

# FAMILY OF CHAOTIC MAPS FROM GAME THEORY

THIPARAT CHOTIBUT, FRYDERYK FALNIOWSKI, MICHAŁ MISIUREWICZ,  
AND GEORGIOS PILIOURAS

ABSTRACT. From a two-agent, two-strategy congestion game where both agents apply the multiplicative weights update algorithm, we obtain a two-parameter family of maps of the unit square to itself. Interesting dynamics arise on the invariant diagonal, on which a two-parameter family of bimodal interval maps exhibits periodic orbits and chaos. While the fixed point  $b$  corresponding to a Nash equilibrium of such map  $f$  is usually repelling, it is globally *Cesàro attracting* on the diagonal, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f^k(x) = b$$

for every  $x$  in the minimal invariant interval. This solves a known open question whether there exists a nontrivial smooth map other than  $x \mapsto axe^{-x}$  with centers of mass of all periodic orbits coinciding. We also study the dependence of the dynamics on the two parameters.

## 1. INTRODUCTION

We will consider a  $2 \times 2$  (two agents/players, two pure strategies) *congestion game* (see [7, 5]) and examine its dynamics under a specific model of learning behavior for both agents known as *multiplicative weights update* [1].

We will start off with the description of the two agent game. Let  $x$  be the probability that the first player (agent) chooses the first strategy (he chooses the second strategy with probability  $1 - x$ ). Similarly, let  $y$  be the probability that the second player (agent) chooses the first strategy (he chooses the second strategy with probability  $1 - y$ ). We denote a tuple of such randomized strategies as  $(x, y)$ . If the agents choose the same strategy, this leads to congestion, and the cost increases. We will assume that the cost of a strategy is proportional to its load, i.e., to the number of agents choosing this strategy. If we denote by  $c(i, j)$  the expected cost of the player number  $i$  playing the strategy number  $j$ , and the coefficients of proportionality are  $\alpha, \beta$ , then we get

$$(1) \quad \begin{aligned} c(1, 1) &= \alpha(1 + x), & c(2, 1) &= \alpha(1 + x), \\ c(1, 2) &= \beta(1 + (1 - y)) = \beta(2 - y), & c(2, 2) &= \beta(1 + (1 - x)) = \beta(2 - x). \end{aligned}$$

A strategy profile/tuple  $(x, y)$  is a *Nash equilibrium* if and only if no agent can strictly decrease their expected cost by unilaterally deviating to another strategy.

---

**2010 Mathematics Subject Classification:** Primary: 37E05, Secondary: 91A05

**Keywords:** Chaos; Interval maps; Center of mass; Multiplicative weights; Congestion game.

**1.1. Multiplicative Weights Update.** We will assume that both agents participate in the game by applying the *multiplicative weights update* (MWU) algorithm [1]. At time  $n$ , the first (second) player/agent chooses the first strategy with probability  $x_n$  ( $y_n$ ). We will study MWU in the full informational setting, where each agent gets to experience the cost of all strategies available to them. Moreover, the (expected) cost of strategy  $j$  for agent 1 (respectively, 2) at time  $n$  is computed given the randomized choice of agent 2 (respectively, 1). In this case, the update of the mixed/randomized strategies of each agent when applying MWU with learning rate  $\varepsilon \in (0, 1)$  is as follows:

$$(2) \quad \begin{aligned} x_{n+1} &= \frac{x_n(1 - \varepsilon)^{c(1,1)}}{x_n(1 - \varepsilon)^{c(1,1)} + (1 - x_n)(1 - \varepsilon)^{c(1,2)}}, \\ y_{n+1} &= \frac{y_n(1 - \varepsilon)^{c(2,1)}}{y_n(1 - \varepsilon)^{c(2,1)} + (1 - y_n)(1 - \varepsilon)^{c(2,2)}}. \end{aligned}$$

Observe that in this way large cost at time  $n$  leads to the decrease of the probability of doing the same at time  $n + 1$ . The learning rate  $\varepsilon$  can be thought of as capturing the patience of the agents. For small  $\varepsilon$  the agents adapt slowly to the costs whereas for large  $\varepsilon$  they respond more aggressively. We will study the effects of this learning rate  $\varepsilon$  to the stability of the MWU dynamics. In [6] it was shown that there exist  $2 \times 2$  congestion games where MWU with large enough  $\varepsilon$  can lead to limit cycles or chaos. We will establish the emergence of chaos for all  $2 \times 2$  congestion games for large enough learning rate  $\varepsilon$ .

## 2. DYNAMICAL MODEL

Let us plug into (2) the values of the cost functions from (1):

$$(3) \quad \begin{aligned} x_{n+1} &= \frac{x_n(1 - \varepsilon)^{\alpha(1+y_n)}}{x_n(1 - \varepsilon)^{\alpha(1+y_n)} + (1 - x_n)(1 - \varepsilon)^{\beta(2-y_n)}} \\ &= \frac{x_n}{x_n + (1 - x_n)(1 - \varepsilon)^{\beta(2-y_n) - \alpha(1+y_n)}}, \\ y_{n+1} &= \frac{y_n(1 - \varepsilon)^{\alpha(1+x_n)}}{y_n(1 - \varepsilon)^{\alpha(1+x_n)} + (1 - y_n)(1 - \varepsilon)^{\beta(2-x_n)}} \\ &= \frac{y_n}{y_n + (1 - y_n)(1 - \varepsilon)^{\beta(2-x_n) - \alpha(1+x_n)}}. \end{aligned}$$

We introduce new variables

$$(4) \quad a = (\alpha + \beta) \ln \frac{1}{1 - \varepsilon}, \quad b = \frac{2\beta - \alpha}{\alpha + \beta}.$$

Then formulas (3) become

$$(5) \quad \begin{aligned} x_{n+1} &= \frac{x_n}{x_n + (1 - x_n) \exp(a(y_n - b))}, \\ y_{n+1} &= \frac{y_n}{y_n + (1 - y_n) \exp(a(x_n - b))}. \end{aligned}$$

