tag{3.2.5}$$

all the matrices $\Lambda_k, k = -r, \ldots, 0$, are diagonal, and the matrix Λ_{-r} is defined in (3.2.3). All the coefficients Ψ_k as well as the diagonal exponent $\Lambda(\xi)$ in (3.2.4) can be determined recursively, via the Laurent series (1.1.38), as polynomials of the matrix coefficients A_k, $k \geq -r$.

One may conjecture that the formal solution $\Psi_f(\lambda)$ (3.2.4) represents the asymptotics of a genuine fundamental solution of (3.2.1) as $\lambda \to \lambda_0$. This is indeed the case provided λ belongs to one of certain sectors of D_{λ_0}. In order to understand how these sectors can be singled out let us discuss the question of uniqueness of the genuine fundamental solution $\Psi(\lambda)$ assuming that

$$\Psi(\lambda) \sim \Psi_f(\lambda), \quad \lambda \to \lambda_0, \quad \lambda \in \Omega, \tag{3.2.6}$$

where the sector Ω is defined as

$$\Omega = \{\xi \in \mathbb{C} : \quad 0 < |\xi| < \rho, \quad \theta_1 < \arg\xi < \theta_2\}.$$

Suppose $\tilde{\Psi}(\lambda)$ is another solution of (3.2.1) satisfying the same asymptotic condition (3.2.6). The matrix functions $\Psi(\lambda)$ and $\tilde{\Psi}(\lambda)$ satisfy the same linear ODE; moreover, in virtue of the asymptotics (3.2.6) they are the fundamental solutions of this equation. Hence their matrix ratio

$$C \equiv \Psi^{-1}(\lambda)\tilde{\Psi}(\lambda),$$

is well defined and does not depend on λ. Thus

$$C = \lim_{\lambda \to \lambda_0} \Psi^{-1}(\lambda)\tilde{\Psi}(\lambda), \quad \lambda \in \Omega,$$

Since the asymptotic equation (1.1.45) is satisfied by both the Ψ-functions, equation (1.1.47) can be rewritten as

$$C = \lim_{\xi \to 0} e^{-\Lambda(\xi)}\{I + O(\xi)\}e^{\Lambda(\xi)}, \quad \xi \in \Omega,$$

or, componentwise,

$$C_{ij} = \lim_{\xi \to 0} e^{(\Lambda(\xi))_{jj} - (\Lambda(\xi))_{ii}} \left\{\delta_{ij} + O(\xi)\right\}, \quad \xi \in \Omega. \tag{3.2.7}$$

Hence in order to be able to conclude that $C = I$, which would imply the uniqueness of the solution (3.2.6), we need the sector Ω to contain at least one ray

$$\mathrm{Re}\left\{(\alpha_j - \alpha_i)\,\xi^{-r}\right\} = 0.$$

for each pair $(i, j), i < j$. Indeed, in this case for each pair $(i, j), i \neq j$, we can choose a path terminating at $\xi = 0$ along which

$$\mathrm{Re}\left\{(\Lambda(\xi))_{jj} - (\Lambda(\xi))_{ii}\right\} < 0,$$

and hence the exponential term in (3.2.7) decays. Assuming that ξ in (3.2.7) belongs to this path, we have that

$$C_{ij} = 0, \quad \forall i \neq j.$$

This, together with the equations

$$C_{jj} = 1, \quad \forall j = 1, \ldots, N,$$

(which follow from (3.2.7)), yield

$$C = I.$$

We can summarize the observations above in the following theorem.

Theorem. *Let $A(\lambda)$ be a $N \times N$ matrix-valued function holomorphic in the punctured disk $D_{\lambda_0} \backslash \{\lambda_0\}$, and let $\lambda_0 \in \mathbb{CP}^1$ be a multiple pole of $A(\lambda)d\lambda$ of order $r+1, r>0$*

$$\mathrm{A}(\lambda)d\lambda = \left(\sum_{k=-r-1}^{\infty} A_{k+1}\xi^k\right) d\xi, \quad \lambda \in D_{\lambda_0} \backslash \{\lambda_0\}\,,$$

$$r>0, \quad A_{-r} \neq 0.$$

Suppose that the leading coefficient A_{-r} of the Laurent series (1.1.53) has distinct eigenvalues (generic case). Suppose also that the sector

$$\Omega = \{\xi \in \mathbb{C}: \quad 0<|\xi|<\rho, \quad \theta_1 < \arg \xi < \theta_2\} \subset D_{\lambda_0}$$

is a Stokes sector at the point λ_0. Then in Ω there exists a unique fundamental solution $\Psi(\lambda)$ of equation (3.2.1) satisfying the asymptotic condition,

$$\Psi(\lambda) \sim \Psi_f(\lambda), \quad \lambda \to \lambda_0, \quad \lambda \in \Omega,$$

where $\Psi_f(\lambda)$ is the formal series (3.2.2), and the principal branch of $\ln \xi$ in (3.2.5) is chosen.

3.2.2 Isomonodromic deformations and Painlevé II

We start with the system with only one irregular singularity of order 3 at $\lambda = \infty$:

$$\frac{d\Psi}{d\lambda} = \mathrm{A}(\lambda)\Psi, \quad \mathrm{A}(\lambda) = A_3\lambda^2 + A_2\lambda + A_1, \quad A_i \in \mathrm{Mat}(2,\mathbb{C}).$$

In the generic situation one can always reduce this system to the normal form

$$A(\lambda) = -4i\lambda^2\sigma_3 + 4i\lambda \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} + \begin{pmatrix} a & -2w \\ -2z & -a \end{pmatrix}, \tag{3.2.8}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The corresponding canonical solutions $\Psi_k(\lambda)$ (see Section 3.2.1) in the neighborhood of $\lambda = \infty$ are characterized by their asymptotics

$$\Psi_k(\lambda) = (I + O(1/\lambda)) \exp\left\{-\frac{4}{3} i\lambda^3\sigma_3 - ix\lambda\sigma_3 - \nu \log \lambda\sigma_3\right\},$$
$$\frac{k-2}{3}\pi < \arg\lambda < \frac{\pi k}{3}, \quad \lambda \to \infty,$$
$$k = 1, \ldots, 7,$$

where

$$x = ia - 2uv, \quad \nu = vw - uz.$$

The monodromy data $\mathfrak{m}$ are given by the parameter ν and the set of Stokes matrices S_k defined as

$$\mathrm{S}_k = \Psi_k^{-1}\Psi_{k+1}, \quad k = 1, \ldots, 6.$$

These matrices have a triangular structure,

$$\mathrm{S}_{2l} = \begin{pmatrix} 1 & s_{2l} \\ 0 & 1 \end{pmatrix}, \quad \mathrm{S}_{2l+1} = \begin{pmatrix} 1 & 0 \\ s_{2l+1} & 1 \end{pmatrix},$$

and satisfy the cyclic relation

$$\mathrm{S}_1\ \mathrm{S}_2 \ldots \mathrm{S}_6 = e^{-2\pi i\nu\sigma_3}.$$

Therefore,

$$\dim \mathcal{M} = 4 = \dim \mathcal{A} - 1,$$

and again we expect nontrivial isomonodromy deformations. In fact (cf. [35], [48]), if one takes the quantity x as a parameter for the isomonodromy curve, so that u, v, z and w become functions of x, then the condition $\mathfrak{m} \equiv$ const. is equivalent to the system

$$\begin{cases} w &= u_x, \quad u_{xx} - xu - 2u^2v = 0, \\ z &= v_x, \quad v_{xx} - xv - 2v^2u = 0. \end{cases} \tag{3.2.9}$$

Similarly to the Fuchsian case , the proof of (3.2.9) is based on the fact that the x-independence of the Stokes matrices $S_1, \dots, S_6$, implies that the function $\Psi \equiv \Psi_1$ satisfies the system

$$\frac{\partial \Psi}{\partial \lambda} = A(\lambda)\Psi, \qquad (3.2.10)$$
$$\frac{\partial \Psi}{\partial x} = U(\lambda)\Psi,$$

where the matrix $A(\lambda)$ is the one from (3.2.8)

$$U(\lambda) = -i\lambda\sigma_3 + i\lambda \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}.$$

This time it is straightforward to show that the associated zero-curvature relation,

$$A_x(\lambda) - U_\lambda(\lambda) + [A(\lambda), U(\lambda)] = 0 \quad \text{(identically in } \lambda\text{)},$$

is equivalent to system (3.2.9). Under the reduction,

$$u = v,$$

system (3.2.9) becomes a particular case ($\alpha = 0$) of the second Painlevé equation:

$$u_{xx} - xu - 2u^3 = 0.$$

3.2.3 Riemann-Hilbert problem and asymptotics

We have seen in the previous section that to obtain the second Painlevé equation:

$$u_{xx} - xu - 2u^3 = 0. \qquad (3.2.11)$$

we have to consider the reduction $u = v$. This condition implies the restrictions

$$s_{k+3} = -s_k, \quad \nu = 0,$$

so that $\mathcal{M} = \{(s_1, s_2, s_3) : s_2 - s_1 - s_3 - s_1 s_2 s_3 = 0\}$ and hence

$$\dim \mathcal{M} = 2.$$

This means that for generic (s_1, s_2, s_3), one can take s_1 and s_3 as independent coordinates on $\mathcal{M}$.

Reversing the arguments used earlier relating this equation to the isomonodromy deformations of the system (3.2.10), we can now say that the monodromy data,

$$s_1 = s_1\left(x, u, u_x\right), \quad s_3 = s_3\left(x, u, u_x\right),$$

are the first integrals of equation (3.2.11). Formally this means that we have solved equation (3.2.11)! Indeed, we have found a complete set of independent first integrals. However, since these integrals are not explicit, the important question is: Can we make this fact an efficient tool for the study of the PII equation? It turns out that the answer is positive.

The starting point of this approach is, following ideas of Birkhoff [6], to consider the basic monodromy relation, $\mathrm{S}_k = \Psi_k^{-1}\Psi_{k+1}$, as a jump condition for the sectionally analytic function $Y(\lambda)$:

$$Y(\lambda) = \Psi_k(\lambda)e^{\theta(\lambda)\sigma_3} \equiv Y_k(\lambda), \quad \theta(\lambda) = \frac{4i}{3}\lambda^3 + ix\lambda,$$
$$\frac{\pi}{3}(k-2) + \frac{\pi}{6} \leq \arg\lambda \leq \frac{\pi}{3}k - \frac{\pi}{6}, \quad k = 1, \ldots, 6,$$

on the anti-Stokes rays, $\arg\lambda = \pi k/3 - \pi/6, k = 1, \ldots, 6$. This gives rise to a special type of oscillatory Riemann-Hilbert problem, which is now understood as a factorization or jump problem:

$$\begin{gathered} Y_{k+1}(\lambda) = Y_k(\lambda)e^{-\theta(\lambda)\sigma_3}S_k e^{\theta(\lambda)\sigma_3}, \\ \arg\lambda = \frac{\pi k}{3} - \frac{\pi}{6}, \quad k = 1, \ldots, 6, \\ \lim_{\lambda\to\infty} Y(\lambda) = I. \end{gathered} \tag{3.2.12}$$

(The last normalization equation is actually the reason for the introduction of the function $Y(\lambda)$.

The problem (3.2.12) is depicted in Figure 3.2. The corresponding solution $u(x, s_1, s_3)$ of the second Painlevé equation (3.2.11) can be determined from $Y(\lambda)$ by the equation

$$u(x; s_1, s_3) = 2 \lim_{\lambda\to\infty} (\lambda Y_{12}(\lambda)) .$$

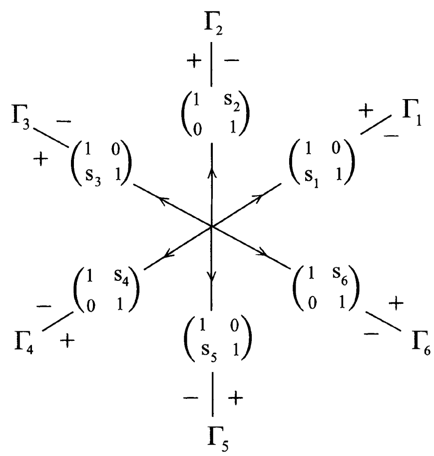

Figure 3.2. The contour and the jump matrices for Painlevé II

The problem (3.2.12), as well as the Fuchsian inverse monodromy problem , are particular cases of a Riemann-Hilbert factorization problem. This problem involves finding an analytic (matrix valued) function having prescribed jumps across a given contour. Following the tradition developed in mathematical physics, it is such more general factorization problems (and not just the inverse monodromy Fuchsian and/or the Birkhoff problems) that we will call *Riemann-Hilbert problems*.

The investigation of the Cauchy problem for the Painlevé II equation based on the analysis of the RH problem (3.2.12), as well as the investigation of its discontinuous generalization for $\alpha \neq 0$ (complete Painlevé II equation) was studied in [39, 40].

The last important step was taken in the works of P. Deift and X. Zhou [21], [20], who introduced an elegant scheme of the asymptotic analysis of this type of oscillatory factorization problems; their method can be thought of as a nonlinear steepest descent method.

The asymptotic method of Deift and Zhou, in complete analogy with the classical method, exploiting the analytic structure of the relevant jump matrices deforms the original contours of the RH problem to contours where the relevant oscillatory factors become exponentially small as $x \to \infty$. In this way, the original RH problem reduces to a collection of local RH problems associated with the relevant saddle points. However, the noncommutative nature of the RH problem requires the development of several totally new and rather sophisticated technical ideas, which, in particular, allow us to solve the local RH problems in closed form. The remarkable fact is that the final result of the analysis is as efficient as the asymptotic evaluation of the classical oscillatory integrals.

In the next section we will apply this technique to obtain the asymptotics of the negative tt^* equation.

3.3 The monodromy data

First we start with the Stokes phenomena for the negative tt^* equation (3.1.5). Recall that we have the Lax pair representation for this equation given by (3.1.4).

Notice that the first equation of (3.1.4) implies that both $\zeta = 0$ and $\zeta = \infty$ are the irregular singularities of Poincaré rank 1 and so the monodromy data consists of Stokes matrices defined around these points and connection matrices which relates solutions near 0 with ones near ∞.

Using the general monodromy theory of ODEs described in Chapter 1 of [41], we shall give the Stokes data for (3.1.4) at $\zeta = 0$ and $\zeta = \infty$ with the connection matrices.

3.3.1 Formal solutions

First, we will find the formal solution at $\zeta = 0$. We start by checking if the coefficient matrix of the leading term of the Lax pair given in (3.1.4),

$$\Psi_\zeta = \left(-\frac{1}{\zeta^2} W - \frac{x}{\zeta} w_x - x^2 W^T \right) \Psi, \tag{3.3.1}$$

is diagonalizable. By direct computation, all eigenvalues of W are distinct and they are $1,\ \omega,\ \omega^2$ where $\omega = e^{i\frac{2\pi}{3}}$. So, one can obtain a non-singular matrix P_0 which diagonalizes W into d_3, where

$$P_0 = \begin{pmatrix} e^{-w_0} & e^{-w_0} & e^{-w_0} \\ 1 & \omega & \omega^2 \\ e^{w_0} & \omega^2 e^{w_0} & \omega e^{w_0} \end{pmatrix},\ d_3 = diag(1, \omega, \omega^2),$$

and they satisfy $W P_0 = P_0 d_3$. We can further write P_0 by decomposing it as

$$P_0 = \begin{pmatrix} e^{-w_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{w_0} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = e^{-w} \Omega,$$

where we defined

$$\Omega = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

By Proposition 1.1 of [41], one can obtain a unique formal solution of (3.3.1) at $\zeta = 0$ of the form

$$\Psi_f^{(0)}(\zeta) = P_0(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3}. \tag{3.3.2}$$

Note that the formal monodromy exponent is 0 in this case.

Similarly, we can obtain the formal solution at $\zeta = \infty$. Let $\zeta \mapsto \frac{1}{\zeta}$. Then, (3.3.1) turns out to be

$$\Psi_\zeta = \left(\frac{x^2}{\zeta^2} W^T + \frac{x}{\zeta} w_x + W \right) \Psi.$$

The coefficient matrix of the leading term is $x^2 W^T$, and W^T is diagonalized as follows:

$$W^T P_\infty = P_\infty d_3$$

where $P_\infty = e^w \Omega^{-1}$. In virtue of Proposition 1.1 of [41], one can obtain a unique formal solution of (3.3.1) at $\zeta = \infty$ of the form

$$\Psi_f^{(\infty)}(\zeta) = P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}. \tag{3.3.3}$$

Note that the formal monodromy exponent is 0 in this case also.

Note. Since $\omega^2 + \omega + 1 = 0$, the product of Ω with itself can be expressed by

$$\Omega^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} =: 3C,$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We also note that $C^2 = I$. Thus, we have

$$\Omega^{-1} = \frac{1}{3}C\Omega = \frac{1}{3}\Omega C. \tag{3.3.4}$$

3.3.2 Stokes sectors

Next, we shall give some details of Stokes rays and Stokes sectors at $\zeta = 0$. As we saw in the previous subsection, eigenvalues of $-W$ are -1, $-\omega$, and $-\omega^2$. Recall that Stokes sectors are defined so that the fundamental solution is uniquely determined there by the asymptotic solution $\Psi_f^{(0)}$. In other words, when two fundamental solutions $\Psi^{(0)}$ and $\tilde{\Psi}^{(0)}$ with the same asymptotic condition are given, the following matrix C tends to the identity matrix I:

$$\begin{aligned} C &= \Psi^{(0)}(\zeta)^{-1}\tilde{\Psi}^{(0)}(\zeta) \sim e^{-\frac{1}{\zeta}d_3}(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3} \\ &= \begin{pmatrix} 1+* & *e^{-\frac{1}{\zeta}(1-\omega)} & *e^{-\frac{1}{\zeta}(1-\omega^2)} \\ *e^{\frac{1}{\zeta}(1-\omega)} & 1+* & *e^{-\frac{1}{\zeta}(\omega-\omega^2)} \\ *e^{\frac{1}{\zeta}(1-\omega^2)} & *e^{\frac{1}{\zeta}(\omega-\omega^2)} & 1+* \end{pmatrix} \to I \quad \text{as } \zeta \to 0, \end{aligned}$$

if ζ is in a Stokes sector. Thus the Stokes rays at $\zeta = 0$ should be the following:

$$\begin{aligned} l_n^{(1,2)} &= \left\{ \zeta \in \mathbb{C} \cup \{\infty\} \;\middle|\; \arg\zeta = -\frac{2\pi}{3} - n\pi \right\}, \\ l_n^{(1,3)} &= \left\{ \zeta \in \mathbb{C} \cup \{\infty\} \;\middle|\; \arg\zeta = -\frac{\pi}{3} - n\pi \right\}, \\ l_n^{(2,3)} &= \left\{ \zeta \in \mathbb{C} \cup \{\infty\} \;\middle|\; \arg\zeta = -n\pi \right\}. \end{aligned}$$

Recall that the Stokes sectors contain exactly one Stokes ray for each superscript (i, j) with $i < j$. Thus, one can take the following Stokes sectors:

$$\Omega_1^{(0)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, -\frac{2\pi}{3} < \arg\zeta < \frac{2\pi}{3}\right\},$$
$$\Omega_2^{(0)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, -\frac{5\pi}{3} < \arg\zeta < -\frac{\pi}{3}\right\},$$
$$\Omega_3^{(0)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, -\frac{8\pi}{3} < \arg\zeta < -\frac{4\pi}{3}\right\} = \Omega_1^{(0)}(\zeta e^{2\pi i}).$$

So, in principle, we have only two distinct Stokes sectors at $\zeta = 0$.

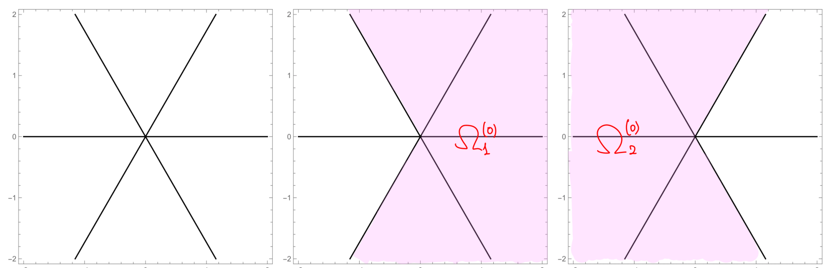

Figure 3.3. Stokes rays and Stokes sectors at $\zeta = 0$.

In a similar way, we can define the Stokes sectors at $\zeta = \infty$. First, we need Stokes rays at $\zeta = \infty$:

$$l_n^{(1,2)} = \left\{\zeta \in \mathbb{C} \cup \{\infty\} \,\middle|\, \arg\zeta = \frac{2\pi}{3} + n\pi\right\},$$
$$l_n^{(1,3)} = \left\{\zeta \in \mathbb{C} \cup \{\infty\} \,\middle|\, \arg\zeta = \frac{\pi}{3} + n\pi\right\},$$
$$l_n^{(2,3)} = \left\{\zeta \in \mathbb{C} \cup \{\infty\} \,\middle|\, \arg\zeta = n\pi\right\}.$$

Then, one can take the following Stokes sectors so that they contain exactly one Stokes ray for each superscript (i, j) with $i < j$.

$$\Omega_1^{(\infty)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, -\frac{2\pi}{3} < \arg\zeta < \frac{2\pi}{3}\right\},$$
$$\Omega_2^{(\infty)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, \frac{\pi}{3} < \arg\zeta < \frac{5\pi}{3}\right\},$$
$$\Omega_3^{(\infty)}(\zeta) = \left\{\zeta \in \mathbb{C}^* \,\middle|\, \frac{4\pi}{3} < \arg\zeta < \frac{8\pi}{3}\right\} = \Omega_1^{(\infty)}(\zeta e^{-2\pi i}).$$

Moreover, we observe that

$$\Omega_n^{(0)}(\zeta) = \Omega_n^{(\infty)}(\zeta^{-1}) \text{ for } n \in \mathbf{Z}. \tag{3.3.5}$$

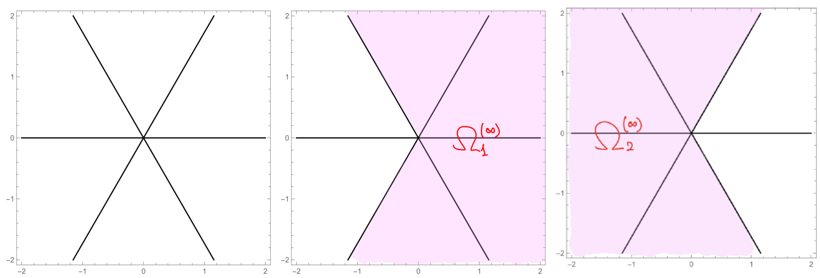

Figure 3.4. Stokes rays and Stokes sectors at $\zeta = \infty$.

3.3.3 Stokes matrices

As a step toward the Stokes matrices, we shall define canonical solutions. According to Theorem 1.4 in [41], in each sector $\Omega_n^{(\infty,0)}$ for $n = 1, 2, 3$, there exists a unique solution $\Psi_n^{(\infty,0)}$ of (3.3.1) satisfying the asymptotic condition,

$$\Psi_n^{(\infty,0)}(\zeta) \sim \Psi_f^{(\infty,0)}(\zeta),\ \zeta \to \infty, 0,\ \zeta \in \Omega_n^{(\infty,0)},\ n = 1, 2, 3.$$

Moreover, we have

$$\Psi_3^{(\infty)}(\zeta) = \Psi_1^{(\infty)}(\zeta e^{-2\pi i}) \text{ and } \Psi_3^{(0)}(\zeta) = \Psi_1^{(0)}(\zeta e^{2\pi i}).$$

So, such solutions, $\Psi_n^{(\infty)}$ and $\Psi_n^{(0)}$, have analytic continuations to the universal covering $\tilde{\mathbb{C}}^*$, and we continue to denote them by $\Psi_n^{(\infty)}$ and $\Psi_n^{(0)}$ for the extended solution.

Stokes matrices are defined by

$$S_n^{(\infty,0)} = [\Psi_n^{(\infty,0)}(\zeta)]^{-1}\Psi_{n+1}^{(\infty,0)}(\zeta) \tag{3.3.6}$$

for $n = 1, 2$.

Lemma 3.3.1. Stokes matrices admit the following structure,

$$S_1^{(\infty)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}, \; S_2^{(\infty)} = \begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$
$$S_1^{(0)} = \begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \; S_2^{(0)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

Proof. We will prove here for $S_1^{(\infty)}$ only. The same approach can be applied to determine the structure of other Stokes matrices.

We start with $\Psi_{n+1}^{(\infty)}\Psi_{n+1}^{(\infty)-1} = I$. By definition of Stokes matrices (3.3.6), it follows that $\Psi_n^{(\infty)}S_n^{(\infty)}\Psi_{n+1}^{(\infty)-1} = I$. For $\zeta \in \Omega_n^{(\infty)} \cap \Omega_{n+1}^{(\infty)}$, it can be further written as

$$\begin{aligned}
&\lim_{\zeta\to\infty} \Psi_f^{(\infty)}S_n^{(\infty)}\Psi_f^{(\infty)-1} = I \\
&\Longrightarrow \lim_{\zeta\to\infty} P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}S_n^{(\infty)}e^{x^2\zeta d_3}(I + \mathcal{O}(\zeta^{-1}))^{-1}P_\infty^{-1} = I \\
&\Longrightarrow \lim_{\zeta\to\infty} e^{-x^2\zeta d_3}S_n^{(\infty)}e^{x^2\zeta d_3} = I.
\end{aligned}$$

Therefore, if we write

$$S_n^{(\infty)} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

then we should have

$$\lim_{\zeta\to\infty} \begin{pmatrix} * & *e^{-x^2\zeta(1-\omega)} & *e^{-x^2\zeta(1-\omega^2)} \\ *e^{x^2\zeta(1-\omega)} & * & *e^{-x^2\zeta(\omega-\omega^2)} \\ *e^{x^2\zeta(1-\omega^2)} & *e^{x^2\zeta(\omega-\omega^2)} & * \end{pmatrix} = I, \tag{3.3.7}$$

for $\zeta \in \Omega_n^{(\infty)} \cap \Omega_{n+1}^{(\infty)}$.