Clearly, if  $(x_n, y_n) \in [0, 1]^2$  then also  $(x_{n+1}, y_{n+1}) \in [0, 1]^2$ . Therefore we will be studying the family of maps  $F_{a,b} : [0, 1]^2 \rightarrow [0, 1]^2$ , given by

$$(6) \quad F_{a,b}(x, y) = \left( \frac{x}{x + (1-x)\exp(a(y-b))}, \frac{y}{y + (1-y)\exp(a(x-b))} \right).$$

The diagonal  $x = y$  is invariant for  $F_{a,b}$ , so we can restrict this map to the diagonal and we get the family of maps  $f_{a,b} : [0, 1] \rightarrow [0, 1]$ , given by

$$(7) \quad f_{a,b}(x) = \frac{x}{x + (1-x)\exp(a(x-b))}.$$

In the context of game theory, this restriction means that both players start with the same mixed strategy (the same probability distributions).

Our aim is to investigate the long-term behavior of the orbits of  $F_{a,b}$  and  $f_{a,b}$ . In particular, we can consider periodic orbits, their stability, or chaos. When speaking of chaos, we will use its most popular kind, *Li-Yorke chaos*. Namely, if  $X$  is a compact space with metric  $\rho$  and  $f : X \rightarrow X$  is a continuous map, we say that the pair of points  $x, y \in X$  is a *Li-Yorke pair* if

$$\begin{aligned} \liminf_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) &= 0, \\ \limsup_{n \rightarrow \infty} \rho(f^n(x), f^n(y)) &> 0. \end{aligned}$$

The map  $f$  is *Li-Yorke chaotic* if there is an uncountable set  $S \subset X$  (called *scrambled set*) such that every pair  $(x, y)$  with  $x, y \in S$  and  $x \neq y$  is a Li-Yorke pair.

Before starting detailed investigation of the families of maps  $F_{a,b}$  and  $f_{a,b}$ , let us determine what we should assume about  $a$  and  $b$ . By (4),  $a > 0$  and as we mentioned, we will be interested most at large values of  $a$ . While (4) does not provide any restrictions for  $b$ , let us think what happens if  $b < 0$ .

Then  $F_{a,b}$  has four fixed points:  $(0, 0), (0, 1), (1, 0), (1, 1)$ . Moreover,  $x_{n+1} < x_n$  unless  $x_n = 0$  or  $x_n = 1$ . Similarly,  $y_{n+1} < y_n$  unless  $y_n = 0$  or  $y_n = 1$ . Therefore, trajectories of all points of  $(0, 1)^2$  converge to  $(0, 0)$ . Similarly, if  $b > 1$  then trajectories of all points of  $(0, 1)^2$  converge to  $(1, 1)$ . This dynamics is not interesting. If  $b = 0$  or  $b = 1$ , there are additionally segments of fixed points on the boundary of the square, but still the dynamics is not worth studying. Therefore, we will assume that  $b \in (0, 1)$ .

### 3. ON THE DIAGONAL

Let us start investigating the dynamics of the maps  $f_{a,b}$ , given by (7), with  $a > 0$ ,  $b \in (0, 1)$  (see Figure 1). This map has three fixed points: 0,  $b$  and 1.

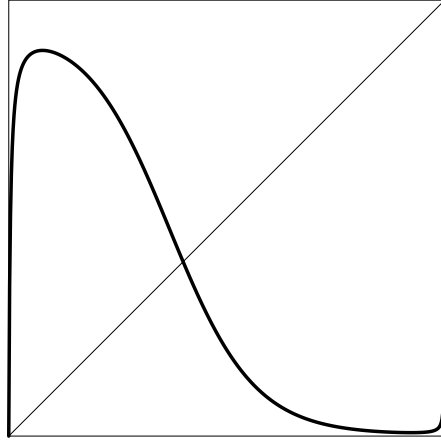
The derivative of  $f_{a,b}$  is given by

$$(8) \quad f'_{a,b}(x) = \frac{(ax^2 - ax + 1)\exp(a(x-b))}{(x + (1-x)\exp(a(x-b)))^2}.$$

Thus,

$$f'_{a,b}(0) = \exp(ab), \quad f'_{a,b}(1) = \exp(a(1-b)), \quad f'_{a,b}(b) = ab^2 - ab + 1.$$

We see that the fixed points 0 and 1 are always repelling, while  $b$  is repelling if  $a > \frac{2}{b(1-b)}$ .

FIGURE 1. The map  $f_{a,b}$  with  $a = 14$ ,  $b = 0.4$ .

The critical points of  $f_{a,b}$  are solutions to  $ax^2 - ax + 1 = 0$ . Thus, if  $0 < a \leq 4$ , then  $f_{a,b}$  is strictly increasing. If  $a > 4$ , it has two critical points

$$(9) \quad c_l = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{a}}, \quad c_r = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{a}},$$

so the map  $f_{a,b}$  is bimodal.

Our family has some symmetry. Consider the flip  $\varphi : [0, 1] \rightarrow [0, 1]$ ,  $\varphi(x) = 1 - x$ . Then

$$\begin{aligned} \varphi(f_{a,b}(x)) &= 1 - \frac{x}{x + (1-x)\exp(a(x-b))} = \frac{(1-x)\exp(a(x-b))}{x + (1-x)\exp(a(x-b))} \\ &= \frac{(1-x)}{(1-x) + x\exp(a((1-x)-(1-b)))} = f_{a,1-b}(\varphi(x)). \end{aligned}$$

This can be written as

$$(10) \quad \varphi \circ f_{a,b} = f_{a,1-b} \circ \varphi, \quad \text{where } \varphi(x) = 1 - x.$$

While  $[0, 1]$  is the natural space on which  $f_{a,b}$  acts, it will be sometimes easier to think of a smaller interval.

**Lemma 3.1.** *For every  $a > 0$ ,  $b \in (0, 1)$  there exists a closed interval  $I_{a,b} \subset (0, 1)$  such that  $f_{a,b}(I_{a,b}) \subset I_{a,b}$  and  $f_{a,b}$ -trajectories of all points of  $(0, 1)$  enter  $I_{a,b}$ .*

*Proof.* Since 0 and 1 are repelling fixed points of  $f_{a,b}$ , there exist  $\delta_1 > 0$  and  $\lambda > 1$  such that if  $x < \delta_1$  or  $1 - x < \delta_1$  then  $\lambda x < f_{a,b}(x) < 1 - \lambda x$ . We have  $f_a([\delta_1, 1 - \delta_1]) \subset (0, 1)$ , so there exists  $\delta_2 > 0$  such that  $f_a([\delta_1, 1 - \delta_1]) \subset (\delta_2, 1 - \delta_2)$ . Set  $\delta = \min(\delta_1, \delta_2)$  and  $I_{a,b} = [\delta, 1 - \delta]$ . Then  $I_{a,b}$  is mapped by  $f_{a,b}$  to itself, and all  $f_{a,b}$ -trajectories of points from  $(0, 1)$  sooner or later enter  $I_{a,b}$ .  $\square$

Let us investigate regularity of  $f_{a,b}$ . It is clear that it is analytic. However, nice properties of interval maps are guaranteed not by analyticity, but by the negative Schwarzian derivative. Let us recall that the Schwarzian derivative of  $f$  is given by

the formula

$$Sf = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

A “metatheorem” states that almost all natural noninvertible interval map have negative Schwarzian derivative. Note that if  $a \leq 4$  then  $f_{a,b}$  is a homeomorphism, so we should not expect negative Schwarzian derivative for that case (and it would be useless, anyway).

**Proposition 3.2.** *If  $a > 4$  then the map  $f_{a,b}$  has negative Schwarzian derivative.*

*Proof.* Instead of making very complicated computations, we will use the formula  $S(h \circ f) = (f')^2((Sh) \circ f) + Sf$  and the fact that Möbius transformations have zero Schwarzian derivative. Set  $h(x) = \exp(ax) \frac{1-x}{x}$  and  $g(x) = h \circ f_{a,b}(x) = \frac{1-x}{x} \exp(ax)$ ; then  $Sg = Sf$ .

Now, simple computations yield

$$Sg(x) = -\frac{a(-12 + 6a + 4a^2(-1+x)x + a^3(-1+x)^2x^2)}{2(1+a(-1+x)x)^2}.$$

Thus, the Schwarzian derivative of  $f_{a,b}$  is negative if and only if

$$-12 + 6a + 4a^2x(x-1) + a^3x^2(1-x)^2 > 0.$$

Set  $t = x(1-x)$ . Then the above inequality becomes

$$(11) \quad P_a(t) > 0, \quad \text{where} \quad P_a(t) = a^3t^2 - 4a^2t + 6a - 12.$$

Clearly,  $t \in [0, 1/4]$ .

The quadratic polynomial  $P_a$  attains its minimum at  $t = 2/a$ . If  $2/a \leq 1/4$ , that is, if  $a \geq 8$ , we have  $P_a(2/a) = 2a - 12 > 0$ , so (11) holds. If  $a \in (0, 8)$ , then (11) holds whenever  $P_a(1/4) > 0$ . We have  $P_a(1/4) = (a-4)(\frac{1}{16}a^2 - \frac{3}{4}a + 3)$ . The second factor is always positive, and thus  $P_a(1/4) > 0$  for  $a > 4$ .  $\square$

For maps with negative Schwarzian derivative each attracting or neutral periodic orbit has a critical point in its immediate basin of attraction. Thus, we know that if  $a > 4$  then  $f_{a,b}$  can have at most two attracting or neutral periodic orbits.

**3.1. Average behavior.** While we know that the fixed point  $b$  is often repelling, especially for large values of  $a$ , we can show that it is attracting in a certain sense.

**Definition 3.3.** For an interval map  $f$  a point  $p$  is *Cesàro attracting* if it has a neighborhood  $U$  such that for every  $x \in U$  the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} f^k(x)$$

converge to  $p$ .

We will show that  $b$  is globally Cesàro attracting. Here by “globally” we mean that the set  $U$  from the definition is the interval  $(0, 1)$ .

**Theorem 3.4.** *For every  $a > 0$ ,  $b \in (0, 1)$  and  $x \in (0, 1)$  we have*

$$(12) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{a,b}^k(x) = b.$$

*Proof.* By Lemma 3.1 there is a closed interval  $I_{a,b} \subset (0, 1)$  which is invariant for  $f_{a,b}$ . Thus, there is  $\delta \in (0, 1)$  such that  $I_{a,b} \subset (\delta, 1 - \delta)$ .

Fix  $x = x_0 \in [0, 1]$  and use our notation  $x_n = f_{a,b}^n(x_0)$ . By induction, we get

$$(13) \quad x_n = \frac{x}{x + (1 - x) \exp \left( a \sum_{k=0}^{n-1} (x_k - b) \right)}.$$

Assume that  $x = x_0 \in I_{a,b}$ . Since  $\delta < x_n < 1 - \delta$ , we have

$$\frac{x}{1 - \delta} < x + (1 - x) \exp \left( a \sum_{k=0}^{n-1} (x_k - b) \right) < \frac{x}{\delta},$$

so

$$\delta^2 < x \frac{\delta}{1 - \delta} < (1 - x) \exp \left( a \sum_{k=0}^{n-1} (x_k - b) \right) < x \frac{1 - \delta}{\delta} < \frac{1}{\delta}.$$

Therefore

$$(14) \quad \delta^2 < \exp \left( a \sum_{k=0}^{n-1} (x_k - b) \right) < \frac{1}{\delta^2},$$

so

$$\left| a \sum_{k=0}^{n-1} (x_k - b) \right| < 2 \log(1/\delta).$$

This inequality can be rewritten as

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} x_k - b \right| < \frac{2 \log(1/\delta)}{an},$$

and (12) follows.