Suppose $n = 1$. Then, ζ belongs to $\Omega_1^{(\infty)} \cap \Omega_2^{(\infty)}$, which means

$$\zeta \in \left\{ \zeta \in \mathbb{C} \cup \{\infty\} \,\middle|\, \frac{\pi}{3} < \arg \zeta < \frac{2\pi}{3} \right\}.$$

And it follows that

$$\begin{aligned} &\frac{\pi}{6} < \arg \zeta(1-\omega) < \frac{\pi}{2}, \\ &\frac{\pi}{2} < \arg \zeta(1-\omega^2) < \frac{5\pi}{6}, \\ &\frac{5\pi}{6} < \arg \zeta(\omega-\omega^2) < \frac{7\pi}{6}, \end{aligned}$$

which implies that equality (3.3.7) holds if $S_1^{(\infty)}$ has the following structure

$$S_1^{(\infty)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

□

Our equation (3.3.1) has many symmetries , that we describe in the next section. These symmetries allow to reduce the number of the parameters (Stokes data) that are needed to parameterize Stokes matrices.

3.3.4 Anti-symmetry relations

Anti-symmetry relation at $\zeta = \infty$

Let $A(\zeta)$ be the coefficient matrix of (3.3.1):

$$A(\zeta) = -\frac{1}{\zeta^2} W - \frac{x}{\zeta} w_x - x^2 W^T.$$

Then, one can observe the following symmetry equation:

$$\Delta A^T(-\zeta)\Delta = A(\zeta), \tag{3.3.8}$$

where

$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Moreover, (3.3.8) implies the following lemma:

Lemma 3.3.2. $\Psi(\zeta)$ is a solution of (3.3.1), then so is $\tilde{\Psi}(\zeta) := \Delta\Psi^{T-1}(-\zeta)$.

One can expect that the asymptotics of $\tilde{\Psi}(\zeta)$ as $\zeta \to \infty$ will be represented by the formal solution. But when we take into account (3.3.3), it turns out some normalization is needed, because it holds that

$$\Delta P_\infty^{T-1}\frac{d_3}{3} = P_\infty.$$

This leads the Anti-symmetry relation for the formal solution:

$$\Delta[\Psi_f^{(\infty)}(-\zeta)]^{T-1}\frac{d_3}{3} = \Psi_f^{(\infty)}(\zeta). \tag{3.3.9}$$

Indeed,

$$\begin{aligned}
\Delta[\Psi_f^{(\infty)}(-\zeta)]^{T-1}\frac{d_3}{3} &= \Delta[P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{x^2\zeta d_3}]^{T-1}\frac{d_3}{3} \\
&= \Delta P_\infty^{T-1}(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}\frac{d_3}{3} \\
&= \Delta P_\infty^{T-1}\frac{d_3}{3}(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3} \\
&= P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3} \\
&= \Psi_f^{(\infty)}(\zeta).
\end{aligned}$$

Moreover, considering the Stokes sectors of each canonical solution (see Figure 3.4), we observe that (3.3.9) implies the Anti-symmetry of canonical solutions at $\zeta = \infty$:

$$\begin{aligned} \Delta[\Psi_2^{(\infty)}(e^{i\pi}\zeta)]^{T-1}\frac{d_3}{3} &= \Psi_1^{(\infty)}(\zeta) \\ \Delta[\Psi_3^{(\infty)}(e^{i\pi}\zeta)]^{T-1}\frac{d_3}{3} &= \Psi_2^{(\infty)}(\zeta) \end{aligned} \tag{3.3.10}$$

In terms of Stokes matrices, they can be interpreted as follows:

$$S_2^{(\infty)} = d_3^{-1}[S_1^{(\infty)}]^{T-1}d_3. \tag{3.3.11}$$

The proof is straightforward:

$$\begin{aligned} S_1^{(\infty)} &= [\Psi_1^{(\infty)}]^{-1}[\Psi_2^{(\infty)}] = 3d_3^{-1}[\Psi_2^{(\infty)}]^T\Delta\Delta[\Psi_3^{(\infty)}]^{T-1}\frac{d_3}{3} \text{ by (3.3.10)} \\ &= d_3^{-1}(\Psi_3^{(\infty)-1}\Psi_2^{(\infty)})^T d_3 = d_3^{-1}[S_2^{(\infty)}]^{T-1}d_3. \\ \Leftrightarrow S_2^{(\infty)} &= d_3^{-1}[S_1^{(\infty)}]^{T-1}d_3. \end{aligned}$$

Anti-symmetry relation at $\zeta = 0$

Furthermore, $A(\zeta)$ admits another symmetry relation,

$$\frac{1}{x^2\zeta^2}\Delta A\left(-\frac{1}{x^2\zeta}\right)\Delta = A(\zeta).$$

Lemma 3.3.3. If $\Psi(\zeta)$ is a solution of (3.3.1), then so is $\tilde{\Psi}(\zeta) := \Delta\Psi\left(-\frac{1}{x^2\zeta}\right)$.

Again, in the limit of $\zeta \to \infty$, this solution $\tilde{\Psi}(\zeta)$ with some normalization is asymptotically represented by the formal solution:

$$\Delta\Psi_f^{(\infty)}\left(-\frac{1}{x^2\zeta}\right)3d_3^{-1} = \Psi_f^{(0)}(\zeta). \tag{3.3.12}$$

A version of canonical solutions follows from (3.3.12) and (3.3.5):

$$\Delta\Psi_{n+1}^{(\infty)}\left(e^{i\pi}\frac{1}{x^2\zeta}\right)3d_3^{-1} = \Psi_n^{(0)}(\zeta) \tag{3.3.13}$$

for $n = 1, 2$. Thus, the Anti-symmetry relation of canonical solutions at $\zeta = 0$ is

$$\Psi_2^{(0)}(\zeta) = \Delta \left[\Psi_1^{(0)}(e^{i\pi}\zeta)\right]^{T-1} 3d_3^{-1}. \tag{3.3.14}$$

In terms of Stokes matrices, they can be interpreted as follows:

$$\begin{aligned} S_n^{(0)} &= d_3 S_{n+1}^{(\infty)} d_3^{-1}, \\ S_2^{(0)} &= d_3 [S_1^{(0)}]^{T-1} d_3^{-1}. \end{aligned} \tag{3.3.15}$$

3.3.5 Cyclic symmetry relations

The coefficient matrix $A(\zeta)$ has one more symmetry, namely

$$\omega d_3^{-1} A(\omega\zeta) d_3 = A(\zeta).$$

Using this symmetry, we obtain the next lemma:

Lemma 3.3.4. If $\Psi(\zeta)$ is a solution of (3.3.1), then so is $\tilde{\Psi}(\zeta) := d_3^{-1}\Psi(\omega\zeta)$.

Cyclic symmetry relations of the formal solutions

In the limit of $\zeta \to \infty$, we claim that this solution $\tilde{\Psi}(\zeta)$ with some normalization is asymptotically represented by the formal solution:

$$d_3^{-1}\Psi_f^{(\infty)}(\omega\zeta)\Pi = \Psi_f^{(\infty)}(\zeta), \tag{3.3.16}$$

where

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In order to prove this equation, we need to have

$$d_3^{-1} P_\infty \Pi = P_\infty \text{ and } \Pi^{-1} e^{x^2 \omega \zeta d_3} \Pi = e^{x^2 \zeta d_3},$$

which can be checked by the direct computation.

Similarly, in the limit of $\zeta \to 0$, we now claim that the solution $\tilde{\Psi}(\zeta)$ in Lemma 3.3.4 with the following normalization is asymptotically represented by the formal solution:

$$d_3^{-1} \Psi_f^{(0)}(\omega\zeta) \Pi^{-1} = \Psi_f^{(0)}(\zeta). \tag{3.3.17}$$

In order to prove this equation, we need to have

$$d_3^{-1} P_0 \Pi^{-1} = P_0 \text{ and } \Pi e^{\frac{1}{\omega\zeta} d_3} \Pi = e^{\frac{1}{\zeta} d_3},$$

which also can be checked by the direct computation.

Cyclic symmetry relations of the fundamental solutions

Using Anti-symmetry relations of the formal solutions we were able to deduce symmetries of the canonical solutions. But in the case of the cyclic symmetry relations it is not straightforward. To overcome this difficulty, we will define some rotated sectors of $\Omega_n^{(\infty,0)}$ and consider fundamental solutions defined on each sector. Then, for such fundamental solutions, we shall give the cyclic symmetry relations.

First, we work on the case of $\zeta = \infty$. For $n \in \frac{1}{3}\mathbf{Z}$, define

$$\Omega_{n+\frac{1}{3}}^{(\infty)} := e^{i\frac{\pi}{3}} \Omega_n^{(\infty)}.$$

Then, we call fundamental solutions of (3.3.1) defined on each sector having the asymptotic condition as $\zeta \to \infty$ as

$$\Psi_n^{(\infty)}(\zeta) \text{ on } \Omega_n^{(\infty)} \text{ for } n \in \frac{1}{3}\mathbf{Z}. \tag{3.3.18}$$

Paying attention to which sector ζ belongs to, we observe the following cyclic symmetry relations of (3.3.18) by (3.3.16):

$$d_3^{-1}\Psi^{(\infty)}_{n+\frac{2}{3}}(\omega\zeta)\Pi = \Psi^{(\infty)}_n(\zeta) \quad \text{on } \Omega^{(\infty)}_n \tag{3.3.19}$$

for $n \in \frac{1}{3}\mathbf{Z}$.

Next we consider $\zeta = 0$. As we checked in (3.3.5), Stokes sectors at $\zeta = 0$ and Stokes sectors at $\zeta = \infty$ are related by $\Omega^{(0)}_n(\zeta) = \Omega^{(\infty)}_n(\zeta^{-1})$ for $n \in \mathbf{Z}$. Similarly, we can define the sectors,

$$\Omega^{(0)}_n(\zeta) := \Omega^{(\infty)}_n(\zeta^{-1})$$

for $n \in \frac{1}{3}\mathbf{Z}$, so that fundamental solutions $\Psi^{(0)}_n(\zeta)$ defined on each sector $\Omega^{(\infty)}_n$ have the unique asymptotic condition to the formal solution $\Psi^{(0)}_f(\zeta)$. Then, the cyclic symmetry relations of such fundamental solutions follow by (3.3.17):

$$d_3^{-1}\Psi^{(0)}_{n-\frac{2}{3}}(\omega\zeta)\Pi^{-1} = \Psi^{(0)}_n(\zeta) \quad \text{on } \Omega^{(0)}_n \tag{3.3.20}$$

for $n \in \frac{1}{3}\mathbf{Z}$.

Cyclic symmetry relations of Jump matrices

Once we have the cyclic symmetry relations, (3.3.19) and (3.3.20), we would like to find the corresponding symmetries of the Stokes matrices. However, we have obtained the cyclic relations for *fundamental solutions* defined on rotated Stokes sectors, not for the canonical solutions. Thus, we begin with computing new jump matrices defined on the intersection of two sectors. We start with the case of $\zeta = \infty$.

Lemma 3.3.5. Let $Q_n^{(\infty)} := \Psi_n^{(\infty)-1}\Psi_{n+\frac{1}{3}}^{(\infty)}$ defined on $\Omega_n^{(\infty)} \cap \Omega_{n+\frac{1}{3}}^{(\infty)}$ for $n \in \frac{1}{3}\mathbf{Z}$. Then, jump matrices have the following structure,

$$Q_1^{(\infty)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_{1\frac{1}{3}}^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix},\ Q_{1\frac{2}{3}}^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix},$$
$$Q_2^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_{2\frac{1}{3}}^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},\ Q_{2\frac{2}{3}}^{(\infty)} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. We will prove here for $Q_1^{(\infty)}$ only. The same approach can be applied to determine the structure of other jump matrices.

Consider a jump matrix,

$$Q_1^{(\infty)} := \Psi_1^{(\infty)-1}\Psi_{1\frac{1}{3}}^{(\infty)} \text{ on } \Omega_1^{(\infty)} \cap \Omega_{1\frac{1}{3}}^{(\infty)} = \left\{-\frac{\pi}{3} < \arg\zeta < \frac{2\pi}{3}\right\}. \tag{3.3.21}$$

Apply (3.3.21) to the equation $\Psi_{1\frac{1}{3}}^{(\infty)}\Psi_{1\frac{1}{3}}^{(\infty)-1} = I$, and we have $\Psi_1^{(\infty)}Q_1^{(\infty)}\Psi_{1\frac{1}{3}}^{(\infty)-1} = I$. For $\zeta \in \Omega_1^{(\infty)} \cap \Omega_{1\frac{1}{3}}^{(\infty)}$, it can be further written as

$$\begin{aligned}
&\lim_{\zeta\to\infty} \Psi_f^{(\infty)}Q_1^{(\infty)}\Psi_f^{(\infty)-1} = I \\
&\Longrightarrow \lim_{\zeta\to\infty} P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}Q_1^{(\infty)}e^{x^2\zeta d_3}(I + \mathcal{O}(\zeta^{-1}))^{-1}P_\infty^{-1} = I \\
&\Longrightarrow \lim_{\zeta\to\infty} e^{-x^2\zeta d_3}Q_1^{(\infty)}e^{x^2\zeta d_3} = I.
\end{aligned}$$

Therefore, if we write

$$Q_1^{(\infty)} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

we should have

$$\lim_{\zeta\to\infty}\begin{pmatrix} * & *e^{-x^2\zeta(1-\omega)} & *e^{-x^2\zeta(1-\omega^2)} \\ *e^{x^2\zeta(1-\omega)} & * & *e^{-x^2\zeta(\omega-\omega^2)} \\ *e^{x^2\zeta(1-\omega^2)} & *e^{x^2\zeta(\omega-\omega^2)} & * \end{pmatrix} = I, \tag{3.3.22}$$

for $\zeta \in \Omega_1^{(\infty)} \cap \Omega_{1\frac{1}{3}}^{(\infty)}$. Since ζ satisfies $-\frac{\pi}{3} < \arg\zeta < \frac{2\pi}{3}$, it follows that

$$\begin{aligned} -\frac{\pi}{2} &< \arg\zeta(1-\omega) < \frac{\pi}{2} \\ -\frac{\pi}{6} &< \arg\zeta(1-\omega^2) < \frac{5\pi}{6} \\ \frac{\pi}{6} &< \arg\zeta(\omega-\omega^2) < \frac{7\pi}{6}, \end{aligned}$$

which implies that equality of (3.3.22) holds if $Q_1^{(\infty)}$ has the following structure

$$Q_1^{(\infty)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

The cyclic symmetry relations of jump matrices $Q_n^{(\infty)}$ for $n \in \frac{1}{3}\mathbf{Z}$ follows from (3.3.19). For example, for $\zeta \in \Omega_1^{(\infty)} \cap \Omega_{1\frac{1}{3}}^{(\infty)}$, we have

$$\begin{aligned} Q_1^{(\infty)} &= \Psi_1^{(\infty)-1}(\zeta)\Psi_{1\frac{1}{3}}^{(\infty)}(\zeta) \\ &= (d_3^{-1}\Psi_{1\frac{2}{3}}^{(\infty)}(\omega\zeta)\Pi)^{-1} d_3^{-1}\Psi_2^{(\infty)}(\omega\zeta)\Pi \\ &= \Pi^{-1}\Psi_{1\frac{2}{3}}^{(\infty)-1}(\omega\zeta)\Psi_2^{(\infty)}(\omega\zeta)\Pi \\ &= \Pi^{-1}Q_{1\frac{2}{3}}^{(\infty)}\Pi \end{aligned}$$

Similarly, for any $n \in \frac{1}{3}\mathbf{Z}$, it holds that

$$Q_n^{(\infty)} = \Pi^{-1}Q_{n+\frac{2}{3}}^{(\infty)}\Pi. \tag{3.3.23}$$

Moreover, Stokes matrices $S_1^{(\infty)}$ and $S_2^{(\infty)}$ can be expressed using $Q_n^{(\infty)}$'s for $n \in \frac{1}{3}\mathbf{Z}$.

Proposition 3.3.6. $S_n^{(\infty)} = Q_n^{(\infty)} Q_{n+\frac{1}{3}}^{(\infty)} Q_{n+\frac{2}{3}}^{(\infty)}$ for $n = 1, 2$.

Proof. For $\zeta \in \Omega_1^{(\infty)} \cap \Omega_2^{(\infty)}$, we have

$$\Psi_2^{(\infty)} = \Psi_{1\frac{2}{3}}^{(\infty)} Q_{1\frac{2}{3}}^{(\infty)} = \Psi_{1\frac{1}{3}}^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} Q_{1\frac{2}{3}}^{(\infty)} = \Psi_1^{(\infty)} Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} Q_{1\frac{2}{3}}^{(\infty)}.$$

Thus, the Stokes matrix $S_1^{(\infty)}$ is expressed as the product, $S_1^{(\infty)} = Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} Q_{1\frac{2}{3}}^{(\infty)}$. Similarly, we have $S_2^{(\infty)} = Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} Q_{2\frac{2}{3}}^{(\infty)}$. □

Repeating this discussion to Lemma 3.3.5 for $Q_n^{(0)} := \Psi_n^{(0)-1} \Psi_{n+\frac{1}{3}}^{(0)}$ which is defined on the intersection $\Omega_n^{(0)} \cap \Omega_{n+\frac{1}{3}}^{(0)}$, we can show that $Q_n^{(0)}$ has the same structure as $Q_{n+1}^{(\infty)}$ for each $n \in \frac{1}{3}\mathbf{Z}$. Then, we have a decomposition of $S_n^{(0)}$:

$$S_n^{(0)} = Q_n^{(0)} Q_{n+\frac{1}{3}}^{(0)} Q_{n+\frac{2}{3}}^{(0)} \tag{3.3.24}$$

for $n \in \frac{1}{3}\mathbf{Z}$. Moreover, making use of (3.3.20), it follows that

$$Q_n^{(0)} = \Pi^{-1} Q_{n+\frac{2}{3}}^{(0)} \Pi \tag{3.3.25}$$

for $n \in \frac{1}{3}\mathbf{Z}$.

3.3.6 Computation of Stokes matrices

So far, we got Anti-symmetry relations of Stokes matrices, (3.3.11) and (3.3.15), and cyclic symmetry relations of $Q_n^{(\infty,0)}$'s, (3.3.23) and (3.3.25), to parameterize Stokes matrices with the smallest number of parameters. Before doing actual computation, let us rephrase Anti-symmetry relations of the Stokes matrices in terms of $Q_n^{(\infty,0)}$ also.

$$\begin{aligned} Q_{n+1}^{(\infty)} &= d_3^{-1} [Q_n^{(\infty)}]^{T-1} d_3 \\ Q_{n+1}^{(0)} &= d_3 [Q_n^{(0)}]^{T-1} d_3^{-1} \\ Q_n^{(0)} &= d_3 Q_{n+1}^{(\infty)} d_3^{-1} \end{aligned} \tag{3.3.26}$$

for $n \in \frac{1}{3}\mathbf{Z}$. They can be proved in the similar way of getting (3.3.11) and (3.3.15) in the subsection 3.3.4.

In summary, we have the following symmetry equations:

- Anti-symmetry of formal solutions ((3.3.9), (3.3.12)):

$$\Delta[\Psi_f^{(\infty)}(-\zeta)]^{T-1}\frac{d_3}{3} = \Psi_f^{(\infty)}(\zeta)$$
$$\Delta\Psi_f^{(\infty)}\left(-\frac{1}{x^2\zeta}\right)3d_3^{-1} = \Psi_f^{(0)}(\zeta)$$
$$\Delta[\Psi_f^{(0)}(-\zeta)]^{T-1}3d_3^{-1} = \Psi_f^{(0)}(\zeta)$$

- Anti-symmetry of canonical solutions ((3.3.10), (3.3.13),(3.3.14)):

$$\Delta[\Psi_{n+1}^{(\infty)}(e^{i\pi}\zeta)]^{T-1}\frac{d_3}{3} = \Psi_n^{(\infty)}(\zeta)$$
$$\Delta\Psi_{n+1}^{(\infty)}\left(e^{i\pi}\frac{1}{x^2\zeta}\right)3d_3^{-1} = \Psi_n^{(0)}(\zeta)$$
$$\Delta[\Psi_{n-1}^{(0)}(e^{i\pi}\zeta)]^{T-1}3d_3^{-1} = \Psi_n^{(0)}(\zeta)$$

- Anti-symmetry of Jump matrices ((3.3.26)):

$$Q_{n+1}^{(\infty)} = d_3^{-1}[Q_n^{(\infty)}]^{T-1}d_3$$
$$Q_{n+1}^{(0)} = d_3[Q_n^{(0)}]^{T-1}d_3^{-1}$$
$$Q_n^{(0)} = d_3Q_{n+1}^{(\infty)}d_3^{-1}$$

- Cyclic symmetry of formal solutions ((3.3.16), (3.3.17)):

$$d_3^{-1}\Psi_f^{(\infty)}(\omega\zeta)\Pi = \Psi_f^{(\infty)}(\zeta)$$
$$d_3^{-1}\Psi_f^{(0)}(\omega\zeta)\Pi^{-1} = \Psi_f^{(0)}(\zeta)$$

- Cyclic symmetry of fundamental solutions ((3.3.19), (3.3.20)):

$$d_3^{-1}\Psi^{(\infty)}_{n+\frac{2}{3}}(\omega\zeta)\Pi = \Psi^{(\infty)}_n(\zeta)$$
$$d_3^{-1}\Psi^{(0)}_{n-\frac{2}{3}}(\omega\zeta)\Pi^{-1} = \Psi^{(0)}_n(\zeta)$$

- Cyclic symmetry of Jump matrices ((3.3.23), (3.3.25)):

$$Q^{(\infty)}_n = \Pi^{-1}Q^{(\infty)}_{n+\frac{2}{3}}\Pi$$
$$Q^{(0)}_n = \Pi^{-1}Q^{(0)}_{n+\frac{2}{3}}\Pi$$

Parametrization of $S^{(\infty)}_n$

Let us start with

$$Q_1^{(\infty)} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the Anti-symmetry relation, it follows that

$$Q_2^{(\infty)} = d_3^{-1}[Q_1^{(\infty)}]^{T-1}d_3 = \begin{pmatrix} 1 & 0 & 0 \\ -a\omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we use the cyclic symmetry to have

$$Q_{1\frac{1}{3}}^{(\infty)} = \Pi^{-1}Q_2^{(\infty)}\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a\omega^2 & 1 \end{pmatrix}.$$

Using the cyclic symmetry relation once more, we can express $Q^{(\infty)}_{1\frac{2}{3}}$ using a:

$$Q^{(\infty)}_{1\frac{2}{3}} = \Pi Q^{(\infty)}_{1} \Pi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}.$$

Proposition 3.3.6 implies that

$$S^{(\infty)}_{1} = Q^{(\infty)}_{1} Q^{(\infty)}_{1\frac{1}{3}} Q^{(\infty)}_{1\frac{2}{3}} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ a & -a\omega^2 & 1 \end{pmatrix}. \tag{3.3.27}$$

The parameterization of $Q^{(\infty)}_{1\frac{1}{3}}$ with a enables us to do more. Indeed,

$$Q^{(\infty)}_{2\frac{1}{3}} = d_3^{-1} [Q^{(\infty)}_{1\frac{1}{3}}]^{T-1} d_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$Q^{(\infty)}_{2\frac{2}{3}} = d_3^{-1} [Q^{(\infty)}_{1\frac{2}{3}}]^{T-1} d_3 = \begin{pmatrix} 1 & 0 & -a\omega^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in virtue of the Anti-symmetry. Moreover, Proposition 3.3.6 gives a parametrization of $S^{(\infty)}_{2}$ in terms of a:

$$S^{(\infty)}_{2} = Q^{(\infty)}_{2} Q^{(\infty)}_{2\frac{1}{3}} Q^{(\infty)}_{2\frac{2}{3}} = \begin{pmatrix} 1 & 0 & -a\omega^2 \\ -a\omega^2 & 1 & a^2\omega + a \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3.28}$$

Parametrization of $S_n^{(0)}$

By (3.3.26), we can translate all $Q_n^{(0)}$'s with a parameter a:

$$Q_1^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; Q_{1\frac{1}{3}}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a\omega^2 \\ 0 & 0 & 1 \end{pmatrix}, \; Q_{1\frac{2}{3}}^{(0)} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$Q_2^{(0)} = \begin{pmatrix} 1 & a\omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; Q_{2\frac{1}{3}}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a & 1 \end{pmatrix}, \; Q_{2\frac{2}{3}}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a\omega^2 & 0 & 1 \end{pmatrix}.$$

Since (3.3.24) tells us how to compute $S_1^{(0)}$ and $S_2^{(0)}$ using $Q_n^{(0)}$'s, we have

$$S_1^{(0)} = \begin{pmatrix} 1 & 0 & -a \\ -a & 1 & a^2 + a\omega^2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } S_2^{(0)} = \begin{pmatrix} 1 & a\omega^2 & 0 \\ 0 & 1 & 0 \\ a\omega^2 & -a & 1 \end{pmatrix}. \tag{3.3.29}$$

Therefore, we find that each Stokes matrix is parameterized by a single parameter a — Stokes data, as in (3.3.27), (3.3.28), and (3.3.29).