If  $x \in (0, 1) \setminus I_{a,b}$ , then by the definition of  $I_{a,b}$  (Lemma 3.1) there is  $n_0$  such that  $f_{a,b}^{n_0}(x) \in I_{a,b}$ , so (12) also holds.  $\square$

**Corollary 3.5.** *For every periodic orbit  $\{x_0, x_1, \dots, x_{n-1}\}$  of  $f_{a,b}$  in  $(0, 1)$  its center of mass*

$$\frac{x_0 + x_1 + \dots + x_{n-1}}{n}$$

*is equal to  $b$ .*

Applying the Birkhoff Ergodic Theorem, we get a stronger corollary.

**Corollary 3.6.** *For every probability measure  $\mu$ , invariant for  $f_{a,b}$  and such that  $\mu(\{0, 1\}) = 0$ , we have*

$$\int_{[0,1]} x \, d\mu = b.$$

Corollary 3.5 solves a known question whether there are nontrivial smooth maps other than  $x \mapsto axe^{-x}$  with centers of mass of all periodic orbits coinciding [4]. Here “nontrivial” means that there are periodic orbits of infinitely many periods; we will show later that for many parameters  $f_{a,b}$  has this property.

**Problem 3.7.** Find all nontrivial smooth (analytic) maps for which centers of mass of all periodic orbits coincide.

**3.2. Symmetric case.** Now we explore what happens as we fix  $b$  and let  $a$  go to infinity. First we study the symmetric case,  $b = 1/2$ , which corresponds to equal coefficients of the cost functions,  $\alpha = \beta$ . To simplify notation we denote  $f_a = f_{a,1/2}$ , so

$$f_a(x) = \frac{x}{x + (1 - x) \exp(a(x - 1/2))}.$$

The interpretation of the formula (10) is very simple in this case: the maps  $f_a$  and  $\varphi$  commute. Set  $g_a = \varphi \circ f_a = f_a \circ \varphi$ . Since  $\varphi$  is an involution, we have  $g_a^2 = f_a^2$ .

We show that the dynamics of  $f_a$  is simple, no matter how large  $a$  is.

**Theorem 3.8.** *The  $g_a$ -trajectory of every point of  $(0, 1)$  converges to a fixed point of  $g_a$ . The  $f_a$ -trajectory of every point of  $(0, 1)$  converges to a fixed point or a periodic orbit of period 2 of  $f_a$ , other than 0 and 1.*

*Proof.* We want to find fixed points and points of period 2 of  $f_a$  and  $g_a$ . Clearly,

$$f_a(0) = 0, \quad f_a(1) = 1, \quad g_a(0) = 1, \quad g_a(1) = 0.$$

By (13) we have

$$f_a^2(x) = \frac{x}{x + (1 - x) \exp(a(x + f_a(x) - 1))},$$

so the fixed points of  $f_a^2$  are 0, 1 and the solutions to  $x + f_a(x) - 1 = 0$ , that is, to  $g_a(x) = x$ . Thus, the fixed points of  $g_a^2$  (which, as we noticed, is equal to  $f_a^2$ ) are the fixed points of  $g_a$  and 0 and 1.

We can choose the invariant interval  $I_a = I_{a,1/2}$  symmetric, so that  $\varphi(I_a) = I_a$ . Let us look at  $G_a = g_a|_{I_a} : I_a \rightarrow I_a$ . All fixed points of  $G_a^2$  are also fixed points of  $G_a$ , so  $G_a$  has no periodic points of period 2. By the Sharkovsky Theorem,  $G_a$  has no periodic points other than fixed points. For such maps it is known (see, e.g., [9]) that the  $\omega$ -limit set of every trajectory is a singleton of a fixed point, that is, every trajectory converges to a fixed point. If  $x \in (0, 1) \setminus I_a$ , then the  $g_a$ -trajectory of  $x$  after a finite time enters  $I_a$ , so  $g_a$ -trajectories of all points of  $(0, 1)$  converge to a fixed point of  $g_a$  in  $I_a$ . Observe that a fixed point of  $g_a$  can be a fixed point of  $f_a$  (other than 0, 1) or a periodic point of  $f_a$  of period 2. Thus the  $f_a$ -trajectory of every point of  $(0, 1)$  converges to a fixed point or a periodic orbit of period 2 of  $f_a$ , other than 0 and 1.  $\square$

Let us identify the possible limits from the above theorem.

**Theorem 3.9.** *If  $0 < a \leq 8$  then  $f_a$ -trajectories of all points of  $(0, 1)$  converge to the fixed point  $1/2$ . If  $a > 8$  then  $f_a$  has a periodic attracting orbit  $\{\sigma_a, 1 - \sigma_a\}$ , where  $0 < \sigma_a < 1/2$ . This orbit attracts trajectories of all points of  $(0, 1)$ , except countably many points, whose trajectories eventually fall into the repelling fixed point  $1/2$ .*

*Proof.* Observe first that  $1/2$  is a fixed point of both  $f_a$  and  $g_a$ . Then, let us look for the fixed points of  $g_a$  in  $[0, 1/2]$ . They are the solutions of the equation  $g_a(x) = x$ , that is,  $f_a(x) = 1 - x$ , which is equivalent to

$$x^2 = (1 - x)^2 \exp(a(x - 1/2)).$$

Since  $x$ ,  $1 - x$  and  $\exp(a(x - 1/2))$  are non-negative, this equation is equivalent to  $\gamma_a(x) = 0$ , where

$$\gamma_a(x) = x - (1 - x) \exp((a/2)(x - 1/2)).$$



We have

$$\begin{aligned}\gamma'_a(x) &= 1 + (1 - (a/2) + (a/2)x) \exp((a/2)(x - 1/2)), \\ \gamma''_a(x) &= (a/2)(2 - (a/2) + (a/2)x) \exp((a/2)(x - 1/2)).\end{aligned}$$

For all  $a > 0$  we have  $\gamma_a(0) < 0$  and  $\gamma_a(1/2) = 0$ . If  $a \geq 8$  then  $\gamma''_a \leq 0$  on  $[0, 1/2]$ , so  $\gamma_a$  is concave there. We have  $\gamma'_8(1/2) = 0$ , so since  $\gamma_a$  is real analytic, we have  $\gamma_8(x) < 0$  for all  $x \in [0, 1/2)$ . For all  $a \in (0, 8]$  and  $x \in [0, 1/2)$  we have  $\gamma_a(x) \leq \gamma_8(x) < 0$ , so in this case  $g_a$  has no fixed points in  $[0, 1/2)$ . However, if  $a > 8$  then  $\gamma'_a(1/2) = 2 - a/4 < 0$ , so  $\gamma_a$  has exactly one zero in  $[0, 1/2)$ . Thus, in this case  $g_a$  has exactly one fixed point in  $[0, 1/2)$ . We denote it by  $\sigma_a$ .

The situation for  $x \in (1/2, 1]$  is the same, because  $\varphi$  conjugates  $g_a$  with itself and maps  $[0, 1/2)$  to  $(1/2, 1]$ . Observe that  $f_a(\sigma_a) = 1 - g_a(\sigma_a) = 1 - \sigma_a$  and similarly,  $f_a(1 - \sigma_a) = \sigma_a$ .

Now, if  $0 < a \leq 8$ , then  $1/2$  is the only fixed point of  $g_a$  in  $(0, 1)$ , so by Theorem 3.8, the  $g_a$ -trajectory of every point of  $(0, 1)$  converges to  $1/2$ . Thus, the  $f_a$ -trajectory of every point of  $(0, 1)$  also converges to  $1/2$ .

If  $a > 8$ , then  $|g'_a(1/2)| = |f'_a(1/2)| > 1$  by (8), so the only way a  $g_a$ -trajectory of  $x$  can converge to  $1/2$  is that  $g_a^n(x) = 1/2$  for some  $n$ . Since the function  $g_a$  is real analytic, there are only countably many such points  $x$ . According to Theorem 3.8, the  $g_a$ -trajectories of all other points of  $(0, 1)$  converge to  $\sigma_a$  or  $1 - \sigma_a$ . Thus, the period 2 orbit  $\{\sigma_a, 1 - \sigma_a\}$  of  $f_a$  attracts  $f_a$ -trajectories of all points of  $(0, 1)$ , except countably many points, whose trajectories eventually fall to the repelling fixed point  $1/2$ .  $\square$

**3.3. Asymmetric case.** Now we proceed with the case when  $b \neq 1/2$ , that is, the cost functions differ. As we noticed in Section 3, if  $a \leq 4$  then  $f_{a,b}$  is strictly increasing and has three fixed points: 0 and 1 repelling and  $b$  attracting. Therefore in this case trajectories of all points of  $(0, 1)$  converge in a monotone way to  $b$ .

We know that the fixed point  $b$  is repelling if and only if  $a > \frac{2}{b(1-b)}$ .

**Conjecture 3.10.** *If  $a \leq \frac{2}{b(1-b)}$  then trajectories of all points of  $(0, 1)$  converge to  $b$ .*

This is a simple situation, so we turn to the case of large  $a$ . We fix  $b \in (0, 1) \setminus \{1/2\}$  and let  $a$  go to infinity. We will show that if  $a$  becomes sufficiently large (but how large, depends on  $b$ ), then  $f_{a,b}$  is Li-Yorke chaotic and has periodic orbits of all periods.

**Theorem 3.11.** *If  $b \in (0, 1) \setminus \{1/2\}$ , then there exists  $a_b$  such that if  $a > a_b$  then  $f_{a,b}$  has periodic orbit of period 3.*

*Proof.* Fix  $a > 0$  and  $b, x \in (0, 1)$  (we will vary  $a$  later). As in the proof of Theorem 3.4, we set  $x_n = f_{a,b}^n(x)$ , and then the formula (13) holds. Hence,  $f_{a,b}(x) > x$  is equivalent to  $x < b$  and  $f_{a,b}^3(x) < x$  is equivalent to  $x + f_{a,b}(x) + f_{a,b}^2(x) > 3b$ .

Assume that  $0 < b < 1/2$ . Then  $3b - 1 < b$ , so we can take  $x > 0$  such that  $3b - 1 < x < b$ . Then  $f_{a,b}(x) > x$ . Moreover,  $\exp(a(x - b))$  goes to 0 as  $a$  goes to infinity, so

$$\lim_{a \rightarrow \infty} f_{a,b}(x) = \lim_{a \rightarrow \infty} \frac{x}{x + (1 - x) \exp(a(x - b))} = 1.$$

Thus, since  $3b - x < 1$ , there exists  $a_b$  such that if  $a > a_b$  then  $f_{a,b}(x) > 3b - x$ , so  $x + f_{a,b}(x) + f_{a,b}^2(x) > 3b$  (note that  $x$  depends only on  $b$ , not on  $a$ ). Hence, if  $a > a_b$  then  $f_{a,b}^3(x) < x$ .



Now, because for  $a > a_b$  there exist  $x$  such that  $f_{a,b}^3(x) < x < f_{a,b}(x)$ , from the theorem from [2] it follows that if  $a > a_b$  then  $f_{a,b}$  has a periodic point of period 3.