3.3.7 Connection matrices

In this section we describe the connection matrices E_k which connects solutions at $\zeta = \infty$ and solutions at $\zeta = 0$ by

$$\begin{aligned} \Psi_1^{(\infty)}(\zeta) &= \Psi_1^{(0)}(\zeta) E_1, \\ \Psi_2^{(\infty)}(\zeta) &= \Psi_2^{(0)}(\zeta e^{-2\pi i}) E_2, \end{aligned} \tag{3.3.30}$$

for all ζ in the universal covering $\tilde{\mathbb{C}}^*$. We claim that the connection matrix E_1 generates E_2 and this can be checked by upcoming lemma where we deal with several symmetry relations. As with the Stokes matrices, our goal is to understand how many parameters are needed to explicitly write E_1.

Lemma 3.3.7. $E_2 = S_1^{(0)-1} E_1 S_2^{(\infty)-1} = S_2^{(0)} E_1 S_1^{(\infty)} = \frac{1}{9} d_3 E_1^{T-1} d_3$.

Note. By the Anti-symmetry relations of the Stokes matrices, (3.3.11) and (3.3.15), one can write

$$S_2^{(\infty)-1} = d_3^{-1}[S_1^{(\infty)}]^T d_3 \quad \text{and} \quad S_1^{(0)-1} = d_3^{-1}[S_2^{(0)}]^T d_3,$$

so that the first symmetry of Lemma 3.3.7 can be written as follows too:

$$E_2 = d_3[S_2^{(0)}]^T d_3^{-1} E_1 d_3^{-1} [S_1^{(\infty)}]^T d_3.$$

Anti-symmetry relation of E_1

Let us call the symmetry in Lemma 3.3.7,

$$S_2^{(0)} E_1 S_1^{(\infty)} = \frac{1}{9} d_3 E_1^{T-1} d_3, \tag{3.3.31}$$

the *Anti-symmetry relation of* E_1. Using (3.3.15), we have

$$S_1^{(\infty)} d_3^{-1} E_1 S_1^{(\infty)} = \frac{1}{9} E_1^{T-1} d_3. \tag{3.3.32}$$

Taking a determinant of the both sides of equation (3.3.32), we get

$$\begin{aligned}
&\det S_1^{(\infty)} \det d_3^{-1} \det E_1 \det S_1^{(\infty)} = \left(\frac{1}{9}\right)^3 \det E_1^{T-1} \det d_3 \\
&\Leftrightarrow (\det E_1)^2 = \left(\frac{1}{3}\right)^6 \Leftrightarrow \det E_1 = \pm \left(\frac{1}{3}\right)^3
\end{aligned}$$

where we used $\det d_3^{\pm 1} = 1$ and $\det S_n^{(\infty)} = 1$. We can also determine which sign should appear in $\det E_1$. Taking a determinant at (3.3.30), we have

$$\det E_1 = \frac{\det \Psi_1^{(\infty)}}{\det \Psi_1^{(0)}}$$

so all we have to know is $\det \Psi_1^{(\infty,0)}$. But E_1 is independent of ζ so one can replace $\det \Psi_n^{(\infty,0)}$ by $\det \Psi_f^{(\infty,0)} = \det P_{\infty,0}$. Since

$$P_0 = P_\infty^{T-1} = e^w \Omega,$$

it follows that $\det P_0 = (\det P_\infty)^{-1} = \det \Omega = -i3\sqrt{3}$ by the direct computation. Thus,

$$\det E_1 = \frac{1}{(\det \Omega)^2} = -\frac{1}{27}.$$

Cyclic symmetry relation of E_1

By definition of the Jump matrices $Q_n^{(\infty,0)}$ for $n \in \frac{1}{3}\mathbf{Z}$ introduced in subsubsection 3.3.5, we have

$$\Psi_{1\frac{2}{3}}^{(\infty,0)}(\zeta) = \Psi_1^{(\infty,0)}(\zeta) Q_1^{(\infty,0)} Q_{1\frac{1}{3}}^{(\infty,0)}.$$

Using the cyclic symmetry relation, (3.3.25) and (3.3.20), it follows that

$$\Psi_1^{(\infty)} = d_3^{-1} \Psi_{1\frac{2}{3}}^{(\infty)}(\omega\zeta)\Pi = d_3^{-1} \Psi_1^{(\infty)}(\omega\zeta) Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi, \tag{3.3.33}$$

and

$$\Psi_1^{(0)} = \Psi_{1\frac{2}{3}}^{(0)}(\zeta) \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \right)^{-1} = d_3^{-1} \Psi_1^{(0)}(\omega\zeta) \Pi^{-1} \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \right)^{-1}. \tag{3.3.34}$$

Substituting (3.3.33) and (3.3.34) into (3.3.30), we have

$$\begin{aligned}
& d_3^{-1} \Psi_1^{(\infty)}(\omega\zeta) Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi = d_3^{-1} \Psi_1^{(0)}(\omega\zeta) \Pi^{-1} \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \right)^{-1} E_1 \\
\Longrightarrow\ & d_3^{-1} \Psi_1^{(0)}(\omega\zeta) E_1 Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi = d_3^{-1} \Psi_1^{(0)}(\omega\zeta) \Pi^{-1} \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \right)^{-1} E_1 \\
\Longrightarrow\ & E_1 Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi = \Pi^{-1} \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \right)^{-1} E_1.
\end{aligned}$$

Thus, we obtain

$$E_1 = \left(Q_1^{(0)} Q_{1\frac{1}{3}}^{(0)} \Pi\right) E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right). \tag{3.3.35}$$

Moreover, using the symmetry relation (3.3.26), equation (3.3.35) turns out to be

$$\begin{aligned} E_1 &= \left(d_3 Q_2^{(\infty)} d_3^{-1} d_3 Q_{2\frac{1}{3}}^{(\infty)} d_3^{-1} \Pi\right) E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right) \\ \Longrightarrow d_3^{-1} E_1 &= \left(Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} d_3^{-1} \Pi d_3\right) d_3^{-1} E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right). \end{aligned}$$

By direct computation, one can get $d_3^{-1} \Pi d_3 = \omega \Pi$. Hence, it follows that

$$d_3^{-1} E_1 = \omega \left(Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} \Pi\right) d_3^{-1} E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right). \tag{3.3.36}$$

We call this relation the *cyclic symmetry relation of* E_1.

In summary,

- Anti-symmetry relation of E_1 of the connection matrices ((3.3.31)):

$$S_2^{(0)} E_1 S_1^{(\infty)} = \frac{1}{9} d_3 E_1^{T-1} d_3$$

- Cyclic symmetry relation of the connection matrices ((3.3.36)):

$$d_3^{-1} E_1 = \omega \left(Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} \Pi\right) d_3^{-1} E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right)$$

3.3.8 Computation of the connection matrix E_1

Using the cyclic symmetry relation (3.3.36)we obtain

$$d_3^{-1} E_1 \left(Q_1^{(\infty)} Q_{1\frac{1}{3}}^{(\infty)} \Pi\right)^{-1} = \omega \left(Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} \Pi\right) d_3^{-1} E_1. \tag{3.3.37}$$

Let us parameterize the entries of the matrix E_1 by

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix},$$

then the LHS of (3.3.37) becomes

$$d_3^{-1} E_1 \Pi^{-1} Q_{1\frac{1}{3}}^{(\infty)-1} Q_1^{(\infty)-1} = \begin{pmatrix} B & a\omega^2 A - aB + C & A \\ \omega^2 E & a\omega D - a\omega^2 E + \omega^2 F & \omega^2 D \\ \omega H & aG - a\omega H + \omega I & \omega G \end{pmatrix},$$

and the RHS of (3.3.37) becomes

$$\omega \left(Q_2^{(\infty)} Q_{2\frac{1}{3}}^{(\infty)} \Pi \right) d_3^{-1} E_1 =$$
$$= \begin{pmatrix} D & E & F \\ -a\omega^2 D + \omega^2 G + a\omega A & -a\omega^2 E + \omega^2 H + a\omega B & -a\omega^2 F + \omega^2 I + a\omega C \\ \omega A & \omega B & \omega C \end{pmatrix}.$$

Therefore, by comparing each matrix, we obtain 9 entry-wise identity equations . They are reduced to the following six equations:

$$D = B,\ E = a\omega^2 A - aB + C,\ F = A,$$
$$H = A,\ I = aA + B - a\omega^2 C,\ G = C.$$

Thus the parametrization of E_1 can be reduced to 3 letters, $A,\ B$, and C:

$$E_1 = \begin{pmatrix} A & B & C \\ B & a\omega^2 A - aB + C & A \\ C & A & aA + B - a\omega^2 C \end{pmatrix}. \tag{3.3.38}$$

Next, rearranging the equation (3.3.32), we get

$$S_1^{(\infty)} d_3^{-1} E_1 S_1^{(\infty)} d_3^{-1} E_1^T = \frac{1}{9} I \Leftrightarrow \tilde{E}_1 \tilde{E}_1 = \frac{1}{9} I, \tag{3.3.39}$$

where we defined $\tilde{E}_1 := S_1^{(\infty)} d_3^{-1} E_1$ and used the fact that $E = E^T$. Let's use parametrization (3.3.27) of $S_1^{(\infty)}$ and (3.3.38) of E_1 to compute more about (3.3.39). In other words, by comparing each entry of $\tilde{E}_1^2 = \frac{1}{9} I$, we will obtain other identity equations. First, compute $\tilde{E}_1$:

$$\tilde{E}_1 = S_1^{(\infty)} d_3^{-1} E_1 = \begin{pmatrix} A + a\omega^2 B & B + a^2\omega A - a^2\omega^2 B + a\omega^2 C & C + a\omega^2 A \\ \omega^2 B & a\omega A - a\omega^2 B + \omega^2 C & \omega^2 A \\ aA - a\omega B + \omega C & aB - a^2 A + a^2 \omega B - a\omega C + \omega A & \omega B \end{pmatrix}.$$

Then, comparing $\tilde{E}_1{}^2$ with $\frac{1}{9} I$, we have

$$(\tilde{E}_1^2)_{3,3} = \frac{1}{9} \Leftrightarrow A^2 + \omega^2 B^2 + \omega C^2 + a\omega^2 AB - a\omega BC + aCA = \frac{1}{9} \tag{3.3.40}$$

and

$$(\tilde{E}_1^2)_{3,1} = 0 \Leftrightarrow aA^2 + AB + \omega^2 BC + \omega AC = 0. \tag{3.3.41}$$

Subtracting the $a\omega^2$ multiple of (3.3.41) from (3.3.40), we have

$$(1 - a^2\omega^2) A^2 + \omega^2 B^2 - 2a\omega BC + \omega C^2 = \frac{1}{9}. \tag{3.3.42}$$

Using (3.3.41) and (3.3.42), one can express B and C in terms of A respectively. For now, we will stop here, but after we talk about the reality condition next section, we shall give more detailed parameterization. So far, we figured out that the connection matrix E_1 is parameterized by two complex numbers: a and A. Recall $\{a\}$ was called the Stokes data. Together with one more parameter A, we call $\{a, A\}$ the *monodromy data*.

3.4 Reality condition

Recall that we consider the real solution $w_0(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. In terms of the first equation of the Lax pair (3.1.4)

$$\Psi_\zeta = \left(-\frac{1}{\zeta^2}W - \frac{x}{\zeta}w_x - x^2W^T\right)\Psi, \tag{3.4.1}$$

we should have that

$$A(\zeta) = -\frac{1}{\zeta^2}W - \frac{x}{\zeta}w_x - x^2W^T,$$

satisfies $\overline{A(\overline{\zeta})} = A(\zeta)$. This implies that if $\Psi(\zeta)$ is a solution of (3.4.1) then so is $\overline{\Psi(\overline{\zeta})}$. We will see how they are related in this section.

3.4.1 Reality condition of the formal solutions

Recall that the formal solution at $\zeta = \infty$ is given by (3.3.3), so that by taking complex conjugate we have

$$\begin{aligned}\overline{\Psi_f^{(\infty)}(\overline{\zeta})} &= \overline{P_\infty}(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta\overline{d_3}} \\ &= \overline{P_\infty}C(I + \mathcal{O}(\zeta^{-1}))Ce^{-x^2\zeta\overline{d_3}},\end{aligned} \tag{3.4.2}$$

where we used the property of $C^2 = I$. First, note that

$$\overline{d_3} = Cd_3C. \tag{3.4.3}$$

Also, since $P_\infty = e^w\Omega^{-1}$, we have

$$\overline{P_\infty} = e^w\frac{1}{3}\Omega \tag{3.4.4}$$

by (3.3.4). Furthermore,

$$\Omega\Omega = 3C \Leftrightarrow \frac{1}{3}\Omega = \Omega^{-1}C^{-1}.$$

Thus, together with (3.4.4), we obtain

$$\overline{P_\infty}C = e^{w}\Omega^{-1} = P_\infty. \tag{3.4.5}$$

Substitution (3.4.3) and (3.4.5) into (3.4.2), it follows that

$$\begin{aligned}\overline{\Psi_f^{(\infty)}(\overline{\zeta})} &= \overline{P_\infty}C(I+\mathcal{O}(\zeta^{-1}))Ce^{-x^2\zeta\overline{d_3}}\\ &= P_\infty(I+\mathcal{O}(\zeta^{-1}))CCe^{-x^2\zeta d_3}C\\ &= P_\infty(I+\mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}C.\end{aligned}$$

Therefore, we obtain the reality condition of the formal solutions:

$$\overline{\Psi_f^{(\infty)}(\overline{\zeta})}C = \Psi_f^{(\infty)}(\zeta).$$

Similarly, the formal solution at $\zeta = 0$ has the same property:

$$\overline{\Psi_f^{(0)}(\overline{\zeta})}C = \Psi_f^{(0)}(\zeta).$$

3.4.2 Reality condition of the Stokes matrices

In this section we discuss how the reality condition affects the Stokes matrices. Recall that the Stokes sectors for the canonical solution at $\zeta = \infty$ are given in the Subsection 3.3.2:

$$\begin{aligned}\Omega_1^{(\infty)} &= \left\{\zeta\in\mathbb{C}^* \;\middle|\; -\frac{2\pi}{3} < \arg\zeta < \frac{2\pi}{3}\right\},\\ \Omega_2^{(\infty)} &= \left\{\zeta\in\mathbb{C}^* \;\middle|\; \frac{\pi}{3} < \arg\zeta < \frac{5\pi}{3}\right\}.\end{aligned}$$

By taking complex conjugate of ζ, i.e. $\arg\overline{\zeta} = 2\pi - \arg\zeta$ above, Stokes sectors for the solution $\overline{\Psi_n^{(\infty)}(\overline{\zeta})}$ for $n = 1, 2$ are

$$\overline{\Omega_1^{(\infty)}} := \left\{\zeta \in \mathbb{C}^* \;\middle|\; \frac{4\pi}{3} < \arg\zeta < \frac{8\pi}{3}\right\},$$
$$\overline{\Omega_2^{(\infty)}} := \left\{\zeta \in \mathbb{C}^* \;\middle|\; \frac{\pi}{3} < \arg\zeta < \frac{5\pi}{3}\right\}.$$

Then, we have the relation

$$\overline{\Omega_1^{(\infty)}} = \Omega_3^{(\infty)},$$
$$\overline{\Omega_2^{(\infty)}} = \Omega_2^{(\infty)},$$

on the level of Stokes sectors and the version of the canonical solutions are

$$\overline{\Psi_1^{(\infty)}(\overline{\zeta})}C = \Psi_3^{(\infty)}(\zeta),$$
$$\overline{\Psi_2^{(\infty)}(\overline{\zeta})}C = \Psi_2^{(\infty)}(\zeta).$$

Thus the reality condition of the Stokes matrices are

$$\overline{S_1^{(\infty)}} = [\overline{\Psi_1^{(\infty)}(\overline{\zeta})}]^{-1}\overline{\Psi_2^{(\infty)}(\overline{\zeta})} = C[\Psi_3^{(\infty)}(\zeta)]^{-1}\Psi_2^{(\infty)}(\zeta)C \; = CS_2^{(\infty)-1}C \tag{3.4.6}$$

This can be interpreted as the reality condition of the Stokes data. Since $S_n^{(\infty)}$ for $n = 1, 2$ are parametrized in (3.3.27) and (3.3.28), we have

$$\overline{S_1^{(\infty)}} = \begin{pmatrix} 1 & \overline{a} & 0 \\ 0 & 1 & 0 \\ \overline{a} & -\overline{a}\omega & 1 \end{pmatrix} \quad \text{and} \quad S_2^{(\infty)-1} = \begin{pmatrix} 1 & 0 & a\omega^2 \\ a\omega^2 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, condition (3.4.6) implies $\overline{a} = a\omega^2$. Thus, one can write a as

$$\bar{a} = a\omega^2 \Longrightarrow a = \omega^2 s^{\mathbb{R}}. \tag{3.4.7}$$

for some real parameter $s^{\mathbb{R}}$.

3.4.3 Reality condition of the connection matrices

In this section we will find the reality condition for the connection matrix E_1. Let $\arg\overline{\zeta} = -\arg\zeta$ this time. Then, the Stokes sectors have the property,

$$\overline{\Omega_1^{(\infty)}} = \Omega_1^{(\infty)} \quad \text{and} \quad \overline{\Omega_1^{(0)}} = \Omega_1^{(0)},$$

which implies the reality condition of the canonical solutions,

$$\begin{aligned} \overline{\Psi_1^{(\infty)}(\overline{\zeta})}C &= \Psi_1^{(\infty)}(\zeta) \\ \overline{\Psi_1^{(0)}(\overline{\zeta})}C &= \Psi_1^{(0)}(\zeta). \end{aligned} \tag{3.4.8}$$

By (3.3.30) and (3.4.8), it follows that

$$E_1 = \left[\Psi_1^{(0)}\right]^{-1}\Psi_1^{(\infty)} = C\left[\overline{\Psi_1^{(0)}}\right]^{-1}\overline{\Psi_1^{(\infty)}}C = C\overline{E_1}C. \tag{3.4.9}$$

3.4.4 More about monodromy data

In the previous section, we parameterized E_1 with three complex parameters A, B, C and the Stokes multiplier a as in (3.3.38) that satisfy two identities, (3.3.40) and (3.3.41). In virtue of the reality condition, (3.4.7) and (3.4.9), we can show that the monodromy data consists of two real numbers.

To begin with, we will compare the both sides of (3.4.9):

$$\text{(LHS)} = \begin{pmatrix} A & B & C \\ B & a\omega^2 A - aB + C & A \\ C & A & aA + B - a\omega^2 C \end{pmatrix}$$

$$\text{(RHS)} = \begin{pmatrix} \overline{A} & \overline{C} & \overline{B} \\ \overline{C} & \overline{a}\overline{A} + \overline{B} - \overline{a}\omega\overline{C} & \overline{A} \\ \overline{B} & \overline{A} & \overline{a}\omega\overline{A} - \overline{a}\overline{B} + \overline{C} \end{pmatrix}.$$

Then, we have

$$A = \overline{A},\ B = \overline{C},\ a\omega^2 A - aB + C = \overline{a}\overline{A} + \overline{B} - \overline{a}\omega\overline{C}.$$

The last equation is trivial if we have $A = \overline{A},\ B = \overline{C}$ and (3.4.7). Let us write A as $A^{\mathbb{R}}$ to highlight that it is a real number.

Recall (3.3.41): $aA^2 + AB + \omega^2 BC + \omega AC = 0$. Since $A = \overline{A}$, $B = \overline{C}$, and $a = \omega^2 s^{\mathbb{R}}$, it further turns out to be

$$s^{\mathbb{R}}(A^{\mathbb{R}})^2 + |B|^2 + A^{\mathbb{R}}(\omega B + \omega^2 \bar{B}) = 0. \tag{3.4.10}$$

By setting $\omega B = x + iy$ where $x, y \in \mathbb{R}$, then (3.4.10) provides us with

$$s^{\mathbb{R}}(A^{\mathbb{R}})^2 + 2A^{\mathbb{R}}x + x^2 + y^2 = 0 \Leftrightarrow y^2 = -s^{\mathbb{R}}(A^{\mathbb{R}})^2 - 2A^{\mathbb{R}}x - x^2, \tag{3.4.11}$$

Next, recall (3.3.42): $(1 - a^2\omega^2)A^2 + \omega^2 B^2 - 2a\omega BC + \omega C^2 = \frac{1}{9}$. Since $A = \overline{A}$, $B = \overline{C}$, and $a = \omega^2 s^{\mathbb{R}}$, it turns out to be

$$(1 - (s^{\mathbb{R}})^2)(A^{\mathbb{R}})^2 - 2s^{\mathbb{R}}|B|^2 + (\omega^2 B^2 + \omega \bar{B}^2) = \frac{1}{9}. \tag{3.4.12}$$

Since $\omega^2 B^2 = (\omega B)^2 = x^2 - y^2 + 2ixy$ and $\omega \bar{B}^2 = \overline{\omega^2 B^2} = x^2 - y^2 - 2ixy$, (3.4.12) turns out to be

$$(1 - (s^{\mathbb{R}})^2)(A^{\mathbb{R}})^2 - 2s^{\mathbb{R}}(x^2 + y^2) + 2(x^2 - y^2) = \frac{1}{9}, \tag{3.4.13}$$

Substituting (3.4.11) into (3.4.13), we have

$$\left(2x + (1 + s^{\mathbb{R}})A^{\mathbb{R}}\right)^2 = \frac{1}{9} \Leftrightarrow x = \frac{-3(s^{\mathbb{R}} + 1)A^{\mathbb{R}} \pm 1}{6}. \tag{3.4.14}$$

And inserting (3.4.14) into (3.4.11) gives

$$y = \pm\sqrt{\frac{-9(A^{\mathbb{R}})^2((s^{\mathbb{R}})^2 + 2s^{\mathbb{R}} - 3) \pm 6A^{\mathbb{R}}(s^{\mathbb{R}} - 1) - 1}{36}}.$$

Therefore, the expression of B in terms of $s^{\mathbb{R}}$ and $A^{\mathbb{R}}$ is,

$$\begin{aligned} B = \omega^2 \Bigg(& \frac{-3(s^{\mathbb{R}}+1)A^{\mathbb{R}} \pm 1}{6} \pm \\ & \pm i\sqrt{\frac{-9(A^{\mathbb{R}})^2((s^{\mathbb{R}})^2 + 2s^{\mathbb{R}} - 3)6A^{\mathbb{R}}(s^{\mathbb{R}} - 1) - 1}{36}} \Bigg). \end{aligned} \tag{3.4.15}$$

So, we proved that E_1 is parametrized by $s^{\mathbb{R}}$ and $A^{\mathbb{R}}$ only:

$$E_1 = \begin{pmatrix} A^{\mathbb{R}} & B & \overline{B} \\ B & \omega s^{\mathbb{R}} A^{\mathbb{R}} - \omega^2 s^{\mathbb{R}} B + \overline{B} & A^{\mathbb{R}} \\ \overline{B} & A^{\mathbb{R}} & \omega^2 s^{\mathbb{R}} A^{\mathbb{R}} + B - \omega s^{\mathbb{R}} \overline{B} \end{pmatrix}$$

where B is given by (3.4.15). As a corollary, we give the updated version of the monodromy data, $\{s^{\mathbb{R}}, A^{\mathbb{R}}\}$.

3.5 Riemann-Hilbert setting

Recall the functions $\Psi_n^{(\infty,0)}$ on the universal covering $\tilde{\mathbb{C}}^*$ of $\mathbb{C}^*$ that we discussed in the subsection 3.3.3. This functions $\Psi_n^{(\infty,0)}$ were originally defined on the Stokes sector $\Omega_n^{(\infty,0)}$ and then extended to $\tilde{\mathbb{C}}^*$ by analytic continuation. But hereafter we would like to consider $\Psi_n^{(\infty,0)}$ as holomorphic functions only on the projected $\Omega_n^{(\infty,0)}$ on $\mathbb{C}^*$. In other words, these functions are sectionally holomorphic on $\mathbb{C}^*$ whose jumps are given by using the monodromy data. This gives rise to the Riemann-Hilbert problem for the negative tt*-Toda equations.

3.5.1 Ψ-problem

Let us choose the sectionally holomorphic function shown in Figure 3.5. Altogether, we call these functions $\hat{\Psi}$. The two rays in this diagram have arguments $\frac{\pi}{2} + \pi\mathbf{Z}$ and the circle is the unit circle. We call the oriented contour Γ_1.