If  $1/2 < b < 1$  then by (10) we can reduce it to the case  $0 < b < 1/2$ .  $\square$

By the Sharkovsky Theorem ([8], see also [3]), existence of a periodic orbit of period 3 implies existence of periodic orbits of all periods, and by the result of [3], it implies that the map is Li-Yorke chaotic.

Thus, we get the following corollary.

**Corollary 3.12.** *If  $b \in (0, 1) \setminus \{1/2\}$ , then there exists  $a_b$  such that if  $a > a_b$  then  $f_{a,b}$  has periodic orbits of all periods and is Li-Yorke chaotic.*

This result has an interesting interpretation in the context of game theory and learning in games. Parameter  $a$  can be treated as measuring the aggressiveness of a player. Corollary 3.12 implies that if players have different cost functions (but they do not differ too much, that is,  $\alpha < 2\beta$  and  $\beta < 2\alpha$  in (1)), their behavior will be chaotic if only they are aggressive enough. Moreover, having in mind the connection between this parameter and learning rate  $\varepsilon$ , see (4), we can interpret Corollary 3.12 as stating that if players learn fast enough ( $\varepsilon$  is close to 1) then the system may become chaotic.

#### 4. OFF THE DIAGONAL

The natural question which arises is whether the chaotic behavior emerges only if both players share the same initial strategy, or chaos can happen also when their initial strategy profiles are different. To answer this question, we study the family of maps  $F_{a,b}: [0, 1]^2 \rightarrow [0, 1]^2$  defined by (6), with  $a > 0$  and  $b \in (0, 1)$ .

The map  $F_{a,b}$  has five fixed points:  $(0, 0)$ ,  $(1, 1)$ ,  $(0, 1)$ ,  $(1, 0)$  and  $(b, b)$ . Three of them, namely  $(0, 1)$ ,  $(1, 0)$  and  $(b, b)$ , are Nash equilibria of the congestion game. That is, no agent can strictly decrease their cost by unilateral deviations to another strategy.

Indeed, in the case of the  $(b, b)$  randomized strategy profile, the expected cost of both agents (in the game defined by (1)) is equal to  $3\alpha\beta/(\alpha + \beta)$ . Moreover, if the first (second) agent were to deviate and choose a different strategy, his expected cost would still remain unchanged. This means that  $(b, b)$  is a Nash equilibrium of the game. By our assumption that  $0 < b < 1$  we derive that  $\alpha < 2\beta$  and  $\beta < 2\alpha$ . This implies that  $(0, 0)$  and  $(1, 1)$  are not Nash equilibria, as each agent would strictly prefer to choose distinct strategies than use the same strategy as the other agent. Applying the same reasoning, states  $(0, 1)$  and  $(1, 0)$  are Nash equilibria, since in these states deviating to another strategy leads to sharing the same strategy as your opponent.

The derivative of  $F_{a,b}$  at  $(0, 1)$  and  $(1, 0)$  is

$$DF_{a,b}(0, 1) = \begin{pmatrix} \exp(-a(1-b)) & 0 \\ 0 & \exp(-ab) \end{pmatrix},$$

$$DF_{a,b}(1, 0) = \begin{pmatrix} \exp(-ab) & 0 \\ 0 & \exp(-a(1-b)) \end{pmatrix},$$

so those points are attracting.

To study the behavior of  $F_{a,b}$  close to the other three fixed points, and in general, close to the diagonal, we compute the derivative of  $F_{a,b}$  on the diagonal:

$$(15) \quad DF_{a,b}(x, x) = \frac{\exp(a(x-b))}{(x + (1-x)\exp(a(x-b)))^2} \begin{pmatrix} 1 & -ax(1-x) \\ -ax(1-x) & 1 \end{pmatrix}.$$

Thus, the eigenvectors of  $DF_{a,b}$  at points  $(x, x)$  on the diagonal are  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . The eigenvalue corresponding to the first one is of course the derivative of  $f_{a,b}$  at  $x$ , given by the formula (8). The eigenvalue corresponding to the second vector, perpendicular to the diagonal, is

$$(16) \quad \lambda_{a,b}(x) = \frac{(1 + ax(1-x))\exp(a(x-b))}{(x + (1-x)\exp(a(x-b)))^2}.$$

We will show that in the long run the diagonal is exponentially repelling. That is, the following theorem holds.

**Theorem 4.1.** *For every  $a > 0$  and  $b \in (0, 1)$  there exist a positive integer  $N$  and a number  $\varkappa > 1$  such that for every  $x \in [0, 1]$  we have*

$$(17) \quad \prod_{k=0}^{N-1} \lambda_{a,b}(f_{a,b}^k(x)) \geq \varkappa.$$

*Proof.* We can rewrite (16) as

$$\lambda_{a,b}(x) = (1 + ax(1-x))\exp(a(x-b)) \left( \frac{f_{a,b}(x)}{x} \right)^2.$$

Take  $\delta > 0$  such that  $I_{a,b} \subset (\delta, 1-\delta)$ . If  $\delta$  is sufficiently small, then the interval  $(\delta, 1-\delta)$  is invariant. Assume that  $x \in (\delta, 1-\delta)$ . Then  $1 + ax(1-x) > 1 + a\delta^2$ . Therefore

$$\lambda_{a,b}(x) > (1 + a\delta^2)\exp(a(x-b)) \left( \frac{f_{a,b}(x)}{x} \right)^2.$$

Taking the product over a piece of the trajectory of  $x$ , and using notation  $f_{a,b}^k(x) = x_k$ , we get