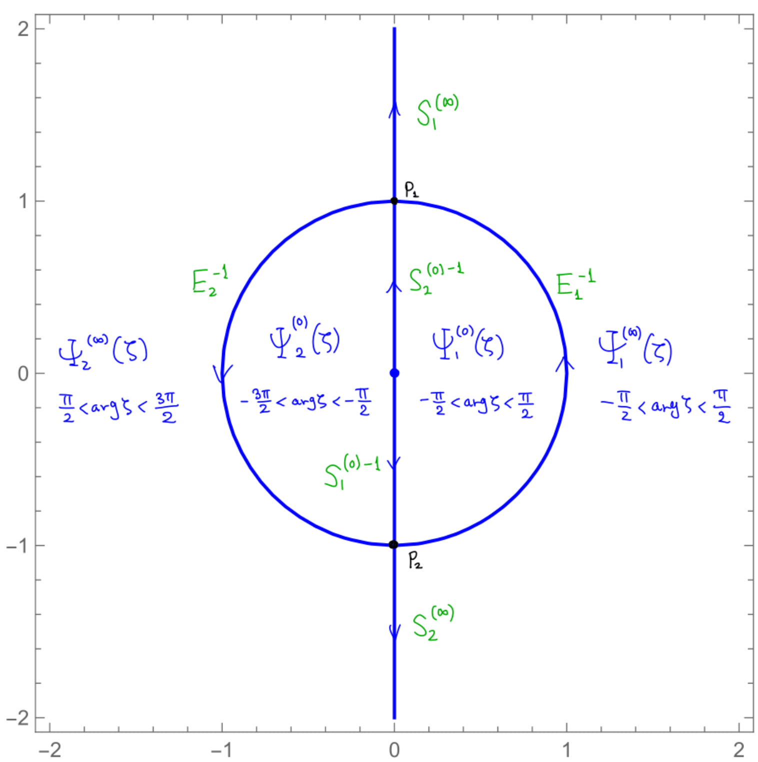

Figure 3.5. Riemann-Hilbert Problem of $\hat{\Psi}$.

Bellow we will prove that the jump matrices for this functions $\Psi_n^{(\infty,0)}$ are indeed as prescribed.

- For ζ on the outer part of the $\arg\zeta = \frac{\pi}{2}$ ray,

$$\hat{\Psi}_+(\zeta) = \Psi_2^{(\infty)}(\zeta) = \Psi_1^{(\infty)}(\zeta)S_1^{(\infty)} = \hat{\Psi}_-(\zeta)S_1^{(\infty)}$$

  by (3.3.6).

- For ζ on the semi-circle in the right half-plane,

$$\hat{\Psi}_+(\zeta) = \Psi_1^{(0)}(\zeta) = \Psi_1^{(\infty)}(\zeta)E_1^{-1} = \hat{\Psi}_-(\zeta)E_1^{-1}$$

  by (3.3.30).

- For ζ on the inner part of the $\arg\zeta = -\frac{\pi}{2}$ ray,

$$\hat{\Psi}_+(\zeta) = \Psi_1^{(0)}(\zeta) = \Psi_2^{(0)}(\zeta)S_1^{(0)-1} = \hat{\Psi}_-(\zeta)S_1^{(0)-1}$$

  by (3.3.6).

- For ζ on the inner part of the $\arg\zeta = -\frac{3\pi}{2}$ ray,

$$\begin{aligned}\hat{\Psi}_+(\zeta) &= \Psi_2^{(0)}(\zeta) = \Psi_3^{(0)}(\zeta)S_2^{(0)-1} = \Psi_1^{(0)}(\zeta e^{2\pi i})S_2^{(0)-1} \\ &\underset{\arg\zeta = \pi/2}{=} \Psi_1^{(0)}(\zeta)S_2^{(0)-1} = \hat{\Psi}_-(\zeta)S_2^{(0)-1}\end{aligned}$$

  by (3.3.6).

- For ζ on the outer part of the $\arg\zeta = -\frac{\pi}{2}$ ray,

$$\begin{aligned}\hat{\Psi}_+(\zeta) &= \Psi_1^{(\infty)}(\zeta) \underset{\arg\zeta = 3\pi/2}{=} \Psi_1^{(\infty)}(\zeta e^{-2\pi i}) = \Psi_3^{(\infty)}(\zeta) \\ &= \Psi_2^{(\infty)}(\zeta)S_2^{(\infty)} = \hat{\Psi}_-(\zeta)S_2^{(\infty)}\end{aligned}$$

  by (3.3.38).

- For ζ on the semi-circle in the left half-plane,

$$\hat{\Psi}_+(\zeta) = \Psi_2^{(0)}(\zeta) \underset{\pi/2 < \arg\zeta < 3\pi/2}{=} \Psi_2^{(0)}(\zeta e^{-2\pi i})$$
$$= \Psi_2^{(\infty)}(\zeta) E_2^{-1} = \hat{\Psi}_-(\zeta) E_2^{-1}$$

  by (3.3.30).

At the self-intersection of Γ_1, p_1 and p_2 in Figure 3.5, one can check the following cyclic relation, since we have no formal monodromy,

$$E_2 = S_2^{(0)} E_1 S_1^{(\infty)} = S_1^{(0)-1} E_1 S_2^{(\infty)-1}.$$

Our next goal is to connect the asymptotic of the radial solution of the negative sign tt*-Toda when $x \to \infty$, with the RHP that we introduced above. We will first simplify this RHP by the sequence of transformations.

3.5.2 Φ-problem

We first transform the Ψ-problem introducing $\Phi(\zeta)$ by Figure 3.6. This transformation comes from the decomposition of the Stokes matrices proved in Proposition 3.3.6 and (3.3.24). We call the new oriented contour Γ_2.

One can check that this change of the contour gives new jump matrices written in green in Figure 3.6 along with the new contour Γ_2. Now, let's write these jump matrices altogether by G_Φ.

3.5.3 $\tilde{\Phi}$-problem

Next, we define $\tilde{\Phi}(\zeta)$ as like $\tilde{\Psi}(\zeta)$:

$$\tilde{\Phi}(\zeta) = \begin{cases} \Phi(\zeta) & \text{if } |\zeta| > 1 \\ \Phi(\zeta)\dfrac{1}{3}C & \text{if } |\zeta| < 1 \end{cases}.$$

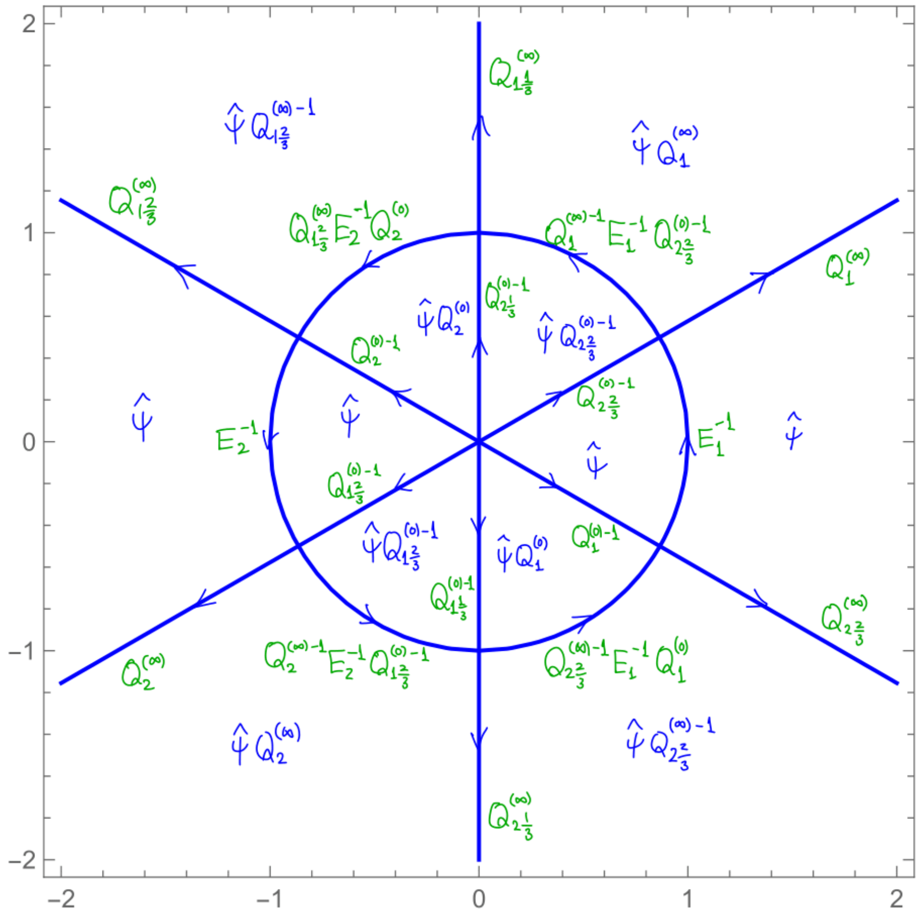

Figure 3.6. Riemann-Hilbert Problem of Φ.

This further transforms the Φ-problem in Figure 3.6 into the $\tilde{\Phi}$-problem. Although this does not change the jump contour, the jump matrices will be modified:

- Jumps on the rays outside the unit circle do not change.
- Jumps on the rays inside the unit circle change from G_Φ to $CG_\Phi C$.
- Jumps on the unit circle change from G_Φ to $\frac{1}{3}G_\Phi C$.

Some simplification of jump matrices

After this transformation the jump matrices inside the unit circle will be simplified. For example,

$$CQ^{(0)-1}_{2\frac{2}{3}}C \underset{(3.3.26)}{=} Cd_3Q^{(\infty)-1}_{1\frac{2}{3}}d_3^{-1}C \underset{(3.3.26)}{=} C\left[Q^{(\infty)}_{2\frac{2}{3}}\right]^T C \underset{\text{computation}}{=} Q^{(\infty)}_2$$

One can repeat the similar discussion for the others.

Next, by setting $\tilde{E}_n^{-1} = \frac{1}{3}E_n^{-1}C$ for $n = 1, 2$, we can simplify the jump matrices on the unit circle as well. For example,

$$\frac{1}{3}Q^{(\infty)-1}_1 E_1^{-1}Q^{(0)-1}_{2\frac{2}{3}}C = Q^{(\infty)-1}_1\left(\frac{1}{3}E_1^{-1}C\right)\left(CQ^{(0)-1}_{2\frac{2}{3}}C\right) = Q^{(\infty)-1}_1\tilde{E}_1^{-1}Q^{(\infty)}_2.$$

One can repeat the similar discussion for the other jump matrices on the unit circle. Thus, the updated jump matrices on Γ_2 are depicted in Figure 3.7.

Asymptotics near the singularities

Lastly, we will check the asymptotic behavior near the singularities. By definition of $\tilde{\Phi}(\zeta)$ and (3.3.3), we have

$$\tilde{\Phi}(\zeta) = P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3},$$

as $\zeta \to \infty$. On the other hand, as $\zeta \to 0$, we have the following asymptotics:

$$\begin{aligned}\tilde{\Phi}(\zeta) &= \frac{1}{3}P_0(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3}C = \frac{1}{3}P_0(I + \mathcal{O}(\zeta))C\left[Ce^{\frac{1}{\zeta}d_3}C\right] \\ &= \frac{1}{3}P_0C(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3^{-1}} = \frac{1}{3}e^{-w}\Omega(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3^{-1}} \\ &= e^{-w}\Omega^{-1}(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3^{-1}}\end{aligned}$$

by definition of $\tilde{\Phi}(\zeta)$ and (3.3.2).

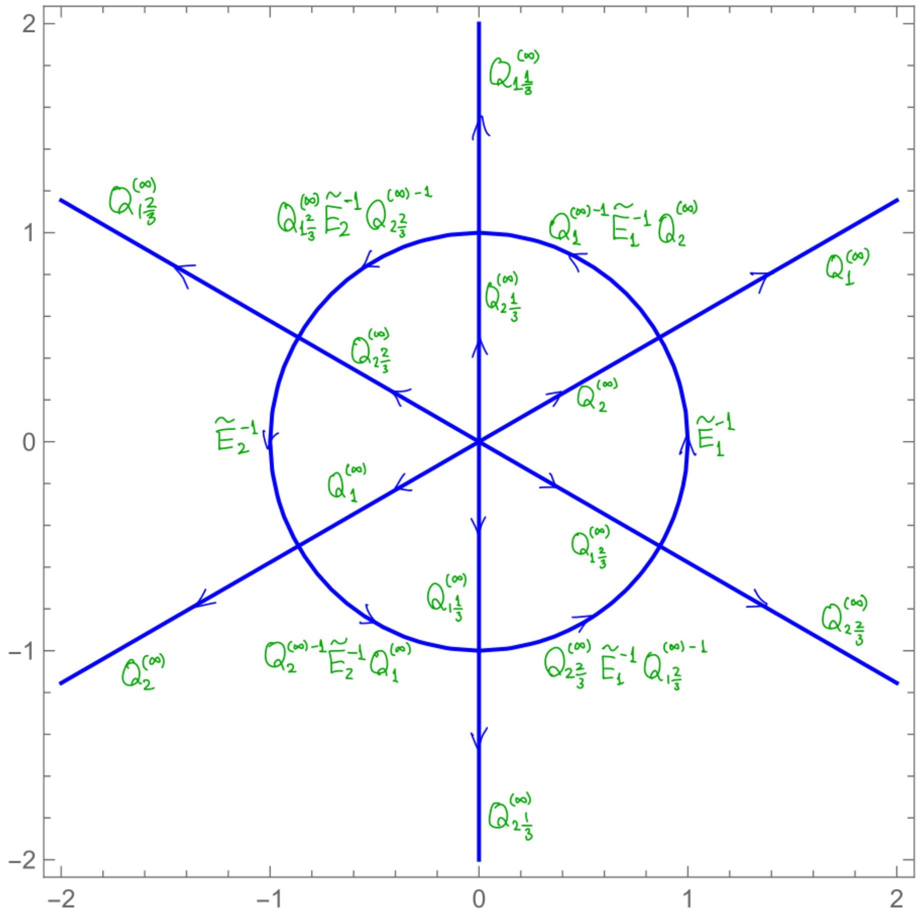

Figure 3.7. Riemann-Hilbert Problem of $\tilde{\Phi}$.

3.5.4 $\hat{\Phi}$-problem

Define $\hat{\Phi}$ in Figure 3.8 so that all jump contours are rotated by $\frac{\pi}{3}$ degree in anti-clockwise. First, notice that original jump contours but the unit circle are not jump contours anymore. For example,

- For ζ on the outer part of the $\arg \zeta = \frac{\pi}{6}$ ray of Figure 3.8,

$$\hat{\Phi}_+(\zeta) = \tilde{\Phi}_+ Q_2^{(\infty)-1} = \tilde{\Phi}_-(\zeta) Q_2^{(\infty)} Q_2^{(\infty)-1} = \tilde{\Phi}_-(\zeta) = \hat{\Phi}_-(\zeta).$$

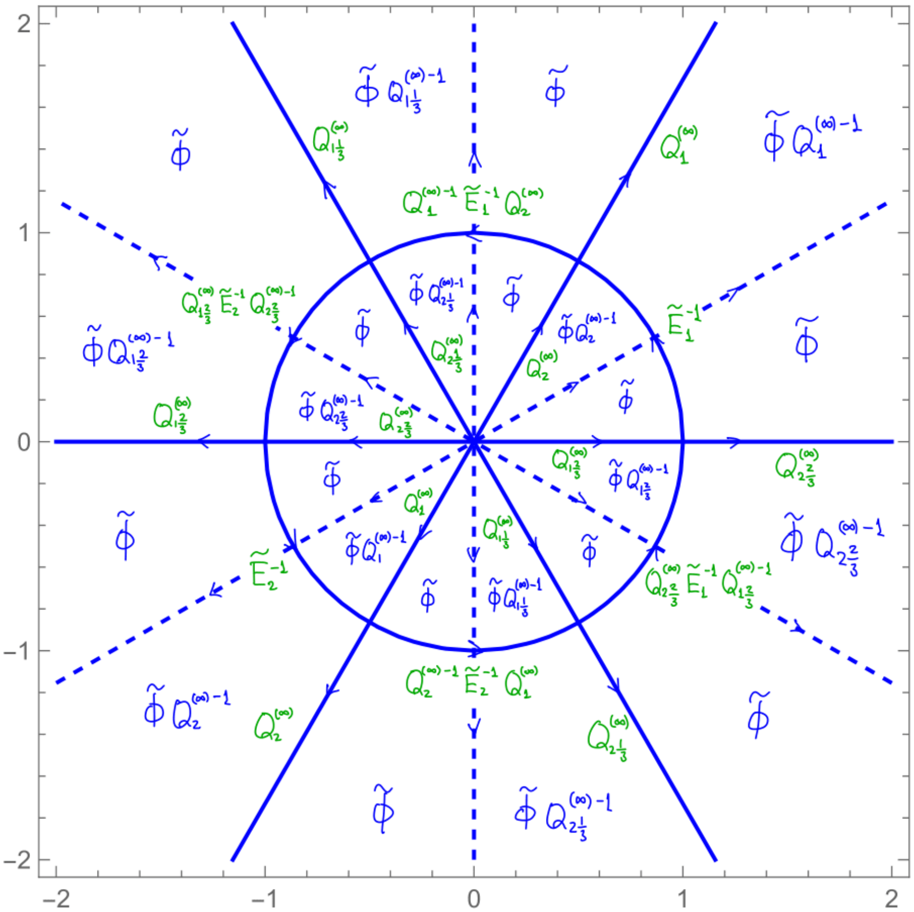

Figure 3.8. Riemann-Hilbert Problem of $\hat{\Phi}$.

- For ζ on the inner part of the $\arg\zeta = \frac{\pi}{6}$ ray of Figure 3.8,

$$\hat{\Phi}_+(\zeta) = \tilde{\Phi}_+ Q_1^{(\infty)-1} = \tilde{\Phi}_-(\zeta) Q_1^{(\infty)} Q_1^{(\infty)-1} = \tilde{\Phi}_-(\zeta) = \hat{\Phi}_-(\zeta).$$

Thus, the ray with $\arg\zeta = \frac{\pi}{6}$ is not the jump contour. Similarly, all the other rays in Figure 3.7 are eliminated. Instead, jump matrices are moved on new rays.

Next, we will check how we get jump matrices on the unit circle in Figure 3.8. For example,

- For ζ on the 1/6-circle with $0 < \arg\zeta < \frac{\pi}{3}$ in the right half-plane of Figure 3.8, we have either

$$\hat{\Phi}_+(\zeta) = \tilde{\Phi}_+ = \tilde{\Phi}_- \tilde{E_1}^{-1} = \hat{\Phi}_+(\zeta)\tilde{E_1}^{-1}$$

or

$$\begin{aligned}\hat{\Phi}_+(\zeta) &= \tilde{\Phi}_+ Q_2^{(\infty)-1} = \tilde{\Phi}_- Q_1^{(\infty)-1}\tilde{E_1}^{-1}Q_2^{(\infty)}Q_2^{(\infty)-1}\\ &= \tilde{\Phi}_- Q_1^{(\infty)-1}\tilde{E_1}^{-1} = \hat{\Phi}_-(\zeta)\tilde{E_1}^{-1}.\end{aligned}$$

- For ζ on the 1/6-circle with $\frac{\pi}{3} < \arg\zeta < \frac{2\pi}{3}$ in the upper half-plane of Figure 3.8, we have either

$$\begin{aligned}\hat{\Phi}_+(\zeta) = \tilde{\Phi}_+ &= \tilde{\Phi}_- Q_1^{(\infty)-1}\tilde{E_1}^{-1}Q_2^{(\infty)}\\ &= \hat{\Phi}_+(\zeta)Q_1^{(\infty)-1}\tilde{E_1}^{-1}Q_2^{(\infty)}\end{aligned}$$

or

$$\begin{aligned}\hat{\Phi}_+(\zeta) &= \tilde{\Phi}_+ Q_{2\frac{1}{3}}^{(\infty)-1} = \tilde{\Phi}_- Q_{1\frac{2}{3}}^{(\infty)}\tilde{E_2}^{-1}Q_{2\frac{2}{3}}^{(\infty)-1}Q_{2\frac{1}{3}}^{(\infty)-1}\\ &= \tilde{\Phi}_- Q_{1\frac{1}{3}}^{(\infty)-1}Q_{1\frac{1}{3}}^{(\infty)}Q_{1\frac{2}{3}}^{(\infty)}\tilde{E_2}^{-1}Q_{2\frac{2}{3}}^{(\infty)-1}Q_{2\frac{1}{3}}^{(\infty)-1}\\ &= \tilde{\Phi}_- Q_{1\frac{1}{3}}^{(\infty)-1}Q_1^{(\infty)-1}S_1^{(\infty)}\tilde{E_2}^{-1}S_2^{(\infty)-1}Q_2^{(\infty)}\\ &= \hat{\Phi}_- Q_1^{(\infty)-1}S_1^{(\infty)}\frac{1}{3}E_2^{-1}CS_2^{(\infty)-1}Q_2^{(\infty)}\\ &= \hat{\Phi}_- Q_1^{(\infty)-1}S_1^{(\infty)}\frac{1}{3}E_2^{-1}S_2^{(0)}CQ_2^{(\infty)}\\ &= \hat{\Phi}_- Q_1^{(\infty)-1}\frac{1}{3}E_1^{-1}CQ_2^{(\infty)} = \hat{\Phi}_- Q_1^{(\infty)-1}\tilde{E_1}^{-1}Q_2^{(\infty)},\end{aligned}$$

  where the last two equality is obtained from Proposition 3.3.7.

This is how jump matrices on the unit circle in Figure 3.8 are obtained. Let us call jump matrices in Figure 3.8 altogether $G_{\hat{\Phi}}$ and the jump contour Γ_3.

The asymptotics near the singularities are not changed from $\tilde{\Phi}(\zeta)$:

$$\begin{aligned} \hat{\Phi}(\zeta) &= P_\infty(I + \mathcal{O}(\zeta^{-1}))e^{-x^2\zeta d_3}, \text{ as } \zeta \to \infty \\ \hat{\Phi}(\zeta) &= e^{-w}\Omega^{-1}(I + \mathcal{O}(\zeta))e^{\frac{1}{\zeta}d_3^{-1}}, \text{ as } \zeta \to 0. \end{aligned} \tag{3.5.1}$$

3.5.5 Y-probelm

Considering the asymptotics (3.5.1), let's introduce the following scaled version:

$$Y(\zeta) := P_\infty^{-1}\hat{\Phi}\left(\frac{\zeta}{x}\right)e^{-x\theta(\zeta)} \tag{3.5.2}$$

where $\theta(\zeta) = -\zeta d_3 + \frac{1}{\zeta}d_3^{-1}$. Then, we interpret the Riemann-Hilbert setting in terms of the Y function. Since the transformation (3.5.2) is just scaling, the jump contour does not change from Γ_3 as in Figure 3.8, but jump matrices do change from $G_{\hat{\Phi}}$ to G_Y:

$$\begin{aligned} Y_+(\zeta) = P_\infty^{-1}\hat{\Phi}_+\left(\frac{\zeta}{x}\right)e^{-x\theta(\zeta)} &= P_\infty^{-1}\hat{\Phi}_-\left(\frac{\zeta}{x}\right)G_{\hat{\Phi}}e^{-x\theta(\zeta)} \\ &= P_\infty^{-1}\hat{\Phi}_-\left(\frac{\zeta}{x}\right)e^{-x\theta(\zeta)}e^{x\theta(\zeta)}G_{\hat{\Phi}}e^{-x\theta(\zeta)} \\ &= Y_-(\zeta)e^{x\theta(\zeta)}G_{\hat{\Phi}}e^{-x\theta(\zeta)}. \end{aligned}$$

Now, the normalization condition of this problem becomes the following:

- As $\zeta \to \infty$, we have

  $$Y(\zeta) = I + \mathcal{O}(\zeta^{-1})$$

  by (3.5.1).

- As $\zeta \to 0$, we have

  $$Y(\zeta) = \mathcal{O}(1) = \Omega e^{-2w}\Omega^{-1} + \mathcal{O}(\zeta)$$

  by (3.5.1) and the fact $P_\infty = e^w\Omega^{-1}$.

So, we ended up trying to solve the following RHP.

Riemann-Hilbert Problem 1. Find a solution Y satisfying

- $Y(\zeta) \in H(\mathbb{C} \setminus \Gamma_2)$ with the contour Γ_3 depicted in Figure 3.8.
- The jump conditions are

$$Y_+(\zeta) = Y_-(\zeta) e^{x\theta(\zeta)} G_{\hat{\Phi}} e^{-x\theta(\zeta)}.$$

  where $G_{\hat{\Phi}}$ is given in Figure 3.8.
- The normalization condition is

$$Y(\zeta) = I + \mathcal{O}(\zeta^{-1}) \text{ as } \zeta \to \infty,$$
$$Y(\zeta) = \mathcal{O}(1) = \Omega e^{-2w} \Omega^{-1} + \mathcal{O}(\zeta) \text{ as } \zeta \to 0.$$

Jump matrices on rays

In the previous section we stated the RHP for $Y(\zeta)$. By the small norm theorem ([41], Theorem 8.1), we know that this problem is solvable if the jump matrices are close to identify in $L^2 \cap L^\infty$ norm. Let us check that this condition is satisfied in our case.

$$G_{\hat{\Phi}} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and set

$$\varphi_1(\zeta) = (1-\omega)\zeta - (1-\omega^2)\frac{1}{\zeta} = \sqrt{3}\left(\zeta e^{-i\frac{\pi}{6}} - \frac{1}{\zeta}e^{i\frac{\pi}{6}}\right)$$
$$\varphi_2(\zeta) = (1-\omega^2)\zeta - (1-\omega)\frac{1}{\zeta} = \sqrt{3}\left(\zeta e^{i\frac{\pi}{6}} - \frac{1}{\zeta}e^{-i\frac{\pi}{6}}\right)$$
$$\varphi_3(\zeta) = (\omega-\omega^2)\zeta + (\omega-\omega^2)\frac{1}{\zeta} = i\sqrt{3}\left(\zeta + \frac{1}{\zeta}\right),$$

then G_Y becomes

$$G_Y = \begin{pmatrix} * & *e^{-x\varphi_1} & *e^{-x\varphi_2} \\ *e^{x\varphi_1} & * & *e^{-x\varphi_3} \\ *e^{x\varphi_2} & *e^{x\varphi_3} & * \end{pmatrix}. \tag{3.5.3}$$

We need to find the stationary points for each exponent. First,we consider φ_1. Since

$$\varphi_1'(\zeta) = \sqrt{3}(e^{-i\frac{\pi}{6}} + \zeta^{-2}e^{i\frac{\pi}{6}}),$$

it equals zero when $\zeta = \pm e^{-i\frac{\pi}{3}}$, i.e., $\varphi_1(\zeta)$ has its stationary points at $\zeta = \pm e^{-i\frac{\pi}{3}}$. Moreover, their values are

$$\varphi_1\left(\pm e^{-i\frac{\pi}{3}}\right) = \mp 2\sqrt{3}i.$$

We also note that $\varphi_1''(\zeta)\Big|_{\zeta=\pm e^{-i\frac{\pi}{3}}} = -2\sqrt{3}\zeta^{-3}e^{i\frac{\pi}{6}}\Big|_{\zeta=\pm e^{-i\frac{\pi}{3}}} = \pm 2\sqrt{3}e^{i\frac{\pi}{6}}$. Let $\zeta = \xi + i\eta$. Then,

$$\varphi_1(\zeta) = \sqrt{3}\left(\frac{\sqrt{3}\xi}{2} + \frac{\eta}{2} - \frac{\sqrt{3}\xi}{2(\xi^2+\eta^2)} - \frac{\eta}{2(\xi^2+\eta^2)}\right) + i\sqrt{3}\left(-\frac{\xi}{2} + \frac{\sqrt{3}\eta}{2} + \frac{\sqrt{3}\eta}{2(\xi^2+\eta^2)} - \frac{\xi}{2(\xi^2+\eta^2)}\right).$$

The level curves of $\mathrm{Re}(\varphi_1(z)) = 0$ are

$$\frac{\sqrt{3}\xi}{2} + \frac{\eta}{2} - \frac{\sqrt{3}\xi}{2(\xi^2+\eta^2)} - \frac{\eta}{2(\xi^2+\eta^2)} = 0.$$

Together with change of the sign of $\text{Re}(\varphi_1(z))$, the level curve of $\text{Re}(\varphi_1(z)) = 0$ is drawn in Figure 3.9.

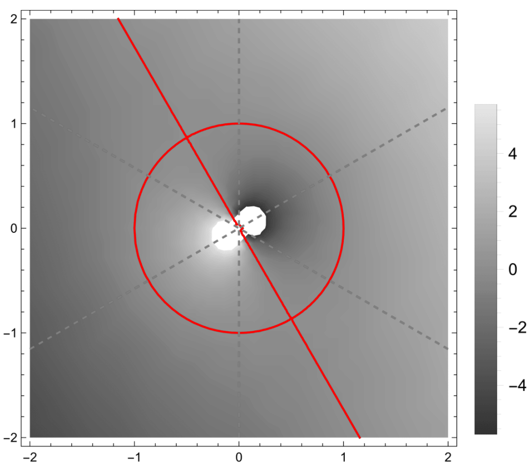

Figure 3.9. The level curves of $\text{Re}(\varphi_1(z)) = 0$.

Next, consider φ_2. Since

$$\varphi_2'(\zeta) = \sqrt{3}(e^{i\frac{\pi}{6}} + \zeta^{-2}e^{-i\frac{\pi}{6}}),$$

it equals zero when $\zeta = \pm e^{i\frac{\pi}{3}}$, i.e., $\varphi_2(\zeta)$ has its stationary points at $\zeta = \pm e^{i\frac{\pi}{3}}$. Moreover, their values are

$$\varphi_2\left(\pm e^{i\frac{\pi}{3}}\right) = \pm 2\sqrt{3}i.$$

We also note that $\varphi_2''(\zeta)\Big|_{\zeta=\pm e^{i\frac{\pi}{3}}} = -2\sqrt{3}\zeta^{-3}e^{-i\frac{\pi}{6}}\Big|_{\zeta=\pm e^{i\frac{\pi}{3}}} = \pm 2\sqrt{3}e^{-i\frac{\pi}{6}}$. Let $\zeta = \xi + i\eta$. Then,

$$\varphi_2(\zeta) = \sqrt{3}\left(\frac{\sqrt{3}\xi}{2} - \frac{\eta}{2} - \frac{\sqrt{3}\xi}{2(\xi^2+\eta^2)} + \frac{\eta}{2(\xi^2+\eta^2)}\right) \\ + i\sqrt{3}\left(\frac{\xi}{2} + \frac{\sqrt{3}\eta}{2} + \frac{\sqrt{3}\eta}{2(\xi^2+\eta^2)} + \frac{\xi}{2(\xi^2+\eta^2)}\right).$$

The level curves of $\text{Re}(\varphi_2(z)) = 0$ are

$$\frac{\sqrt{3}\xi}{2} - \frac{\eta}{2} - \frac{\sqrt{3}\xi}{2(\xi^2+\eta^2)} + \frac{\eta}{2(\xi^2+\eta^2)} = 0.$$

Together with change of the sign of $\mathrm{Re}(\varphi_2(z))$, the level curve of $\mathrm{Re}(\varphi_2(z)) = 0$ is drawn in Figure 3.10.

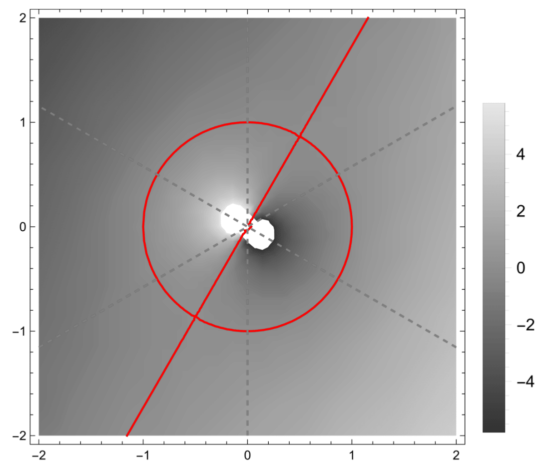

Figure 3.10. The level curves of $\mathrm{Re}(\varphi_2(z)) = 0$.

Finally, consider φ_3. Since

$$\varphi_3'(\zeta) = i\sqrt{3}(1 - \zeta^{-2}),$$

it equals zero when $\zeta = \pm 1$, i.e., $\varphi_3(\zeta)$ has its stationary points at $\zeta = \pm 1$. Moreover, their values are

$$\varphi_3\left(\pm 1\right) = \pm 2\sqrt{3}i.$$

We also note that $\varphi_3''(\zeta)\Big|_{\zeta=\pm 1} = 2\sqrt{3}\zeta^{-3}i\Big|_{\zeta=\pm 1} = \pm 2\sqrt{3}i$. Let $\zeta = \xi + i\eta$. Then,

$$\varphi_3(\zeta) = \sqrt{3}\left(-\eta + \frac{\eta}{\xi^2 + \eta^2}\right) + i\sqrt{3}\left(\xi + \frac{\xi}{\xi^2 + \eta^2}\right).$$

The level curves of $\mathrm{Re}(\varphi_3(z)) = 0$ are

$$-\eta + \frac{\eta}{\xi^2 + \eta^2} = 0.$$

Toghether with change of the sign of $\mathrm{Re}(\varphi_3(z))$, the level curve of $\mathrm{Re}(\varphi_3(z)) = 0$ is drawn in Figure 3.11.

Therefore, jump matrices of Y-problem on each of the rays admits the property of approaching to the identity matrix as $x \to +\infty$. In fact,

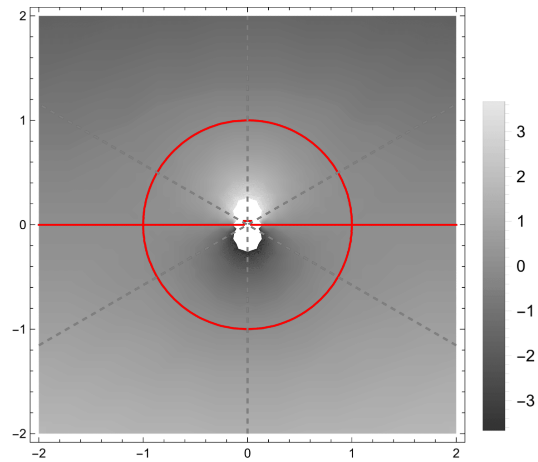

Figure 3.11. The level curves of $\text{Re}(\varphi_3(z)) = 0$.

- If the jump matrix is given by $e^{x\theta}Q_1^{(\infty)}e^{-x\theta}$, then by the parametrization of $Q_1^{(\infty)}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q_1^{(\infty)}e^{-x\theta} = \begin{pmatrix} 1 & ae^{-x\varphi_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  such that $|e^{-x\varphi_1}| \to 0$ by Figure 3.7 and 3.9.

- If the jump matrix is given by $e^{x\theta}Q_{1\frac{1}{3}}^{(\infty)}e^{-x\theta}$, then by the parametrization of $Q_{1\frac{1}{3}}^{(\infty)}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q_{1\frac{1}{3}}^{(\infty)}e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a\omega^2 e^{x\varphi_3} & 1 \end{pmatrix}$$

  such that $|e^{x\varphi_3}| \to 0$ by Figure 3.7 and 3.11.

- If the jump matrix is given by $e^{x\theta}Q^{(\infty)}_{1\frac{2}{3}}e^{-x\theta}$, then by the parametrization of $Q^{(\infty)}_{1\frac{2}{3}}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q^{(\infty)}_{1\frac{2}{3}}e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ae^{x\varphi_2} & 0 & 1 \end{pmatrix}$$

such that $|e^{x\varphi_2}| \to 0$ by Figure 3.7 and 3.10.

- If the jump matrix is given by $e^{x\theta}Q^{(\infty)}_{2}e^{-x\theta}$, then by the parametrization of $Q^{(\infty)}_{2}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q^{(\infty)}_{2}e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ -a\omega^2e^{x\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $|e^{x\varphi_1}| \to 0$ by Figure 3.7 and 3.9.

- If the jump matrix is given by $e^{x\theta}Q^{(\infty)}_{2\frac{1}{3}}e^{-x\theta}$, then by the parametrization of $Q^{(\infty)}_{2\frac{1}{3}}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q^{(\infty)}_{2\frac{1}{3}}e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & ae^{-x\varphi_3} \\ 0 & 0 & 1 \end{pmatrix}$$

such that $|e^{-x\varphi_3}| \to 0$ by Figure 3.7 and 3.11.

- If the jump matrix is given by $e^{x\theta}Q^{(\infty)}_{2\frac{2}{3}}e^{-x\theta}$, then by the parametrization of $Q^{(\infty)}_{2\frac{2}{3}}$ in subsubsection 3.3.6 and (3.5.3), we have

$$e^{x\theta}Q^{(\infty)}_{2\frac{2}{3}}e^{-x\theta} = \begin{pmatrix} 1 & 0 & -a\omega^2e^{-x\varphi_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $|e^{-x\varphi_2}| \to 0$ by Figure 3.7 and 3.10.

Jump matrices on the unit circle

The last piece to be considered is the ones on the unit circle. Recall that G_Y is given by

$$G_Y = \begin{pmatrix} * & *e^{-x\varphi_1} & *e^{-x\varphi_2} \\ *e^{x\varphi_1} & * & *e^{-x\varphi_3} \\ *e^{x\varphi_2} & *e^{x\varphi_3} & * \end{pmatrix}$$

it turns out that each φ_k is purely imaginary for ζ on the unit circle, so jump matrices there are oscillating. We can simplify our problem by decomposing G_Y into 3 matrices where the left and the right matrices have exponential decay at the outside and inside of the unit circle respectively, and the middle matrix is diagonal and constant. Suppose

$$\begin{aligned} &\tilde{E}_1^{-1} = L_1 D_1 R_1,\ Q_1^{(\infty)-1}\tilde{E}_1^{-1}Q_2^{(\infty)} = L_2 D_2 R_2, \\ &Q_{1\frac{2}{3}}^{(\infty)}\tilde{E}_2^{-1}Q_{2\frac{2}{3}}^{(\infty)-1} = L_3 D_3 R_3,\ \tilde{E}_2^{-1} = L_4 D_4 R_4, \\ &Q_2^{(\infty)-1}\tilde{E}_2^{-1}Q_1^{(\infty)} = L_5 D_5 R_5,\ Q_{2\frac{2}{3}}^{(\infty)}\tilde{E}_1^{-1}Q_{1\frac{2}{3}}^{(\infty)-1} = L_6 D_6 R_6, \end{aligned} \tag{3.5.4}$$

such that $e^{x\theta}L_k e^{-x\theta}$ and $e^{x\theta}R_k e^{-x\theta}$ have non-diagonal entries with exponentially decay and D_k are constant diagonal matrices. Then, one can transform the $\hat{\Phi}$-problem to the following $\check{\Phi}$-problem in Figure 3.12 by opening lenses so that new contours intersect with rays and the unit circle by 45 degree. We call the new oriented contour Γ_4.

If we think about the corresponding $\check{Y}$-problem just by replacing $\hat{\Phi}$ with $\check{\Phi}$ in (3.5.2), we observe that all jump matrices of $\check{Y}$-problem but all D_k's tend to the identity matrix. Thus, we will be able to build an model RHP for $\check{Y}^D(\zeta)$ with only a jump matrix D_k's on the unit circle. We call its solution a *global parametrix*. However, such convergence of the jump matrices is not uniform near the stationary points. In order for $\check{Y}(\zeta)$ to take continuous boundary values, the approximation of $\check{Y}$-problem by $\check{Y}^D$-problem is not enough and extra care is needed around each stationary point. So, we have to construct different model RHPs there and call their solutions *local parametrices*. Since we want to make global and local parametrices be a uniformly accurate approximation of $\check{Y}(\zeta)$, we lastly consider a small norm RHP for an error function $R(\zeta)$ by comparing $\check{Y}(\zeta)$ with its parametrices.

Note. In analogy with the classical steepest descent method, the main contribution to the asymptotics of $\check{Y}(\zeta)$ as $\zeta \to +\infty$ comes from the unit circle and from the neighborhood of the stationary points.

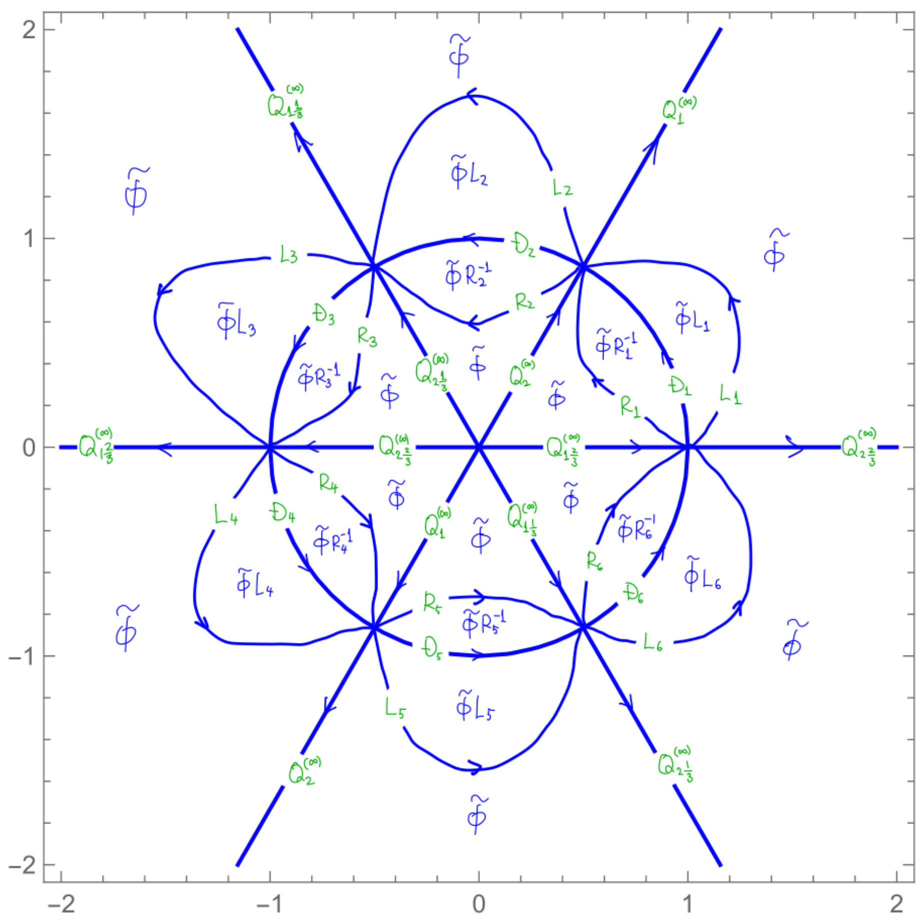

Figure 3.12. Riemann-Hilbert Problem of $\check{\Phi}$.

So, the question becomes how to decompose G_Y on the unit circle. Consider, for instance, $G_Y = e^{x\theta}\tilde{E}_1^{-1}e^{-x\theta}$, it should have the following decomposition,

$$G_Y = \begin{pmatrix} 1 & *e^{-x\varphi_1} & *e^{-x\varphi_2} \\ 0 & 1 & 0 \\ 0 & *e^{x\varphi_3} & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ *e^{x\varphi_1} & 1 & *e^{-x\varphi_3} \\ *e^{x\varphi_2} & 0 & 1 \end{pmatrix},$$

since the left matrix has an exponential decay at non-diagonal entries when ζ belongs to the outside of the unit circle and the right matrix has the same exponential decay at non-diagonal entries when ζ belongs to the inside of the unit circle. But, essentially, the decomposition of $\tilde{E}_1^{-1}$ is rather needed, because

$$G_Y = e^{x\theta} \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & 1 \end{pmatrix} e^{-x\theta}.$$

For the same reason, we need the following decompositions.

$$Q_1^{(\infty)-1} \tilde{E}_1^{-1} Q_2^{(\infty)} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

$$Q_{1\frac{2}{3}}^{(\infty)} \tilde{E}_2^{-1} Q_{2\frac{2}{3}}^{(\infty)-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\tilde{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

$$Q_2^{(\infty)-1} \tilde{E}_2^{-1} Q_1^{(\infty)} = \begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

$$Q_{2\frac{2}{3}}^{(\infty)} \tilde{E}_1^{-1} Q_{1\frac{2}{3}}^{(\infty)-1} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

3.6 Decomposition of Jump matrices on the unit circle

In this section we obtain the desired decomposition of the connections matrices discussed in the previous section. We start with some useful identifies.

3.6.1 Some identities between $A^{\mathbb{R}}$ and B

Here we recall identities (3.3.41) and (3.3.42). By the reality condition, we also have $A = \overline{A}$, $B = \overline{C}$, and $a = \omega^2 s^{\mathbb{R}}$. Then, it follows that

$$\begin{aligned}
[(1+a\omega)A^{\mathbb{R}} + \omega B + \omega^2\overline{B}]^2 &= (1 + a^2\omega^2 + 2a\omega)(A^{\mathbb{R}})^2 + \omega^2 B^2 + \omega\overline{B}^2 \\
&\quad + 2[(1+a\omega)\omega AB + |B|^2 + (1+a\omega)\omega^2 AB] \\
&= (1 + a^2\omega^2)(A^{\mathbb{R}})^2 + \omega^2 B^2 + \omega\overline{B}^2 + 2[a\omega^2 AB + aA\overline{B}] \\
&= (1 - a^2\omega^2)(A^{\mathbb{R}})^2 + \omega^2 B^2 + \omega\overline{B}^2 - 2a\omega|B|^2 \\
&= \frac{1}{9}.
\end{aligned}$$

One can have the following identity by choosing plus sign when taking the square root at both sides.

$$(1+a\omega)A^{\mathbb{R}} + \omega B + \omega^2\overline{B} = \frac{1}{3}. \tag{3.6.1}$$

By (3.4.10) and (3.6.1), it follows that

$$(A^{\mathbb{R}})^2 - \frac{1}{3}A^{\mathbb{R}} = |B|^2. \tag{3.6.2}$$

Moreover, (3.3.41) and (3.6.1) implies

$$\begin{aligned}
a(A^{\mathbb{R}})^2 - \overline{B}^2 + A^{\mathbb{R}}B - a\omega^2 A^{\mathbb{R}}\overline{B} &= -\omega^2|B|^2 - \omega A^{\mathbb{R}}\overline{B} - \overline{B}^2 - a\omega^2 A^{\mathbb{R}}\overline{B} \\
&= -\omega A^{\mathbb{R}}\overline{B} - \frac{1}{3}\omega\overline{B} + \omega A^{\mathbb{R}}\overline{B} \\
&= -\frac{1}{3}\omega\overline{B}.
\end{aligned} \tag{3.6.3}$$

3.6.2 Desired decompositions

Proposition 3.6.1. Recall that L_k and R_k are defined in (3.5.4). Then, we have

$$L_1 = \begin{pmatrix} 1 & -\frac{B}{A^{\mathbb{R}}} - \omega^2 \frac{|B|^2}{(A^{\mathbb{R}})^2} & -\frac{\overline{B}}{A^{\mathbb{R}}} \\ 0 & 1 & 0 \\ 0 & \omega^2 \frac{B}{A^{\mathbb{R}}} & 1 \end{pmatrix}, \; D_1 = \begin{pmatrix} \frac{1}{3A^{\mathbb{R}}} & & \\ & 3A^{\mathbb{R}} & \\ & & 1 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\overline{B}}{A^{\mathbb{R}}} - \omega \frac{|B|^2}{(A^{\mathbb{R}})^2} & 1 & \omega \frac{\overline{B}}{A^{\mathbb{R}}} \\ -\frac{B}{A^{\mathbb{R}}} & 0 & 1 \end{pmatrix}.$$

$$L_2 = \begin{pmatrix} 1 & \omega \frac{\overline{B}}{A^{\mathbb{R}}} & 0 \\ 0 & 1 & 0 \\ -\frac{B}{A^{\mathbb{R}}} & -\omega \frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega s^{\mathbb{R}} - \frac{\overline{B}}{A^{\mathbb{R}}} & 1 \end{pmatrix}, \; D_2 = \begin{pmatrix} 1 & & \\ & 3A^{\mathbb{R}} & \\ & & \frac{1}{3A^{\mathbb{R}}} \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 1 & 0 & -\frac{\overline{B}}{A^{\mathbb{R}}} \\ \omega^2 \frac{B}{A^{\mathbb{R}}} & 1 & -\omega^2 \frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega^2 s^{\mathbb{R}} - \frac{B}{A^{\mathbb{R}}} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega^2 \frac{B}{A^{\mathbb{R}}} & 1 & 0 \\ -\omega^2 \frac{|B|^2}{(A^{\mathbb{R}})^2} - \frac{B}{A^{\mathbb{R}}} & -\frac{\overline{B}}{A^{\mathbb{R}}} & 1 \end{pmatrix}, \; D_3 = \begin{pmatrix} 3A^{\mathbb{R}} & & \\ & 1 & \\ & & \frac{1}{3A^{\mathbb{R}}} \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & \omega \frac{\overline{B}}{A^{\mathbb{R}}} & -\omega \frac{|B|^2}{(A^{\mathbb{R}})^2} - \frac{\overline{B}}{A^{\mathbb{R}}} \\ 0 & 1 & -\frac{B}{A^{\mathbb{R}}} \\ 0 & 0 & 1 \end{pmatrix}.$$

$$L_4 = \begin{pmatrix} 1 & 0 & 0 \\ -\omega\frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega s^{\mathbb{R}} - \frac{\overline{B}}{A^{\mathbb{R}}} & 1 & -\frac{B}{A^{\mathbb{R}}} \\ \omega\frac{\overline{B}}{A^{\mathbb{R}}} & 0 & 1 \end{pmatrix},\ D_4 = \begin{pmatrix} 3A^{\mathbb{R}} & & \\ & \frac{1}{3A^{\mathbb{R}}} & \\ & & 1 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 1 & -\omega^2\frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega^2 s^{\mathbb{R}} - \frac{B}{A^{\mathbb{R}}} & \omega^2\frac{B}{A^{\mathbb{R}}} \\ 0 & 1 & 0 \\ 0 & -\frac{\overline{B}}{A^{\mathbb{R}}} & 1 \end{pmatrix}.$$

$$L_5 = \begin{pmatrix} 1 & 0 & \omega^2\frac{B}{A^{\mathbb{R}}} \\ -\frac{\overline{B}}{A^{\mathbb{R}}} & 1 & -\omega^2\frac{|B|^2}{(A^{\mathbb{R}})^2} - \frac{B}{A^{\mathbb{R}}} \\ 0 & 0 & 1 \end{pmatrix},\ D_5 = \begin{pmatrix} 1 & & \\ & \frac{1}{3A^{\mathbb{R}}} & \\ & & 3A^{\mathbb{R}} \end{pmatrix},$$

$$R_5 = \begin{pmatrix} 1 & -\frac{B}{A^{\mathbb{R}}} & 0 \\ 0 & 1 & 0 \\ \omega\frac{\overline{B}}{A} & -\omega\frac{|B|^2}{(A^{\mathbb{R}})^2} - \frac{\overline{B}}{A^{\mathbb{R}}} & 1 \end{pmatrix}.$$

$$L_6 = \begin{pmatrix} 1 & -\frac{B}{A^{\mathbb{R}}} & -\omega\frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega s^{\mathbb{R}} - \frac{\overline{B}}{A^{\mathbb{R}}} \\ 0 & 1 & \omega\frac{\overline{B}}{A^{\mathbb{R}}} \\ 0 & 0 & 1 \end{pmatrix},\ D_6 = \begin{pmatrix} \frac{1}{3A^{\mathbb{R}}} & & \\ & 1 & \\ & & 3A^{\mathbb{R}} \end{pmatrix}$$

$$R_6 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\overline{B}}{A^{\mathbb{R}}} & 1 & 0 \\ -\omega^2\frac{|B|^2}{(A^{\mathbb{R}})^2} - \omega^2 s^{\mathbb{R}} - \frac{B}{A^{\mathbb{R}}} & \omega^2\frac{B}{A^{\mathbb{R}}} & 1 \end{pmatrix}.$$

Proof. By straightforward calculations using identities (3.6.2) and (3.6.3).