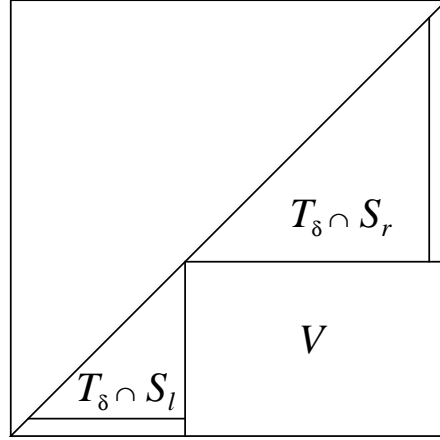
$$\prod_{k=0}^{n-1} \lambda_{a,b}(x_k) > (1 + a\delta^2)^n \exp \left( a \sum_{k=0}^{n-1} (x_k - b) \right) \left( \frac{x_n}{x_0} \right)^2.$$

Using (14), we get

$$(18) \quad \prod_{k=0}^{n-1} \lambda_{a,b}(x_k) > \delta^4 (1 + a\delta^2)^n.$$

We have  $\lambda_{a,b}(0) = \exp(ab) > 1$  and  $\lambda_{a,b}(1) = \exp(a(1-b)) > 1$ . Therefore, if  $\delta > 0$  is sufficiently small then  $\lambda(x) > 1 + a\delta^2$  whenever  $x \leq \delta$  or  $x \geq 1-\delta$ . Clearly, in (18) we can take arbitrarily small  $\delta > 0$ , and the estimate holds whenever  $x \in (\delta, 1-\delta)$ . This means that (18) holds for all  $x \in [0, 1]$ .

Now we take  $N$  such large that  $\delta^4(1 + a\delta^2)^N > 1$  and set  $\varkappa = \delta^4(1 + a\delta^2)^N$ . Then (17) holds.  $\square$

FIGURE 2. Regions  $V$ ,  $T_\delta$ ,  $S_r$  and  $S_\ell$ .

**Corollary 4.2.** *There is a neighborhood  $U$  of the diagonal such that if  $(x, y) \in U$  then the distance of  $F_{a,b}^N(x, y)$  from the diagonal is larger than the distance of  $(x, y)$  from the diagonal.*

Let us fix  $a > 0$  and  $b \in (0, 1)$ . For simplicity, we will now write  $F$  for  $F_{a,b}$  and  $(\hat{x}, \hat{y})$  for  $F(x, y)$  (if we apply  $F$  only once). If we iterate  $F$ , we will continue to write  $(x_n, y_n)$  for  $F^n(x, y)$ .

Let us establish some simple properties of  $F$ . We will consider the triangle below the diagonal in the square  $[0, 1]^2$ . Switching  $x$  and  $y$  will result in switching  $\hat{x}$  and  $\hat{y}$ , so the corresponding results for the triangle above the diagonal will be the same.

Consider the rectangle

$$V = \{(x, y) : 0 \leq y < b < x \leq 1\}$$

(see Figure 2).

**Lemma 4.3.** *If  $(x, y) \in V$  then its trajectory converges to  $(1, 0)$ .*

*Proof.* From the formula for  $F$  (equation (6)) it follows that if  $(x, y) \in V$  then  $\hat{x} > x$  and  $\hat{y} < y$ . In particular, also  $(\hat{x}, \hat{y}) \in V$ . Therefore the sequence  $(x_n)$  increases, while the sequence  $(y_n)$  decreases. The only possible limits for those sequences are 1 and 0 respectively.  $\square$

Elementary calculations result in the next property.

**Lemma 4.4.** *The map  $F$  preserves the triangle below the diagonal. That is, if  $x > y$  then  $\hat{x} > \hat{y}$ .*

For small  $\delta > 0$  we set

$$T_\delta = \{(x, y) : b \leq y < x \leq 1 - \delta\} \cup \{(x, y) : \delta \leq y < x \leq b\}$$

(see Figure 2).

**Lemma 4.5.** *If  $\delta > 0$  is sufficiently small then  $F(T_\delta) \subset T_\delta \cup V$ .*

*Proof.* Set

$$S_r = \{(x, y) : b \leq y \leq x \leq 1\}, \quad S_\ell = \{(x, y) : 0 \leq y \leq x \leq b\}.$$

If  $(x, y) \in S_r$  then

$$\hat{x} = \frac{x}{x + (1 - x) \exp(a(y - b))} \leq \frac{x}{x + (1 - x)} = x.$$

Similarly,  $\hat{y} \leq y$ . The set  $F(S_r)$  is a compact subset of the triangle  $\{(x, y) : x \geq y\}$ , disjoint from the line  $y = 0$  (because if  $\hat{y} = 0$  then  $y = 0$ ). Thus, there is  $\delta_r > 0$  such that if  $(x, y) \in S_r$  then  $\hat{y} > \delta_r$ . By similar arguments, there is  $\delta_\ell > 0$  such that if  $(x, y) \in S_\ell$  then  $\hat{x} < 1 - \delta_\ell$ .

Take  $\delta \in (0, \min(\delta_r, \delta_\ell))$  and  $(x, y) \in T_\delta \cap S_r$ . Then  $\hat{x} \leq x \leq 1 - \delta$ . Since  $x > y$ , by Lemma 4.4 we have  $\hat{x} > \hat{y}$ . Moreover,  $\hat{y} > \delta_r > \delta$ . Therefore,  $(\hat{x}, \hat{y}) \in T_\delta \cup V$ . Similarly, if  $(x, y) \in T_\delta \cap S_\ell$  then  $(\hat{x}, \hat{y}) \in T_\delta \cup V$ . Since  $T_\delta \subset S_r \cup S_\ell$ , this completes the proof.  $\square$

Similarly as in the one-dimensional case, we get by induction the formulas

$$x_n = \frac{x}{x + (1 - x) \exp\left(a \sum_{k=0}^{n-1} (y_k - b)\right)},$$

$$y_n = \frac{y}{y + (1 - y) \exp\left(a \sum_{k=0}^{n-1} (x_k - b)\right)}.$$

Now we can prove the main theorem of this section.