□

3.7 Model Riemann-Hilbert Problems

We denote the arcs of S^1 where the jump matrices D_n are defined by C_n, and denote intersections of the unit circle and the infinite ray by p_n, see Figure 3.12.

This section consists of three steps. First, we solve a model RHP 2 on the unit circle $S^1 = \bigcup_{n=1}^{6} C_n$ to get the global parametrix $\check{Y}^D$. Then, we construct a model RHP near p_1 to get a local parametrix $P^{(1)}(\zeta)$. Lastly, we obtain other local parametrices:

$$P^{(w)}(\zeta), \text{ where } w = -\overline{\omega},\ \omega,\ -1,\ \overline{\omega}, \text{ and } -\omega,$$

from $P^{(1)}(\zeta)$ and using the symmetry relations.

3.7.1 Global parametrix

Consider the following model RHP for the global parametrix.

Riemann-Hilbert Problem 2. Find a solution $\check{Y}^D$ satisfying the following conditions (see Fig. 3.13):

- $\check{Y}^D(\zeta) \in H(\mathbb{C} \setminus S^1)$.
- On the unit circle oriented counterclockwise

  $$\check{Y}^D_+(\zeta) = \check{Y}^D_-(\zeta) G_D,$$

  where the jump matrices are

  $$G_D = D_i \ \text{ if } \zeta \in C_i, \qquad i = 1, \dots 6,$$

  with $D_1, \cdots, D_6$ parametrized in subsection 3.6.2.
- The normalization condition : $\check{Y}^D(z) \to I$ as $\zeta \to \infty$.

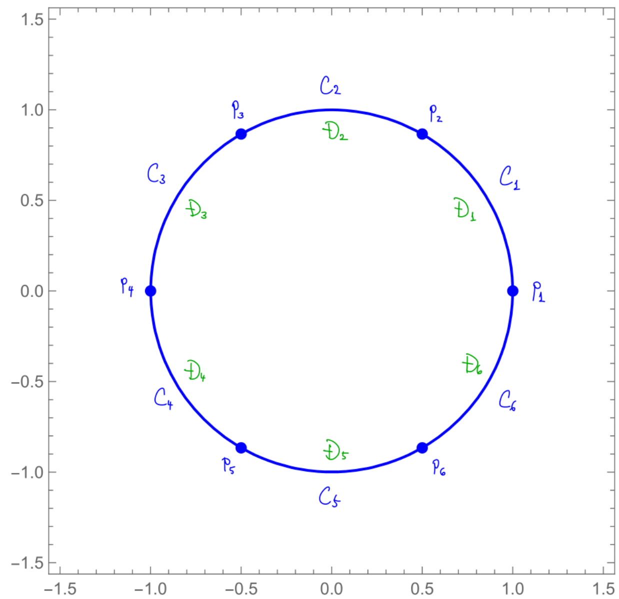

Figure 3.13. Riemann-Hilbert Problem of $\check{Y}^D$.

Theorem 3.7.1. *The solution of the RHP 2 is*

$$\begin{aligned}\check{Y}^D(\zeta) = &\left(\frac{\zeta-1}{\zeta-1/2-i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_1}\left(\frac{\zeta-1/2-i\sqrt{3}/2}{\zeta+1/2-i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_2}\\ &\times\left(\frac{\zeta+1/2-i\sqrt{3}/2}{\zeta+1}\right)^{-\frac{1}{2\pi i}\ln D_3}\left(\frac{\zeta+1}{\zeta+1/2+i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_4}\\ &\times\left(\frac{\zeta+1/2+i\sqrt{3}/2}{\zeta-1/2+i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_5}\left(\frac{\zeta-1/2+i\sqrt{3}/2}{\zeta-1}\right)^{-\frac{1}{2\pi i}\ln D_6}.\end{aligned} \tag{3.7.1}$$

Proof. We will check (3.7.1) satisfies all conditions in the RHP 2.

(i) Let

$$f_1(\zeta) = \left(\frac{\zeta-1}{\zeta-1/2-i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_1},\ f_2(\zeta) = \left(\frac{\zeta-1/2-i\sqrt{3}/2}{\zeta+1/2-i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_2},$$

$$f_3(\zeta) = \left(\frac{\zeta+1/2-i\sqrt{3}/2}{\zeta+1}\right)^{-\frac{1}{2\pi i}\ln D_3},\ f_4(\zeta) = \left(\frac{\zeta+1}{\zeta+1/2+i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_4},$$

$$f_5(\zeta) = \left(\frac{\zeta+1/2+i\sqrt{3}/2}{\zeta-1/2+i\sqrt{3}/2}\right)^{-\frac{1}{2\pi i}\ln D_5},\ f_6(\zeta) = \left(\frac{\zeta-1/2+i\sqrt{3}/2}{\zeta-1}\right)^{-\frac{1}{2\pi i}\ln D_6}.$$

Note that each $f_n(\zeta)$ is a composite function of a certain linear fractional transformation and the power function.

Also, for the arc of the unit circle with two endpoints α and β in the ζ-plane, we can consider the composite function of a square root and a linear fractional transformation,

$$\varphi(\zeta) = \frac{\zeta - \alpha}{\zeta - \beta},$$

such that φ maps that arc to the straight line on the $\varphi(\zeta)$-plane with an angle θ, admits the branch cut. This is because a complex logarithmic function requires a branch cut, for it to be a single-valued function. By writing a principal logarithm by Log_θ after choosing a branch at the angle θ line, one can write $\sqrt{\varphi(\zeta)}$ by

$$\sqrt{\varphi(\zeta)} = e^{\frac{1}{2} Log_\theta \varphi(\zeta)}.$$

That is why $\sqrt{\varphi(\zeta)}$ has a branch cut on the arc of the unit circle with two endpoints α and β, and is holomorphic everywhere except for on the arc.

Each $f_n(\zeta)$ has such a linear fractional transformation and so it has a distinct branch cut, C_n, in ζ-plane. Therefore, their product,

$$\check{Y}^D(\zeta) = f_1(\zeta) f_2(\zeta) \cdots f_6(\zeta),$$

is holomorphic except for on S^1.

(ii) Now, we check the jump condition. Let's focus on the arc C_1 now. From the above branch cut discussion, all $f_n(\zeta)$'s but $f_1(\zeta)$ have no jump on C_1, i.e.,

$$f_{n,+}(\zeta) = f_{n,-}(\zeta) \tag{3.7.2}$$

for $n = 2, \cdots, 6$.

Let

$$\varphi_1(\zeta) = \frac{\zeta - 1}{\zeta - 1/2 - i\sqrt{3}/2}.$$

Then, C_1 is mapped into the straight line starting from the origin with an angle $5\pi/6$. Since approaching from the positive side of C_1 gives an angle $-7\pi/6$ to $\varphi_1(\zeta)$ and approaching from the negative side of C_1 gives $\varphi_1(\zeta)$ an angle $5\pi/6$. Thus, we know that $+$-side limit of φ_1 and $-$-side of limit of φ_1 differs by $e^{-2\pi i}$:

$$\varphi_{1,+}(\zeta) = \varphi_{1,-}(\zeta)e^{-2\pi i}.$$

Taking $-\frac{1}{2\pi i}\ln D_1$ powers on the both sides, we have

$$f_{1,+}(\zeta) = f_{1,-}(\zeta)e^{\ln D_1} = f_{1,-}(\zeta)D_1. \tag{3.7.3}$$

By (3.7.2) and (3.7.3), we have

$$\begin{aligned} \check{Y}_+^D(\zeta) &= f_{1,+}(\zeta)f_{2,+}(\zeta)\cdots f_{6,+}(\zeta) \\ &= f_{1,-}(\zeta)D_1 f_{2,-}(\zeta)\cdots f_{6,-}(\zeta) \\ &= \check{Y}_-^D(\zeta)D_1 \end{aligned}$$

on C_1. Note that we use the diagonality of all matrices to interchange them so as to get the last equality. This jump condition holds for other arcs too.

(iii) The normalization condition is easy; as $\zeta \to \infty$, we have each linear fractional transformation in $f_n(\zeta)$ goes to 1 and so (3.7.1) tends to I. □

Remark. In the neighborhood of p_1, we consider a small ball U_1 centered at 1. Let

$$\begin{aligned} R_1^{(1)} &= U_1 \cap (\text{outside of the unit circle}) \\ R_1^{(2)} &= U_1 \cap (\text{inside of the unit circle}) \cap (\text{upper half plane}) \\ R_1^{(3)} &= U_1 \cap (\text{inside of the unit circle}) \cap (\text{lower half plane}). \end{aligned}$$

Then, we observe the following asymptotic behavior of $\check{Y}^D(\zeta)$ in the neighborhood of $p_1 = 1$,

$$\check{Y}^D(\zeta) = P_1\Theta_1(\zeta)\begin{pmatrix}1 & 0 & 0\\ 0 & (\zeta-1)^{\nu} & 0\\ 0 & 0 & (\zeta-1)^{-\nu}\end{pmatrix} \times \begin{cases} I & \text{for } \zeta \in R_1^{(1)} \\ D_1 & \text{for } \zeta \in R_1^{(2)} \\ D_6 & \text{for } \zeta \in R_1^{(3)}, \end{cases} \tag{3.7.4}$$

where $P_1\Theta_1(\zeta)$ is a Taylor expansion of $\check{Y}^D(\zeta)/(\zeta-1)^{\frac{1}{2\pi i}\ln D_1 D_6^{-1}}$,

$$P_1 = \begin{pmatrix} e^{-\pi i\nu} & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{e^{\frac{\pi i \nu}{2}}\left(2\sqrt{3}\right)^{-\nu\sigma_3}} \end{pmatrix}$$

$$\begin{aligned} P_1 = &\left(\frac{1}{1+\overline{\omega}}\right)^{-\frac{1}{2\pi i}\ln D_1}\left(\frac{1+\overline{\omega}}{1-\omega}\right)^{-\frac{1}{2\pi i}\ln D_2}\left(\frac{1-\omega}{2}\right)^{-\frac{1}{2\pi i}\ln D_3} \\ &\times\left(\frac{2}{1-\overline{\omega}}\right)^{-\frac{1}{2\pi i}\ln D_4}\left(\frac{1-\overline{\omega}}{1+\omega}\right)^{-\frac{1}{2\pi i}\ln D_5}(1+\omega)^{-\frac{1}{2\pi i}\ln D_6}, \end{aligned}$$

and $\nu = -\frac{1}{2\pi i}\ln 3A^{\mathbb{R}}$ (provided that $A^{\mathbb{R}} > 0$).

3.7.2 Local parametrix near p_1

Now we consider the model RHP for the local parametrix near p_1 of $\check{Y}$-problem whose contour is depicted in Figure 3.12. In the ζ-plane, take a small disk U_1 around p_1, which we later rigorously define in (3.7.5). We call a function defined locally on U_1 by $\Phi^{(0)}(\zeta)$. Jump matrices of this problem are written in Figure 3.14. More precisely, they are

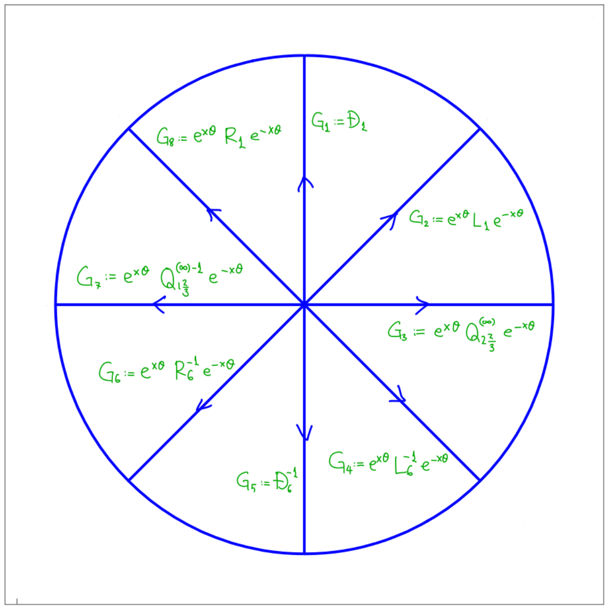

Figure 3.14. $\Phi^{(0)}(\zeta)$-problem.

$$G_1 = e^{x\theta} D_1 e^{-x\theta} = \begin{pmatrix} 1/3A^{\mathbb{R}} & & \\ & 3A^{\mathbb{R}} & \\ & & 1 \end{pmatrix},$$

$$G_2 = e^{x\theta} L_1 e^{-x\theta} = \begin{pmatrix} 1 & \left(-\frac{B}{A^{\mathbb{R}}} - \omega^2 \frac{|B|^2}{(A^{\mathbb{R}})^2}\right) e^{-x\varphi_1} & -\frac{\overline{B}}{A^{\mathbb{R}}} e^{-x\varphi_2} \\ 0 & 1 & 0 \\ 0 & \omega^2 \frac{B}{A^{\mathbb{R}}} e^{x\varphi_3} & 1 \end{pmatrix},$$

$$G_3 = e^{x\theta} Q^{(\infty)}_{2\frac{2}{3}} e^{-x\theta} = \begin{pmatrix} 1 & 0 & -\omega s^{\mathbb{R}} e^{-x\varphi_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_4 = e^{x\theta} L_6^{-1} e^{-x\theta} = \begin{pmatrix} 1 & \frac{B}{A^{\mathbb{R}}} e^{-x\varphi_1} & \left(\omega s^{\mathbb{R}} + \frac{\overline{B}}{A^{\mathbb{R}}}\right) e^{-x\varphi_2} \\ 0 & 1 & -\omega \frac{\overline{B}}{A^{\mathbb{R}}} e^{-x\varphi_3} \\ 0 & 0 & 1 \end{pmatrix},$$

$$G_5 = e^{x\theta} D_6^{-1} e^{-x\theta} = \begin{pmatrix} 3A^{\mathbb{R}} & & \\ & 1 & \\ & & 1/3A^{\mathbb{R}} \end{pmatrix},$$

$$G_6 = e^{x\theta} R_6^{-1} e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\overline{B}}{A^{\mathbb{R}}} e^{x\varphi_1} & 1 & 0 \\ \left(\omega^2 s^{\mathbb{R}} + \frac{B}{A^{\mathbb{R}}}\right) e^{x\varphi_2} & -\omega^2 \frac{B}{A^{\mathbb{R}}} e^{x\varphi_3} & 1 \end{pmatrix},$$

$$G_7 = e^{x\theta} Q_{1\frac{2}{3}}^{(\infty)-1} e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\omega^2 s^{\mathbb{R}} e^{x\varphi_2} & 0 & 1 \end{pmatrix},$$

$$G_8 = e^{x\theta} R_1 e^{-x\theta} = \begin{pmatrix} 1 & 0 & 0 \\ \left(-\frac{\overline{B}}{A^{\mathbb{R}}} - \omega\frac{|B|^2}{(A^{\mathbb{R}})^2}\right) e^{x\varphi_1} & 1 & \omega\frac{\overline{B}}{A^{\mathbb{R}}} e^{-x\varphi_3} \\ -\frac{B}{A^{\mathbb{R}}} e^{x\varphi_2} & 0 & 1 \end{pmatrix}.$$

Next, define $\Phi^{(1)}(\zeta)$ by multiplying $\Phi^{(0)}(\zeta)$ by certain matrices as in Figure 3.15. Matrices L and R used in this definition are

$$L = \begin{pmatrix} 1 & -\frac{B}{A^{\mathbb{R}}} e^{-x\varphi_1} & -\frac{\overline{B}}{A^{\mathbb{R}}} e^{-x\varphi_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\overline{B}}{A^{\mathbb{R}}} e^{x\varphi_1} & 1 & 0 \\ -\frac{B}{A^{\mathbb{R}}} e^{x\varphi_2} & 0 & 1 \end{pmatrix}.$$

Notice that these multiplications does not affect the asymptotic behavior on the boundary of the disk.

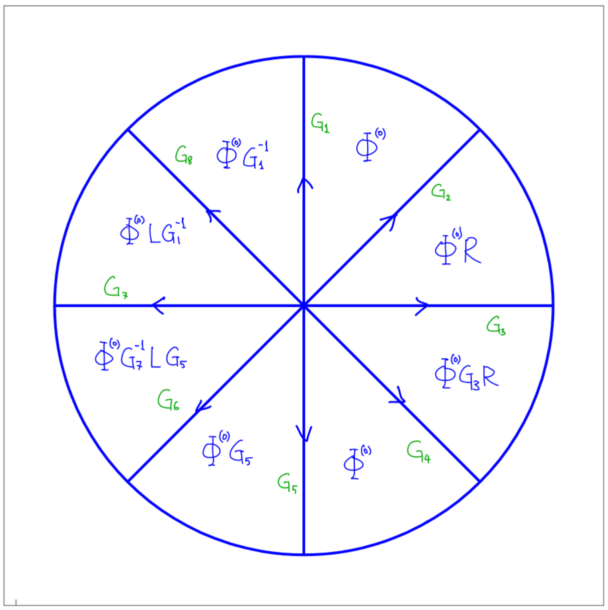

Figure 3.15. Definition of the $\Phi^{(1)}(\zeta)$-problem..

One can compute jump matrices H_i for the $\Phi^{(1)}$-problem as they are written in Figure 3.16. More precisely,

$$H_1 = R^{-1}G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_1 e^{x\varphi_3} & 1 \end{pmatrix}, \; H_2 = G_1 G_8 L G_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3A^{\mathbb{R}}\overline{s_1} e^{-x\varphi_3} \\ 0 & 0 & 1 \end{pmatrix},$$

$$H_0 = G_5 G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3A^{\mathbb{R}} & 0 \\ 0 & 0 & \frac{1}{3A^{\mathbb{R}}} \end{pmatrix}, \; H_3 = G_5^{-1} L^{-1} G_7 G_6 G_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3A^{\mathbb{R}} s_1 e^{x\varphi_3} & 1 \end{pmatrix},$$

$$H_4 = G_4 G_3 R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\overline{s_1} e^{-x\varphi_3} \\ 0 & 0 & 1 \end{pmatrix},$$

where $s_1 = \omega^2 \frac{B}{A^{\mathbb{R}}}$.

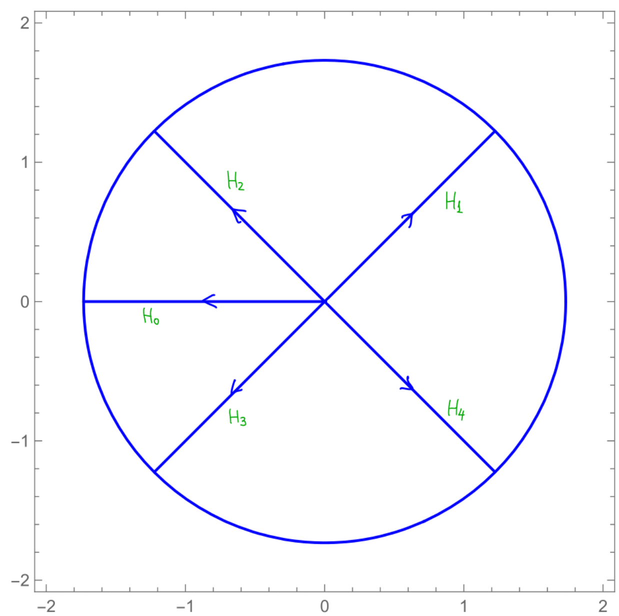

Figure 3.16. $\Phi^{(1)}(\zeta)$-problem.

Let us now introduce a new variable $z(\zeta)$ by

$$z(\zeta) := \sqrt{2x} e^{i\frac{\pi}{2}} \sqrt{\varphi_3(\zeta) - i2\sqrt{3}}.$$

Let ε satisfy $\frac{1}{2} < \varepsilon < \frac{2}{3}$. Then, the map $\zeta \mapsto z$ is conformal on

$$U_1 = \left\{ \zeta \in \mathbb{C} \;\middle|\; |\zeta - 1| < \frac{1}{\sqrt{2}\, 3^{1/4}} x^{\varepsilon - 1} \right\}, \tag{3.7.5}$$

since $z'(\zeta) \neq 0$ there, i.e. the map z just rotates the contours of Figure 3.16 and preserves the angle among them. Since $\varepsilon < 1$ by choice, $z(\zeta)$ behaves

$$z(\zeta) \sim \sqrt{2x} e^{i\frac{3\pi}{4}} 3^{1/4} (\zeta - 1) \tag{3.7.6}$$

in $\overline{U_1}$. On ∂U_1, $|z| = x^{\varepsilon - \frac{1}{2}} \to \infty$ as $x \to \infty$. Furthermore, it holds that

$$x\varphi_3(\zeta) = x2\sqrt{3}i - \frac{z^2}{2} + \mathcal{O}(x^{3\varepsilon - 2}). \tag{3.7.7}$$

When we write $\Psi^{(1)}(z) = \Phi^{(1)}(\zeta)$, we can consider everything in the z-plane. The jump contour and jump matrices of $\Psi^{(1)}(z)$-problem are expressed in Figure 3.17. Moreover, these jump matrices admit a 2×2 block structure,

$$K_1(z) = H_1(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{S_1}} \\ 0 & & \end{pmatrix}, \quad K_2(z) = H_2(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{S_2}} \\ 0 & & \end{pmatrix},$$

$$K_0 = H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{D_0}} \\ 0 & & \end{pmatrix}, \quad K_3(z) = H_3(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{S_3}} \\ 0 & & \end{pmatrix},$$

$$K_4(z) = H_4(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{S_4}} \\ 0 & & \end{pmatrix},$$

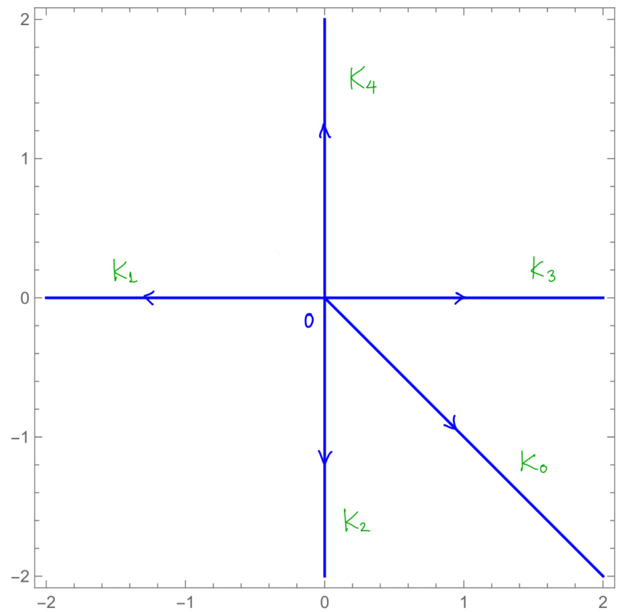

Figure 3.17. $\Psi^{(1)}(z)$-problem.

where

$$
\begin{aligned}
S_1 &= \begin{pmatrix} 1 & 0 \\ s_1 e^{x\varphi_3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \widehat{s_1} e^{-\frac{z^2}{2}} & 1 \end{pmatrix}, \\
S_2 &= \begin{pmatrix} 1 & 3A^{\mathbb{R}} \overline{s_1} e^{-x\varphi_3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-2\pi i\nu} \overline{\widehat{s_1}} e^{\frac{z^2}{2}} \\ 0 & 1 \end{pmatrix}, \\
D_0 &= \begin{pmatrix} 3A^{\mathbb{R}} & 0 \\ 0 & \frac{1}{3A^{\mathbb{R}}} \end{pmatrix} = e^{-2\pi i\nu\sigma_3}, \\
S_3 &= \begin{pmatrix} 1 & 0 \\ -3A^{\mathbb{R}} s_1 e^{x\varphi_3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{-2\pi i\nu} \widehat{s_1} e^{-\frac{z^2}{2}} & 1 \end{pmatrix}, \\
S_4 &= \begin{pmatrix} 1 & -\overline{s_1} e^{-x\varphi_3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\overline{\widehat{s_1}} e^{\frac{z^2}{2}} \\ 0 & 1 \end{pmatrix},
\end{aligned}
\tag{3.7.8}
$$

where we used (3.7.7) and set

$$\widehat{s_1} = s_1 e^{i2\sqrt{3}x} \text{ and } \nu = -\frac{1}{2\pi i}\ln 3A^{\mathbb{R}} = \frac{1}{2\pi i}\ln(1-|s_1|^2).$$

Thus, what we need to find is a 2×2 matrix valued function $\phi^{(0)}(z)$, with the jump contour in Figure 3.17 and the jump matrices (3.7.8). Let us formulate the following RHP:

Riemann-Hilbert Problem 3. Find a solution $\phi^{(0)}(z)$ satisfying

- $\phi^{(0)}(z)$ is holomorphic in $\mathbb{C}\setminus\Gamma_5$ where Γ_5 is the jump contour defined in Figure 3.17.
- $\phi^{(0)}_+(z) = \phi^{(0)}_-(z)G_{\phi^{(0)}}$, where the jump matrix $G_{\phi^{(0)}}$ is given by (3.7.8).
- $\phi^{(0)}(z) = (I + \mathcal{O}(1/z))z^{\nu\sigma_3}$ as $z\to\infty$.

One can solve this problem by the use of parabolic cylinder function. (See [41], Chapter 2, Section 1.5). The asymptotic behaviour of $\phi^{(0)}(z)$ is

$$\phi^{(0)}(z) = \left(I + \frac{1}{z}\begin{pmatrix} 0 & -\alpha \\ \nu/\alpha & 0 \end{pmatrix} + \mathcal{O}(z^{-2})\right) z^{\nu\sigma_3}, \tag{3.7.9}$$

where

$$\alpha = -\frac{i}{\widehat{s_1}}\frac{\sqrt{2\pi}e^{2\pi i\nu}}{\Gamma(-\nu)},$$

as $z\to\infty$.

Thus, the asymptotic behaivor of $\Psi^{(1)}(z)$ on the boundary of U_1 is

$$\Psi^{(1)}(z) = \left(I + \frac{1}{z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & m_1^{(0)} & \end{pmatrix} + \cdots\right)\begin{pmatrix} 1 & & \\ & z^{\nu} & \\ & & z^{-\nu}\end{pmatrix},$$

where

$$m_1^{(0)} = \begin{pmatrix} 0 & -\alpha \\ \nu/\alpha & 0 \end{pmatrix}$$

by (3.7.9). Going backward, one can find the asymptotic behavior of $\Phi^{(1)}(\zeta)$ and then that of $\Phi^{(0)}(\zeta)$. In fact, on the boundary of U_1, it holds that

$$\Phi^{(0)}(\zeta) = \left(I + \frac{1}{z(\zeta)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$m_1^{(0)}$}} \\ 0 & & \end{pmatrix} + \cdots \right) \begin{pmatrix} 1 & & \\ & z(\zeta)^{\nu} & \\ & & z(\zeta)^{-\nu} \end{pmatrix} \times \begin{cases} I & \text{for } \zeta \in R_1^{(1)} \\ D_1 & \text{for } \zeta \in R_1^{(2)} \\ D_6 & \text{for } \zeta \in R_1^{(3)}. \end{cases} \tag{3.7.10}$$

where

$$\begin{aligned} R_1^{(1)} &= U_1 \cap (\text{outside of the unit circle}) \\ R_1^{(2)} &= U_1 \cap (\text{inside of the unit circle}) \cap (\text{upper half plane}) \\ R_1^{(3)} &= U_1 \cap (\text{inside of the unit circle}) \cap (\text{lower half plane}). \end{aligned}$$

INtroducing

$$E(\zeta) = \check{Y}^D(\zeta) \begin{pmatrix} 1 & & \\ & z(\zeta)^{-\nu} & \\ & & z(\zeta)^{\nu} \end{pmatrix} \times \begin{cases} I & \text{for } \zeta \in R_1^{(1)} \\ D_1^{-1} & \text{for } \zeta \in R_1^{(2)} \\ D_6^{-1} & \text{for } \zeta \in R_1^{(3)} \end{cases} \tag{3.7.11}$$

where $z(\zeta)$ is given by (3.7.6), we can define a local parametrix near p_1 by

$$P^{(1)}(\zeta) = E(\zeta)\Phi^{(0)}(\zeta).$$

Notice that $E(\zeta)$ is holomorphic in U_1. Indeed, it admits the following Taylor expansion,

$$E(\zeta) = P_1\Theta_1(\zeta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & (\sqrt{2x}e^{i\frac{3\pi}{4}}3^{1/4})^{-\nu} & 0 \\ 0 & 0 & (\sqrt{2x}e^{i\frac{3\pi}{4}}3^{1/4})^{\nu} \end{pmatrix}$$

by (3.7.4) and (3.7.11). Moreover, on the boundary of U_1, $\Phi^{(0)}(\zeta)$ has the asymptotic expansion (3.7.10), so we have the matching up condition,

$$\begin{aligned} P^{(1)}(\zeta) &= \check{Y}^D(\zeta) \\ &\times \left(I + \frac{1}{z(\zeta)} \begin{Bmatrix} I \\ D_1^{-1} \\ D_6^{-1} \end{Bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & z^{-\nu\sigma_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & m_1^{(0)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & z^{\nu\sigma_3} \end{pmatrix} \begin{Bmatrix} I \\ D_1 \\ D_6 \end{Bmatrix} + \cdots \right) \qquad (3.7.12) \\ &= \check{Y}^D(\zeta)(I + \mathcal{O}(1/\sqrt{x})). \end{aligned}$$

Therefore, we could construct a local parametrix $P^{(1)}(\zeta)$ which matches up with the global parametrix $\check{Y}^D(\zeta)$ on the boundary of U_1.

3.7.3 Other local parametrices

Proposition 3.7.2. Other local parametrices are expressed by $P^{(1)}(\zeta)$ essentially. In fact, we have

$$P^{(-1)}(\zeta) = d_3^{-1}\left[P^{(1)}(-\zeta)\right]^{T-1} d_3. \qquad (3.7.13)$$

$$P^{(-\overline{\omega})}(\zeta) = \Pi^{-1}P^{(-1)}(\omega\zeta)\Pi. \qquad (3.7.14)$$

$$P^{(\omega)}(\zeta) = \Pi P^{(1)}(\overline{\omega}\zeta)\Pi^{-1}. \qquad (3.7.15)$$

$$P^{(\overline{\omega})}(\zeta) = d_3^{-1}\left[P^{(-\overline{\omega})}(-\zeta)\right]^{T-1} d_3. \qquad (3.7.16)$$

$$P^{(-\omega)}(\zeta) = d_3^{-1}\left[P^{(\omega)}(-\zeta)\right]^{T-1} d_3. \qquad (3.7.17)$$

Proof. Let $G^{(w)}(\zeta)$ denote the jump matrices for $P^{(w)}(\zeta)$-problem. If (3.7.13) is true, we have that

$$\begin{aligned}
&P_+^{(-1)}(\zeta) = P_-^{(-1)}(\zeta)G^{(-1)}(\zeta) \\
&\Leftrightarrow d_3^{-1}\left[P_+^{(1)}(-\zeta)\right]^{T-1} d_3 = d_3^{-1}\left[P_-^{(1)}(-\zeta)\right]^{T-1} d_3 G^{(-1)}(\zeta) \\
&\Leftrightarrow \left[P_+^{(1)}(-\zeta)\right]^{T-1} = \left[P_-^{(1)}(-\zeta)\right]^{T-1} d_3 G^{(-1)}(\zeta) d_3^{-1} \\
&\Leftrightarrow \left[G^{(1)}(-\zeta)\right]^{T-1} = d_3 G^{(-1)}(\zeta) d_3^{-1} \\
&\Leftrightarrow G^{(-1)}(\zeta) = d_3^{-1}\left[G^{(1)}(-\zeta)\right]^{T-1} d_3.
\end{aligned}$$

So, we need to check if all jump matrices of the RHP near p_4, that is $\zeta = -1$, are related to the local parametrix in (3.7.13) as above. We saw the anti-symmetry for $Q_n^{(\infty)}$ already,

$$Q_{n+1}^{(\infty)} = d_3^{-1}\left[Q_n^{(\infty)}\right]^{T-1} d_3.$$

Moreover, it holds that

$$L_3^{-1} = d_3^{-1}\left[L_6\right]^{T} d_3,\ L_4 = d_3^{-1}\left[L_1\right]^{T-1} d_3,\ D_4 = D_1^{-1}.$$

Note that the normalization condition is good in this case too.

Next, we check the matching up condition. By (3.7.12),

$$P^{(-1)}(\zeta) = d_3^{-1}\left[\check{Y}^D(-\zeta)\right]^{T-1} d_3 (I + \mathcal{O}(1/\sqrt{x})).$$

But $d_3^{-1}\left[\check{Y}^D(-\zeta)\right]^{T-1} d_3 = \left[\check{Y}^D(-\zeta)\right]^{-1} = \check{Y}^D(\zeta)$. Thus,

$$P^{(-1)}(\zeta) = \check{Y}^D(\zeta)(I + \mathcal{O}(1/\sqrt{x})).$$

So, our $P^{(-1)}(\zeta)$ satisfies all the local parametrix conditions. Thus, (3.7.13) holds.

The similar type of proof can be applied to show the other relations, (3.7.14) – (3.7.17).

□

3.8 The approximate solution of the original RHP as $x \to \infty$

Let $U_w = \left\{\zeta \in \mathbb{C} \mid |\zeta - w| < 2^{-1/2}3^{-1/4}x^{\varepsilon-1}\right\}$ for $\frac{1}{2} < \varepsilon < \frac{2}{3}$. At each p_i, we shall define a small disk of this type. Let us call their union by U_{all}:

$$U_{all} = U_1 \cup U_{-\overline{\omega}} \cup U_\omega \cup U_{-1} \cup U_{\overline{\omega}} \cup U_{-\omega}.$$

Then, define a piecewise holomorphic function,

$$\check{Y}^{(as)}(\zeta) = \begin{cases} \check{Y}^D(\zeta) & \text{if } \zeta \in \mathbb{C} \setminus U_{all} \\ P^{(1)}(\zeta) & \text{if } \zeta \in U_1 \\ P^{(-\overline{\omega})}(\zeta) & \text{if } \zeta \in U_{-\overline{\omega}} \\ P^{(\omega)}(\zeta) & \text{if } \zeta \in U_\omega \\ P^{(-1)}(\zeta) & \text{if } \zeta \in U_{-1} \\ P^{(\overline{\omega})}(\zeta) & \text{if } \zeta \in U_{\overline{\omega}} \\ P^{(-\omega)}(\zeta) & \text{if } \zeta \in U_{-\omega} \end{cases}$$

which is discontinuous on the boundary of each disk and on the unit circle with holes at $1, -\overline{\omega}, \ldots, -\omega$.

3.8.1 The error Riemann-Hilbert Problem

In the previous sections we have constructed asymptotic solution $\check{Y}^{(as)}(\zeta)$. Let us introduce the error functions $R(\zeta)$ by

$$R(\zeta) := \check{Y}(\zeta)\left(\check{Y}^{(as)}(\zeta)\right)^{-1},$$

where $\check{Y}^{(as)}(\zeta)$ is the exact solution of the RHP. Then the error function $R(\zeta)$ solves the following RHP.

Riemann-Hilbert Problem 4. Find a solution $R(\zeta)$ satisfying

- $R(\zeta)$ is holomorphic in $\mathbb{C} \setminus \Gamma_6$ where Γ_6 is the jump contour defined in Figure 3.18.
- $R_+(\zeta) = R_-(\zeta) G_R$, where the jump matrix G_R are

$$G_R = \begin{cases} \check{Y}^D(\zeta) G_{\check{Y}} \left(\check{Y}^D(\zeta)\right)^{-1} & \text{if } \zeta \in \Gamma_4 \setminus U_{all} \\ \check{Y}^D(\zeta) \left(P^{(w)}(\zeta)\right)^{-1} & \text{if } \zeta \in \partial U_w. \end{cases}$$

- $R(\zeta) = I + \mathcal{O}(1/\zeta)$ as $\zeta \to \infty$.

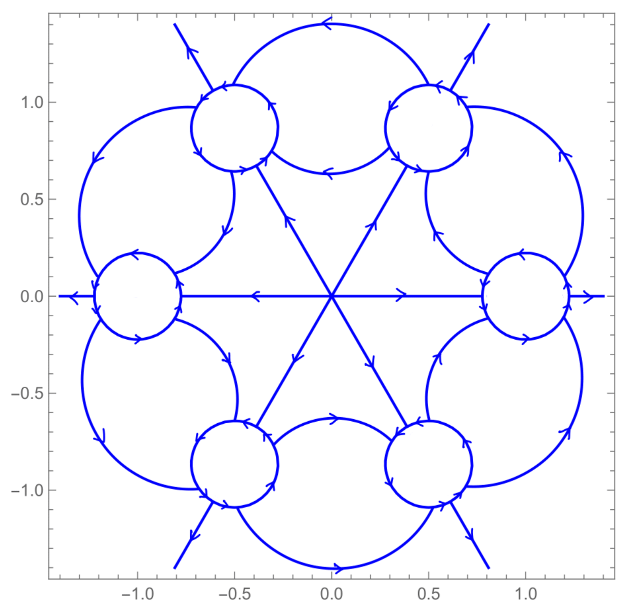

Figure 3.18. The jump contour of $R(\zeta)$-problem.

Moreover, $R(\zeta)$ is uniformly close to I due to the structure of jump matrices of $\check{Y}$-problem on Γ_4 and the matching up conditions such as (3.7.12). Indeed,

$$||G_R - I||_{L^2(\Gamma_6) \cap L^\infty(\Gamma_6)} \leq c_1 \frac{1}{\sqrt{x}}.$$

This small norm condition implies the unique solvability of the RHP 4 for the error function $R(\zeta)$. Indeed, solving this RHP is equivalent to solve the following singular integral equation,

$$R(\zeta) = I + \frac{1}{2\pi i}\int_{\Gamma_6} \frac{\rho(\zeta')(G_R(\zeta') - I)}{\zeta' - \zeta} d\zeta', \ \zeta \notin \Gamma_6. \tag{3.8.1}$$

Then, the small norm theorem (Theorem 8.1 in [41]) implies the unique solvability of $R(\zeta)$. Furthermore, we have

$$||\rho - I||_{L^2(\Gamma_6)} \leq c_2 \frac{1}{\sqrt{x}}. \tag{3.8.2}$$

If we consider the limit $\zeta \to 0$, then the equation (3.8.1) becomes

$$\begin{aligned} R(0) &= I + \frac{1}{2\pi i}\int_{\Gamma_6} \frac{\rho(\zeta')(G_R(\zeta') - I)}{\zeta'} d\zeta' \\ &= I + \frac{1}{2\pi i}\int_{\Gamma_6} \frac{G_R(\zeta') - I}{\zeta'} d\zeta' + \frac{1}{2\pi i}\int_{\Gamma_6} \frac{(\rho(\zeta') - I)(G_R(\zeta') - I)}{\zeta'} d\zeta' \\ &= I + \frac{1}{2\pi i}\int_{\Gamma_6} \frac{G_R(\zeta') - I}{\zeta'} d\zeta' + \mathcal{O}\left(\frac{1}{x}\right), \end{aligned} \tag{3.8.3}$$

where the last equality is obtained by using Cauchy Schwarz inequality, (3.8.2) and

$$\left|\left|\frac{G_R(\cdot) - I}{\cdot}\right|\right|_{L^2(\Gamma_6)} \leq c_3 \frac{1}{\sqrt{x}}.$$

Taking into account

$$p_1 = \{1\}, \ p_2 = \{-\overline{\omega}\}, \ p_3 = \{\omega\}, \ p_4 = \{-1\}, \ p_5 = \{\overline{\omega}\}, \ p_6 = \{-\omega\},$$

let us write the boundary of U_w's as follows:

$$\gamma_1 = \partial U_1, \ \gamma_2 = \partial U_{-\overline{\omega}}, \ \gamma_3 = \partial U_{\omega}, \ \gamma_4 = \partial U_{-1}, \ \gamma_5 = \partial U_{\overline{\omega}}, \ \gamma_6 = \partial U_{-\omega}.$$

Then, since we can ignore the exponentially small contribution of Γ_6, we can transform (3.8.3) into the equation,

$$R(0) = I + \sum_{n=1}^{6} \frac{1}{2\pi i} \int_{\gamma_n} \frac{G_R(\zeta') - I}{\zeta'} d\zeta' + \mathcal{O}\left(\frac{1}{x}\right). \tag{3.8.4}$$

3.8.2 Computation of the asymptotic solution

In this subsection, we will describe the asymptotic behavior of the solution $w_0(x)$ as $x \to \infty$. First, we note that $R(0)$ by definition satisfies

$$\check{Y}(0) = R(0)\check{Y}^D(0). \tag{3.8.5}$$

By (3.7.1), we have $\check{Y}^D(0) = I$. Then, we compute the asymptotics of $\check{Y}(\zeta)$ as $\zeta \to 0$. In section 3.5, we saw that $\check{Y}(0) = \mathcal{O}(1)$. More precisely, it holds that

$$\begin{aligned}
\check{Y}(0) &= \Omega e^{-2w}\Omega^{-1} + \mathcal{O}(\zeta) \\
&= \frac{1}{3}\begin{pmatrix} 1 + e^{2w_0} + e^{-2w_0} & \omega^2 + \omega e^{2w_0} + e^{-2w_0} & \omega + \omega^2 e^{2w_0} + e^{-2w_0} \\ \omega + \omega^2 e^{2w_0} + e^{-2w_0} & 1 + e^{2w_0} + e^{-2w_0} & \omega^2 + \omega e^{2w_0} + e^{-2w_0} \\ \omega^2 + \omega e^{2w_0} + e^{-2w_0} & \omega + \omega^2 e^{2w_0} + e^{-2w_0} & 1 + e^{2w_0} + e^{-2w_0} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{2}{3}w_0(\omega - 1) & \frac{2}{3}w_0(\omega^2 - 1) \\ \frac{2}{3}w_0(\omega^2 - 1) & 1 & \frac{2}{3}w_0(\omega - 1) \\ \frac{2}{3}w_0(\omega - 1) & \frac{2}{3}w_0(\omega^2 - 1) & 1 \end{pmatrix} + \mathcal{O}(w_0^2).
\end{aligned}$$

Thus, from (3.8.5), we have

$$R(0) = \begin{pmatrix} 1 & \frac{2}{3}w_0(\omega - 1) & \frac{2}{3}w_0(\omega^2 - 1) \\ \\ \frac{2}{3}w_0(\omega^2 - 1) & 1 & \frac{2}{3}w_0(\omega - 1) \\ \\ \frac{2}{3}w_0(\omega - 1) & \frac{2}{3}w_0(\omega^2 - 1) & 1 \end{pmatrix} + \mathcal{O}(w_0^2). \tag{3.8.6}$$

We will obtain another expression for $R(0)$ considering the residues of the integrals (3.8.4).

Let's consider jump matrix $G_R(\zeta)$ on ∂U_1. By (3.7.12) and (3.7.4), it holds that

$$\begin{aligned} G_R(\zeta) &= \check{Y}^D \left(P^{(1)}(\zeta)\right)^{-1} \\ &= \check{Y}^D(\zeta) \left(I - \frac{1}{z} \begin{Bmatrix} I \\ D_1^{-1} \\ D_6^{-1} \end{Bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{z^{-\nu\sigma_3}} \\ 0 & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{m_1^{(0)}} \\ 0 & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{z^{\nu\sigma_3}} \\ 0 & & \end{pmatrix} \begin{Bmatrix} I \\ D_1 \\ D_6 \end{Bmatrix} + \cdots \right) \check{Y}^{D-1} \\ &= P_1 \Theta_1(\zeta) \begin{pmatrix} 1 & & \\ & z^{-\nu}(\zeta-1)^{\nu} & \\ & & z^{\nu}(\zeta-1)^{-\nu} \end{pmatrix} \\ &\quad \times \left(I - \frac{1}{z} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{m_1^{(0)}} \\ 0 & & \end{pmatrix} + \cdots \right) \begin{pmatrix} 1 & & \\ & z^{\nu}(\zeta-1)^{-\nu} & \\ & & z^{-\nu}(\zeta-1)^{\nu} \end{pmatrix} \Theta_1^{-1}(\zeta) P_1^{-1} \\ &= I - \frac{1}{z(\zeta)} \widetilde{\Theta_1}(\zeta) P_1 \begin{pmatrix} 1 & & \\ & \kappa^{-1} & \\ & & \kappa \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{m_1^{(0)}} \\ 0 & & \end{pmatrix} \begin{pmatrix} 1 & & \\ & \kappa & \\ & & \kappa^{-1} \end{pmatrix} P_1^{-1} \widetilde{\Theta_1}(\zeta)^{-1} + \mathcal{O}\left(\frac{1}{x}\right), \end{aligned}$$

where we set

$$z^{\nu}(\zeta - 1)^{-\nu} = \kappa\left(1 + \mathcal{O}(\zeta - 1)\right), \ \kappa = e^{\frac{3\pi i}{4}\nu}(2x)^{\nu/2}3^{\nu/4}$$

and $\widetilde{\Theta_1}(\zeta)$ is some Taylor series coming from $\Theta_1(\zeta)$ and $z^\nu(\zeta-1)^{-\nu}$. Since

$$P_1 = \begin{pmatrix} e^{-\pi i\nu} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$e^{\frac{\pi i\nu}{2}}\left(2\sqrt{3}\right)^{-\nu\sigma_3}$}} \\ 0 & & \end{pmatrix},$$

we can compute the residue at $\zeta = 1$ as

$$\operatorname{res}_{\zeta=1}\frac{G_R(\zeta)-I}{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\widehat{m_1^{(0)}}$}} \\ 0 & & \end{pmatrix}, \tag{3.8.7}$$

where

$$\widehat{m_1^{(0)}} = -\frac{1}{\sqrt{2x}}3^{-1/4}e^{-\frac{3\pi i}{4}}(2\sqrt{3})^{-\nu\sigma_3}\kappa^{-\sigma_3}\begin{pmatrix} 0 & -\alpha \\ \nu/\alpha & 0 \end{pmatrix}\kappa^{\sigma_3}(2\sqrt{3})^{\nu\sigma_3}.$$

It then follows from (3.8.7) that

$$\operatorname{res}_{\zeta=1}\frac{G_R(\zeta)-I}{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \widehat{\alpha} \\ 0 & \widehat{\beta} & 0 \end{pmatrix}, \tag{3.8.8}$$

where

$$\begin{aligned} \widehat{\alpha} &= \frac{\alpha}{\sqrt{2x}}(24\sqrt{3}x)^{-\nu}3^{-1/4}e^{-\frac{3\pi i}{4}-\frac{3\pi i}{2}\nu}, \\ \widehat{\beta} &= -\frac{1}{\sqrt{2x}}\frac{\nu}{\alpha}(24\sqrt{3}x)^{\nu}3^{-1/4}e^{-\frac{3\pi i}{4}+\frac{3\pi i}{2}\nu}, \end{aligned}$$

with

$$\alpha = -\frac{i}{s_1 e^{i2\sqrt{3}x}}\frac{\sqrt{2\pi}e^{2\pi i\nu}}{\Gamma(-\nu)},\ \nu = -\frac{1}{2\pi i}\ln 3A^{\mathbb{R}} = \frac{1}{2\pi i}\ln(1-|s_1|^2),\ s_1 = \omega^2\frac{B}{A^{\mathbb{R}}}.$$

Using symmetry relations between local parametrices, (3.7.13) – (3.7.17), one can find all the other residues. By (3.7.13), we have

$$G_R(\zeta) = d_3^{T-1} G_R^{T-1}(-\zeta) d_3$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial U_{-1}} \frac{G_R(\zeta)-I}{\zeta}d\zeta &= \frac{1}{2\pi i}\int_{\partial U_{-1}} \frac{d_3^{T-1}(G_R^{T-1}(-\zeta)-I)d_3}{\zeta}d\zeta \\ &= d_3^{-1}\left[\frac{1}{2\pi i}\int_{\partial U_1} \frac{G_R(\zeta)-I}{\zeta}d\zeta\right]^{T-1} d_3.\end{aligned}$$

This implies that

$$\begin{aligned}\operatorname{res}_{\zeta=-1}\frac{G_R(\zeta)-I}{\zeta} &= d_3^{-1}\left[-\operatorname{res}_{\zeta=1}\frac{G_R(\zeta)-I}{\zeta}\right]^T d_3 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\widehat{\beta}\omega \\ 0 & -\widehat{\alpha}\omega^2 & 0 \end{pmatrix}.\end{aligned}$$

Similarly, the relation (3.7.15) implies

$$\operatorname{res}_{\zeta=\omega}\frac{G_R(\zeta)-I}{\zeta} = \Pi\left[\operatorname{res}_{\zeta=1}\frac{G_R(\zeta)-I}{\zeta}\right]\Pi^{-1} = \begin{pmatrix} 0 & \widehat{\alpha} & 0 \\ \widehat{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The relation (3.7.14) implies

$$\begin{aligned}\operatorname{res}_{\zeta=-\overline{\omega}}\frac{G_R(\zeta)-I}{\zeta} &= \Pi^{-1}\left[\operatorname{res}_{\zeta=-1}\frac{G_R(\zeta)-I}{\zeta}\right]\Pi \\ &= \begin{pmatrix} 0 & 0 & -\widehat{\alpha}\omega^2 \\ 0 & 0 & 0 \\ -\widehat{\beta}\omega & 0 & 0 \end{pmatrix}.\end{aligned}$$

The relation (3.7.16) implies

$$\operatorname{res}_{\zeta=\overline{\omega}} \frac{G_R(\zeta)-I}{\zeta} = d_3^{-1}\left[-\operatorname{res}_{\zeta=-1} \frac{G_R(\zeta)-I}{\zeta}\right]^T d_3 = \begin{pmatrix} 0 & 0 & \widehat{\beta} \\ 0 & 0 & 0 \\ \widehat{\alpha} & 0 & 0 \end{pmatrix}.$$

The relation (3.7.17) implies

$$\begin{aligned}\operatorname{res}_{\zeta=-\omega} \frac{G_R(\zeta)-I}{\zeta} &= d_3^{-1}\left[\operatorname{res}_{\zeta=\omega} \frac{G_R(\zeta)-I}{\zeta}\right]^T d_3 \\ &= \begin{pmatrix} 0 & -\widehat{\beta}\omega & 0 \\ -\widehat{\alpha}\omega^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{3.8.9}$$

Therefore, the residue calculations (3.8.8) – (3.8.9) from (3.8.4) provide another expression of $R(0)$,

$$R(0) = \begin{pmatrix} 1 & \widehat{\alpha}-\omega\widehat{\beta} & \widehat{\beta}-\omega^2\widehat{\alpha} \\ \widehat{\beta}-\omega^2\widehat{\alpha} & 1 & \widehat{\alpha}-\omega\widehat{\beta} \\ \widehat{\alpha}-\omega\widehat{\beta} & \widehat{\beta}-\omega^2\widehat{\alpha} & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{x}\right). \tag{3.8.10}$$

Comparing (3.8.6) with (3.8.10), we obtain the following result:

$$w_0(x) = \frac{\sqrt{3}}{2e^{\frac{5\pi i}{6}}}(\widehat{\alpha}-\omega\widehat{\beta}) + \mathcal{O}\left(\frac{1}{x}\right)$$

where

$$\begin{aligned}\widehat{\alpha} &= \frac{\alpha}{\sqrt{2x}}(24\sqrt{3}x)^{-\nu}3^{-1/4}e^{-\frac{3\pi i}{4}-\frac{3\pi i}{2}\nu} \\ \widehat{\beta} &= -\frac{1}{\sqrt{2x}}\frac{\nu}{\alpha}(24\sqrt{3}x)^{\nu}3^{-1/4}e^{-\frac{3\pi i}{4}+\frac{3\pi i}{2}\nu}\end{aligned}$$

with

$$\alpha = -\frac{i}{s_1 e^{i2\sqrt{3}x}} \frac{\sqrt{2\pi} e^{2\pi i \nu}}{\Gamma(-\nu)}, \ \nu = -\frac{1}{2\pi i} \ln 3A^{\mathbb{R}} = \frac{1}{2\pi i} \ln(1 - |s_1|^2), \ s_1 = \omega^2 \frac{B}{A^{\mathbb{R}}}.$$

Finally, one can write this result in the form of Theorem 3.1.1.