**Theorem 4.6.** *If  $0 \leq y < x \leq 1$  then the trajectory of  $(x, y)$  converges to  $(1, 0)$ .*

*Proof.* Suppose that  $0 < y < x < 1$ , but the trajectory of  $(x, y)$  does not converge to  $(1, 0)$ . By Lemma 4.3, for all  $n$  we have  $(x_n, y_n) \notin V$ . Choose  $\delta > 0$  such that  $\delta < y < x < 1 - \delta$  and the inclusion from Lemma 4.5 holds with this  $\delta$ . Then  $(x_n, y_n) \in T_\delta$  for all  $n$ . Therefore we have  $\delta < y_n < x_n < 1 - \delta$ , so

$$\delta < \frac{y}{y + (1 - y) \exp\left(a \sum_{k=0}^{n-1} (x_k - b)\right)} < \frac{x}{x + (1 - x) \exp\left(a \sum_{k=0}^{n-1} (y_k - b)\right)} < 1 - \delta.$$

In the same way as in the proof of Theorem 3.4, we get

$$\left| \sum_{k=0}^{n-1} (x_k - b) \right| < \frac{2 \log(1/\delta)}{a} \quad \text{and} \quad \left| \sum_{k=0}^{n-1} (y_k - b) \right| < \frac{2 \log(1/\delta)}{a}.$$

Therefore,

$$\sum_{k=0}^{n-1} (x_k - y_k) < \frac{4 \log(1/\delta)}{a}.$$

Since  $x_k > y_k$  for all  $k$ , this means that the series

$$\sum_{k=0}^{\infty} (x_k - y_k)$$

converges, so  $\lim_{k \rightarrow \infty} (x_k - y_k) = 0$ . However, this contradicts Corollary 4.2.

This proves that in the case  $0 < y < x < 1$  the sequence  $(x_n, y_n)$  converges to  $(1, 0)$ .

If  $y = 0$  then  $y_n = 0$  for all  $n$ , and the sequence  $(x_n)$  increases. The limit of this sequence is a number  $z > 0$  such that  $(z, 0)$  is a fixed point of  $F$ , and the only such point is  $(1, 0)$ . Similarly, if  $x = 1$  then  $(x_n, y_n)$  converges to  $(1, 0)$ .  $\square$

Taking into account what we observed about switching  $x$  and  $y$ , we can state a more general corollary.

**Corollary 4.7.** *If a point  $(x, y) \in [0, 1]^2$  is below (respectively, above) the diagonal, its  $F_{a,b}$ -trajectory converges to  $(1, 0)$  (respectively,  $(0, 1)$ ).*

## FUNDING

Thiparat Chotibut gratefully acknowledges the startup research grant SRES15111, and the SUTD-ZJU collaboration research grant ZJURP1600103. Fryderyk Falniowski gratefully acknowledges the support of the National Science Centre, Poland, grant 2016/21/D/HS4/01798 and COST Action CA16228 “European Network for Game Theory”. Research of Michał Misiurewicz was partially supported by grant number 426602 from the Simons Foundation. Georgios Piliouras was partially supported by SUTD grant SRG ESD 2015 097, MOE AcRF Tier 2 Grant 2016-T2-1-170 and NRF 2018 Fellowship NRF-NRFF2018-07.

## REFERENCES

- [1] S. Arora, E. Hazan, and S. Kale, *The multiplicative weights update method: a meta-algorithm and applications*, Theory of Computing, 8(1):121–164, 2012.
- [2] T.-Y. Li, M. Misiurewicz, G. Pianigiani and J. A. Yorke, *Odd chaos*, Phys. Lett. A **87** (1982), 271–273.
- [3] T.-Y. Li and J. A. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992.
- [4] M. Misiurewicz, *Rotation theory*, in Online Proceedings of the RIMS Workshop on “Dynamical Systems and Applications: Recent Progress” (2006), [https://www.math.kyoto-u.ac.jp/~kokubu/RIMS2006/RIMS\\_Online\\_Proceedings.html](https://www.math.kyoto-u.ac.jp/~kokubu/RIMS2006/RIMS_Online_Proceedings.html)
- [5] N. Nisan, T. Roughgarden, E. Tardos and V. Vazirani (Eds.), “Algorithmic Game Theory”, Cambridge University Press, Cambridge, 2007.
- [6] G. Palaiojanos, I. Panageas and G. Piliouras, *Multiplicative Weights Update with Constant Step-Size in Congestion Games: Convergence, Limit Cycles and Chaos*, In Advances in Neural Information Processing Systems (2017), 5874–5884.
- [7] R. W. Rosenthal, *A class of games possessing pure-strategy Nash equilibria*, Internat. J. Game Theory **2** (1973), 65–67.
- [8] A. N. Sharkovsky, *Co-existence of the cycles of a continuous mapping of the line into itself*, Ukrain. Math. Zh. **16** (1964), 61–71 (Russian).
- [9] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, “Dynamics of one-dimensional maps” (Mathematics and its Applications, vol. 407), Kluwer Academic Publishers Group, Dordrecht, 1997.

(T. Chotibut) ENGINEERING SYSTEMS AND DESIGN, SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN, 8 SOMAPAH ROAD, SINGAPORE 487372

*E-mail address:* thiparatc@gmail.com, thiparat\_chotibut@sutd.edu.sg

(F. Falniowski) DEPARTMENT OF MATHEMATICS, CRACOW UNIVERSITY OF ECONOMICS, RAKOWICKA 27, 31-510 KRAKÓW, POLAND

*E-mail address:* fryderyk.falniowski@uek.krakow.pl

(M. Misiurewicz) DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY-PURDUE UNIVERSITY INDIANAPOLIS, 402 N. BLACKFORD STREET, INDIANAPOLIS, IN 46202, USA

*E-mail address:* mmisiure@math.iupui.edu

(G. Piliouras) ENGINEERING SYSTEMS AND DESIGN, SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN, 8 SOMAPAH ROAD, SINGAPORE 487372

*E-mail address:* georgios@sutd.edu.sg