BIBLIOGRAPHY

[1] N. Andrei, K. Furuya, and J. H. Lowenstein, SOLUTION OF THE KONDO PROBLEM, Rev. Mod. Phys. **55** 331, (1983).
doi:https://doi.org/10.1103/RevModPhys.55.331

[2] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., BETHE VECTORS OF GL(3)-INVARIANT INTEGRABLE MODELS, arXiv:1210.0768.

[3] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., BETHE VECTORS OF QUANTUM INTEGRABLE MODELS WITH GL(3) TRIGONOMETRIC R-MATRIX, arXiv:1304.7602.

[4] S. Belliard, E. Ragoucy, THE NESTED BETHE ANSATZ FOR "ALL" CLOSED SPIN CHAINS, J. Phys. A**41**, 295202, (2008).
doi:https://doi.org/10.1088/1751-8113/41/29/295202
https://arxiv.org/abs/0804.2822

[5] H.Bethe, ZUR THEORIE DER METALLE: I. EIGENWERTE UND EIGENFUNKTIONEN DER LINEAREN ATOMKETTE, Z. Phys. 71 (1931), 205226.

[6] G.D. Birkhoff, COLLECTED MATHEMATICAL PAPERS, **3** Amer. Math. Soc., 1950.

[7] A.A.Bolibruch, THE RIEMANN-HILBERT PROBLEM, *Russian Math. Surveys*, **45:2**, 1-47 (1990).

[8] J. Bolton, F. Pedit, and L. Woodward, MINIMAL SURFACES AND THE AFFINE TODA FIELD MODEL, J. reine angew. Math. 459 (1995), 119150.

[9] F. E. Burstall and F. Pedit, HARMONIC MAPS VIA ADLER-KOSTANT-SYMES THEORY, HARMONIC MAPS AND INTEGRABLE SYSTEMS, eds. A. P. Fordy and J. C. Wood, Aspects of Math. E23, Vieweg, 1994, pp. 221272.

[10] S.Cecotti, Int. J. Mod. Phys. **A6** (1991) 1749;

[11] S.Cecotti and C.Vafa, EXACT RESULTS FOR SUPERSYMMETRIC SIGMA MODELS, Preprint HUTP-91/A062.

[12] S.Cecotti and C.Vafa, MASSIVE ORBIFOLDS, Preprint HUTP-92/A013 and SISSA 44/92/EP.

[13] S.Cecotti, C.Vafa, Nucl. Phys. **B367** (1991) 359.

[14] S. Cecotti and C. Vafa, TOPOLOGICALANTI-TOPOLOGICAL FUSION, Nuclear Phys. **B 367**(1991), 359461.

[15] S. Cecotti and C. Vafa, EXACT RESULTS FOR SUPERSYMMETRIC σ MODELS,*Phys. Rev. Lett.* 68 (1992), 903906.

[16] S. Cecotti and C. Vafa, ON CLASSIFICATION OF N = 2 SUPERSYMMETRIC THEORIES, Commun. Math. Phys. 158 (1993), 569644.

[17] K.F.Clancey and I.Gohberg, FACTORIZATION OF MATRIX FUNCTIONS AND SINGULAR INTEGRAL OPERATORS, Operator Theory: Advances and Applications, vol. 3, Birkhäuser Verlag, Basel, 1981. ← 1, 12, 20.

[18] P.Deift, ORTHOGONAL POLYNOMIALS AND RANDOM MATRICES: A RIEMANN-HILBERT APPROACH, Amer. Math.

[19] P.Deift and X.Zhou, A STEEPEST DESCENT METHOD FOR OSCILLATORY RIEMANN-HILBERT PROBLEMS. ASYMPTOTICS FOR THE MKDV EQUATION, Ann. Math. 137 (1993), no. 2, 295368. ←4, 5, 7, 9.

[20] P.A. Deift and X.Zhou, ASYMPTOTICS FOR THE PAINLEVÉ II EQUATION, *Comm. Pure Apple. Math.*, **48** (1995) no 3, 277-337.

[21] P.A. Deift and X.Zhou, A STEEPEST DESCENT METHOD FOR OSCILLATORY RIEMANN-HILBERT PROBLEMS. ASUMPTOTICS FOR THE MKDV EQUATION, *Ann. of Math.*, **137** (1993) 295-368.

[22] R.Dijkgraaf and E.Witten, Nucl. Phys. **B342** (1990) 486.

[23] R.Dijkgraaf, E.Verlinde, and H.Verlinde, Nucl. Phys. **B352** (1991) 59.

[24] R.Dijkgraaf, INTERSECTION THEORY, INTEGRABLE HIERARCHIES AND TOPOLOGICAL FIELD THEORY, Preprint IASSNS-HEP-91/91, December 1991.

[25] J. Dorfmeister, GENERALIZED WEIERSTRASS REPRESENTATIONS OF SURFACES, SURVEYS ON GEOMETRY AND INTEGRABLE SYSTEMS, eds. M. Guest et al., Advanced Studies in Pure Math. 51, Math. Soc. Japan, 2008, pp. 55111. Soc., Providence, RI, 2000. ←1, 31.

[26] B.Dubrovin, INTEGRABLE SYSTEMS IN TOPOLOGICAL FIELD THEORY, Preprint INFN/AE-92/01. To be published in Nucl. Phys. **B**.

[27] B.Dubrovin, GEOMETRY AND INTEGRABILITY OF TOPOLOGICAL-ANTITOPOLOGICAL FUSION
https://arxiv.org/pdf/hep-th/9206037.pdf

[28] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, V. E. Korepin, THE ONE-DIMENSIONAL HUBBARD MODEL, Cambridge: Cambridge Univ. Press, 2005.

[29] P. Etingof and O. Schiffman, LECTURES ON QUANTUM GROUPS, Lectures in Mathematical Physics, International Press, Boston, 1998

[30] L. D. Faddeev, HOW ALGEBRAIC BETHE ANSATZ WORKS FOR INTEGRABLE MODEL, in *Quantum Symmetries*, Les Houches summerschool proceedings **64**, Eds A. Connes, K. Gawedzki and J. Zinn-Justin, North-Holland 149, (1998).
https://arxiv.org/abs/hep-th/9605187

[31] Faddeev L.D., Takhtajan L.A.: Usp. Mat. Nauk 34, 13-63 (1979).

[32] L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan, QUANTUM INVERSE PROBLEM. I, Theor. Math. Phys. **40**, 688, (1979).
doi:https://doi.org/10.1007/BF01018718

[33] L. D. Faddeev and L. A. Takhtajan, THE QUANTUM METHOD OF THE INVERSE PROBLEM AND THE HEISENBERG XYZ MODEL, Russian Math. Surveys **34**, 11, (1979).
doi:https://doi.org/10.1070%2Frm1979v034n05abeh003909

[34] Felder G., Rimányi R., Varchenko A., POINCARÉ–BIRKHOFF–WITT EXPANSIONS OF THE CANONICAL ELLIPTIC DIFFERENTIAL FORM, in Quantum Groups, *Contemp. Math.*, Vol. 433, Amer. Math. Soc., Providence, RI, 2007, 191–208, math.RT/0502296.

[35] H. Flaschka and A.C.Newell, MONODROMY AND SPECTRUM-PRESERVING DEFORMATIONS I, *Comm. Math. Phys* **76** (1980) 65-116.

[36] Gardner, C.S.; Greene, J.M.; Kruskal, M.D.; Miura, R.M (1967), METHOD FOR SOLVING THE KORTEWEGDE VRIES EQUATION, Physical Review Letters, 19 (19): 10951097,

[37] Flaschka H 1974 THE TODA LATTICE. II. EXISTENCE OF INTEGRALS ,*Phys. Rev.* B 9 19245

[38] Flaschka H 1974 ON THE TODA LATTICE. II , *Prog. Theor. Phys.* 51 70316

[39] A.S. Fokas and M.J. Ablowitz, ON THE INITIAL VALUE PROBLEM OF THE SECOND PAINLEVÉ TRANSCEDENT, *Comm. Math Phys.*, **91** (1983) 381-403.

[40] A.S. Fokas and M.J. Ablowitz, LINEARIZATION OF THE KORTEWEG-DE VRIES AND PAINLEVÉ II EQUATIONS, *Phys. Rev. Lett.*, **47** (1981) 1096-1100.

[41] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Y. Novokshenov, PAINLEVE TRANSCENDENTS: THE RIEMANN-HILBERT APPROACH, Mathematical Surveys and Monographs 128, Amer. Math. Soc., 2006.

[42] A.S Fokas and X.Zhou, ON THE SOLVABILITY OF PAINLEVÉ II AND IV, *Comm. Math. Phys.*, **144** (1992) 601-622

[43] Frappat L., Khoroshkin S., Pakuliak S., Ragoucy E., BETHE ANSATZ FOR THE UNIVERSAL WEIGHT FUNCTION, *Ann. Henri Poincaré* **10** (2009), 513–548, arXiv:0810.3135.

[44] B. Gambier. SUR LES ÉQUATIONS DIFFÉRENTIELLES DU SECOND ORDRE ET DU PREMIER DEGRÉ DONT L'INTÉGRALE GÉNÉRALE EST A POINTS CRITIQUES FIXES, Acta Mathematica, 33:1–55, 1910.

[45] M. Gaudin, UN SYSTÈME A UNE DIMENSION DE FERMIONS EN INTERACTION, Phys. Lett. A **24**:1, 55, (1967). doi:https://doi.org/10.1016/0375-9601(67)90193-4

[46] X.-W. Guan, M. T. Batchelor, and C. Lee, FERMI GASES IN ONE DIMENSION: FROM BETHE ANSATZ TO EXPERIMENTS, Rev. Mod. Phys. **85**, 1633, (2013). doi:https://doi.org/10.1103/RevModPhys.85.1633 https://arxiv.org/abs/1301.6446

[47] E.L. Ince, ORDINARY DIFFERENTIAL EQUATIONS, Dover, New York. 1956.

[48] M. Jimbo, T. Miwa, and K. Ueno, MONODROMY PRESERVING DEFORMATION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH RATIONAL COEFFICIENTS I, Physica D 2 (1981) 306352; II: 2 (1981) 407448; III: 4 (1981) 2646.

[49] Khoroshkin S., Pakuliak S., A COMPUTATION OF UNIVERSAL WEIGHT FUNCTION FOR QUANTUM AFFINE ALGEBRA $U_q(\widehat{\mathfrak{gl}}_N)$, *J. Math. Kyoto Univ.* **48** (2008), 277–321, arXiv:0711.2819.

[50] Khoroshkin S., Pakuliak S., GENERATING SERIES FOR NESTED BETHE VECTORS, *SIGMA* **4** (2008), 081, 23 pages, arXiv:0810.3131.

[51] Khoroshkin S., Pakuliak S., Tarasov V., OFF-SHELL BETHE VECTORS AND DRINFELD CURRENTS, *J. Geom. Phys.* **57** (2007), 1713–1732, math.QA/0610517.

[52] A. V. Kitaev, THE METHOD OF ISOMONODROMIC DEFORMATIONS FOR THE DEGENERATE THIRD PAINLEVE EQUATION, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **161** (1987), no. Vopr. Kvant. Teor. Polya i Statist. Fiz. 7, 4553, 175176, 179. MR 918752

[53] Korepin V.E., Bogoliubov N.M., Izergin A.G., QUANTUM INVERSE SCATTERING METHOD AND CORRELATION FUNCTIONS, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 1993.

[54] Korteweg, D. J.; De Vries, G. (1895), ON THE CHANGE OF FORM OF LONG WAVES ADVANCING IN A RECTANGULAR CANAL, AND ON A NEW TYPE OF LONG STATIONARY WAVES, Philosophical Magazine, 39 (240): 422443, doi:10.1080/14786449508620739

[55] Korepin V.E., CALCULATIONS OF NORMS OF BETHE WAVE FUNCTION, *Commun. Math. Phys.* **86** (1982), 391418. p391–418. doi:https://doi.org/10.1007/BF01212176

[56] P. P. Kulish, CLASSICAL AND QUANTUM INVERSE PROBLEM METHOD AND GENERALIZED BETHE ANSATZ, Physica D **3**, 246, (1981).
doi:https://doi.org/10.1016/0167-2789(81)90130-5

[57] Kulish P.P., Reshetikhin N.Y., DIAGONALISATION OF GL(N) INVARIANT TRANSFER MATRICES AND QUANTUM N-WAVE SYSTEM (LEE MODEL), *J. Phys. A: Math. Gen.* **16** (1983), L591–L596.

[58] P. P. Kulish, N. Yu. Reshetikhin, $GL(3)$-INVARIANT SOLUTIONS OF THE YANG–BAXTER EQUATION AND ASSOCIATED QUANTUM SYSTEMS, J. Sov. Math., **34**:5, 1948, (1982).
doi:https://doi.org/10.1007/BF01095104

[59] E. H. Lieb and F. Y. Wu, ABSENCE OF MOTT TRANSITION IN AN EXACT SOLUTION OF THE SHORT-RANGE, ONE-BAND MODEL IN ONE DIMENSION, Phys. Rev. Lett. **20**, 1445, (1968). doi:https://doi.org/10.1103/PhysRevLett.20.1445

[60] E. H. Lieb and F. Y. Wu, THE ONE-DIMENSIONAL HUBBARD MODEL: A REMINISCENCE., Physica A **321**, 1, (2003). doi:https://doi.org/10.1016/S0378-4371(02)01785-5 https://arxiv.org/abs/cond-mat/0207529

[61] G.S.Litvinchuk and I.M.Spitkovskii, FACTORIZATION OF MEASURABLE MATRIX FUNCTIONS, Operator Theory: Advances and Applications, vol. 25, Birkhäuser Basel, Basel, 1987. ← 1, 20.

[62] Manakov S. V., 1975 COMPLETE INTEGRABILITY AND STOCHASTIZATION OF DISCRETE DYNAMICAL SYSTEMS, *Sov. Phys. ZETP* 40 26974

[63] Matsuo A., AN APPLICATION OF AOMOTO–GELFAND HYPERGEOMETRIC FUNCTIONS TO THE SU(n) KNIZHNIK–ZAMOLODCHIKOV EQUATION, *Comm. Math. Phys.* **134** (1990), 65–77.

[64] J. B. McGuire, INTERACTING FERMIONS IN ONE DIMENSION. I. REPULSIVE POTENTIAL, J. Math. Phys. **6**, 432, (1965). doi:https://doi.org/10.1063/1.1704291

[65] J. B. McGuire, INTERACTING FERMIONS IN ONE DIMENSION. II. ATTRACTIVE POTENTIAL, J. Math. Phys. **7**, 123, (1966). doi:https://doi.org/10.1063/1.1704798

[66] I. McIntosh, SPECIAL LAGRANGIAN CONES IN C3 AND PRIMITIVE HARMONIC MAPS, J. Lond. Math. Soc. 67 (2003), 769789.

[67] J. A. Minahan, K. Zarembo THE BETHE-ANSATZ FOR $N = 4$ SUPER YANG–MILLS, JHEP **0303**, 013, (2003). doi:https://doi.org/10.1088/1126-6708/2003/03/013 https://arxiv.org/abs/hep-th/0212208

[68] Miwa T., Takeyama Y., Tarasov V., DETERMINANT FORMULA FOR SOLUTIONS OF THE QUANTUM KNIZHNIK–ZAMOLODCHIKOV EQUATION ASSOCIATED WITH $U_q(\mathrm{sl}_n)$ AT $|q| = 1$, *Publ. Res. Inst. Math. Sci.* **35** (1999), 871–892.

[69] Moser J., 1975 FINITELY MANY MASS POINTS ON THE LINE UNDER THE INFLUENCE OF AN EXPONENTIAL POTENTIALAN INTEGRABLE SYSTEM ,*Dynamical Systems, Theory and Applications (Lecture Notes in Physics vol 38)* (Berlin: Springer) pp 46797

[70] Mukhin E., Tarasov V., Varchenko A., BETHE EIGENVECTORS OF HIGHER TRANSFER MATRICES, *J. Stat. Mech. Theory Exp.* **2006** (2006), P08002, 44 pages, math.QA/0605015.

[71] Mukhin E., Tarasov V., Varchenko A.,MIURA OPERS AND CRITICAL POINTS OF MASTER FUNCTIONS, Cent. Eur. J. Math., vol. 3, no. 2, pp. 155 182, 2005.

[72] Mukhin E., Tarasov V., Varchenko A., SCHUBERT CALCULUS AND REPRESENTATIONS OF THE GENERAL LINEAR GROUP, J. Amer. Math. Soc., vol. 22, no. 4, pp. 909 940, 2009.

[73] Mukhin E., Tarasov V., Varchenko A., SPACES OF QUASI-EXPONENTIALS AND REPRESENTATIONS OF THE YANGIAN $Y(\mathfrak{gl}_N)$, arXiv:1303.1578.

[74] Mukhin E., Tarasov V., Varchenko A., SPACES OF QUASI-EXPONENTIALS AND REPRESENTATIONS OF $\mathfrak{gl}_N$," J. Phys. A, vol. 41, no. 19, 2008, 194017, 28 pp.

[75] E. Mukhin and A. Varchenko, CRITICAL POINTS OF MASTER FUNCTIONS AND FLAG VARIETIES, Commun. Contemp. Math., vol. 6, no. 1, pp. 111 163, 2004.

[76] Os'kin A.F., Pakulyak S.Z., Silant'ev A.V., ON THE UNIVERSAL WEIGHT FUNCTION FOR THE QUANTUM AFFINE ALGEBRA $U_q(\widehat{\mathfrak{gl}}_N)$, *St. Petersburg Math. J.* **21** (2010), 651–680.

[77] P. Painlevé. SUR LES ÉQUATIONS DIFFÉRENTIELLES DU SECOND ORDRE ET D'ORDRE SUPÉRIEUR DONT L'INTÉGRALE GÉNÉRALE EST UNIFORME. Acta Mathematica, 25:1–85, 1902.

[78] N. Yu. Reshetikhin, CALCULATION OF THE NORM OF BETHE VECTORS IN MODELS WITH $SU(3)$-SYMMETRY, Zap. Nauchn. Sem. LOMI **150**, 196, (1986); J. Math. Sci. **46**, 1694, (1989) (Engl. transl.).
doi:https://doi.org/10.1007/BF01099200

[79] Rimányi R., Stevens L., Varchenko A., COMBINATORICS OF RATIONAL FUNCTIONS AND POINCARÉ–BIRCHOFF–WITT EXPANSIONS OF THE CANONICAL $U(\mathfrak{n}_-)$-VALUED DIFFERENTIAL FORM, *Ann. Comb.* **9** (2005), 57–74.

[80] R.Rimanyi, V.Tarasov, A.Varchenko, PARTIAL FLAG VARIETIES, STABLE ENVELOPES AND WEIGHT FUNCTIONS, arXiv:1212.6240v2

[81] Schechtman V.V., Varchenko A.N., HYPERGEOMETRIC SOLUTIONS OF KNIZHNIK–ZAMOLODCHIKOV EQUATIONS, *Lett. Math. Phys.* **20** (1990), 279–283.

[82] Schechtman V.V., Varchenko A.N., ARRANGEMENTS OF HYPERPLANES AND LIE ALGEBRA HOMOLOGY, *Invent. Math.* **106** (1991), 139–194.

[83] P. Schlottmann, INTEGRABLE NARROW-BAND MODEL WITH POSSIBLE RELEVANCE TO HEAVY-FERMION SYSTEMS, Phys. Rev. B **36**, 5177, (1987).
doi:https://doi.org/10.1103/PhysRevB.36.5177

[84] N.A. Slavnov, THE ALGEBRAIC BETHE ANSATZ AND QUANTUM INTEGRABLE SYSTEMS, Russian Math. Surveys, **62**:4, 727, (2007).
doi:http://dx.doi.org/10.1070/RM2007v062n04ABEH004430

[85] N.A. Slavnov, ALGEBRAIC BETHE ANSATZ, arXiv:1804.07350.

[86] B. Sutherland, FURTHER RESULTS FOR THE MANY-BODY PROBLEM IN ONE DIMENSION, Phys. Rev. Lett. **20**:3, 98, (1968). doi:https://doi.org/10.1103/PhysRevLett.20.98

[87] B. Sutherland, A GENERAL MODEL FOR MULTICOMPONENT QUANTUM SYSTEMS, Phys. Rev. B **12**, 3795, (1975). doi:https://doi.org/10.1103/PhysRevB.12.3795

[88] A.Strominger, TOPOLOGY OF SUPERSTRING COMPACTIFICATION in "Unified String Theory" eds. Green and Gross (World Scientific Publishing Co. 1986);

[89] V. Tarasov and A. Varchenko, COMBINATORIAL FORMULAE FOR NESTED BETHE VECTOR, SIGMA **9**, 048, (2013). doi:https://doi.org/10.3842/SIGMA.2013.048 https://arxiv.org/abs/math/0702277v2

[90] V. Tarasov and A. Varchenko, GEOMETRY OF Q-HYPERGEOMETRIC FUNCTIONS, QUANTUM AFFINE ALGEBRAS AND ELLIPTIC QUANTUM GROUPS, *Astérisque* (1997), no. 246, 1–135.

[91] V. Tarasov and A. Varchenko, HYPERGEOMETRIC SOLUTIONS OF THE QUANTUM DIFFERENTIAL EQUATION OF THE COTANGENT BUNDLE OF A PARTIAL FLAG VARIETY, arXiv:1301.2705.

[92] V. Tarasov and A. Varchenko, JACKSON INTEGRAL REPRESENTATIONS FOR SOLUTIONS OF THE KNIZHNIK–ZAMOLODCHIKOV QUANTUM EQUATION, St. Petersburg Math. J. **6**, 275, (1994). https://arxiv.org/abs/hep-th/9311040

[93] V. Tarasov and A. Varchenko, SELBERG-TYPE INTEGRALS ASSOCIATED WITH $\mathfrak{sl}_3$, *Lett. Math. Phys.* **65** (2003), 173–185, math.QA/0302148.

[94] Toda M., 1967 VIBRATION OF A CHAIN WITH NONLINEAR INTERACTION, *J. Phys. Soc. Japan* 22 4316

[95] Toda M., 1967 WAVE PROPAGATION IN ANHARMONIC LATTICES, *J. Phys. Soc. Japan* 23 5016 ,

[96] H.Watanabe, SOLUTIONS OF THE FIFTH PAINLEVÉ EQUATION.I, *Hokkaido Math J.* **24** (1995)

[97] P. B. Wiegmann, TOWARDS AN EXACT SOLUTION OF THE ANDERSON MODEL, Phys. Lett. A. **80**, 163, (1980). doi:https://doi.org/10.1016/0375-9601(80)90212-1

[98] P. B. Wiegmann, EXACT SOLUTION OF THE $s-d$ EXCHANGE MODEL (KONDO PROBLEM), J. Phys. C **14**:10, 1463, (1981). doi:https://doi.org/10.1088/0022-3719/14/10/014

[99] E.Witten, Comm. Math. Phys. **118** (1988) 411; Nucl. Phys. **B340** (1990) 281.

[100] E.Witten, Surv. Diff. Geom. **1** (1991) 243;

[101] C. N. Yang, SOME EXACT RESULTS FOR THE MANY-BODY PROBLEM IN ONE DIMENSION WITH REPULSIVE DELTA-FUNCTION INTERACTION, Phys. Rev. Lett. **19**:23, 1312, (1967). doi:https://doi.org/10.1103/PhysRevLett.19.1312

[102] Y. Yang, SOLITONS IN FIELD THEORY AND NONLINEAR ANALYSIS, Springer, 2001.

[103] A.B.Zamolodchikov, JETP Lett. **43** (1986) 731.

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