

ROTATION SETS FOR UNIMODAL MAPS OF THE INTERVAL

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Abstract. We relate the rotation interval $\rho(f)$ of a unimodal map f of the interval with its kneading invariant $K(f)$. In particular, we show that for any $\mu \in (0, \frac{1}{2})$, there are kneading invariants ν_μ and $\nu_{\mu, \text{hom}}$ such that $\rho(f) = [\mu, \frac{1}{2}]$ if and only if $\nu_\mu \preceq K(f) \preceq \nu_{\mu, \text{hom}}$.

1. Introduction. Let I denote the interval $[a, b]$ for a pair of distinct real numbers $a < b$. A point $x \in I$ is said to be of period n , or n -periodic for a mapping $f : I \rightarrow I$ if $f^n(x) = x$. It has least period n if additionally $f^i(x) \neq x$ for any $0 < i < n$. For the purposes of this paper, points referred to as being n -periodic will be assumed to be least so unless specifically stated otherwise. The orbit X of a period n point x is the collection of points in its image $\cup_{i=0}^{n-1} f^i(x)$. The elements of X are indexed consistently with their ordering in I , $X = \{x_1 < x_2 < \cdots < x_n\}$. The following ordering of the positive integers is called the Sarkovskii ordering:

$$\begin{aligned} 3 \geq_s 5 \geq_s 7 \geq_s \cdots 6 \geq_s 10 \geq_s 14 \geq_s \cdots \geq_s 2 \times (2n+1) \geq_s \cdots \\ \cdots \geq_s 2^m \times 3 \geq_s 2^m \times 5 \geq_s 2^m \times 7 \geq_s \cdots \geq_s 2^m \times (2n+1) \geq_s \cdots \\ \cdots \geq_s 2^{j+1} \geq_s 2^j \geq_s \cdots \geq_s 2 \geq_s 1 \end{aligned}$$

The importance of this ordering is made apparent in the following:

Theorem 1.1 (Sarkovskii [Sar64]). *Let $f : I \rightarrow I$ be a continuous map having a periodic point of period n . If $n >_s m$ in the Sarkovskii ordering then f has a point of period m .*

The Sarkovskii theorem is extremely satisfying in its elegance of statement. Two orbits of the same period, however, may be distinguished from one another in the way that the map permutes their respective members.

Definition 1.2. *A pattern of length n is a cyclic permutation of $\{1, 2, \cdots, n\}$. If x is n -periodic point with orbit $x_1 < x_2 < \cdots < x_n$, then we say that the orbit X has pattern π , or is of type π , if $f(x_i) = x_{\pi(i)}$.*

The pattern of the m -periodic point being forced in Theorem 1.1 is not addressed. The period is simply too coarse an invariant of the orbit to capture this information. The collection of patterns, however, carries with it its own forcing relation.

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Definition 1.3. Let π_1, π_2 be cycles. We say that π_1 forces π_2 ($\pi_1 \rightarrow \pi_2$) if every continuous $f : I \rightarrow I$ which has an orbit of type π_1 has an orbit of type π_2 .

Theorem 1.4 (Baldwin [Bal87]). The relation \rightarrow is a partial ordering on cycles.

We associate a continuous interval map to each pattern. The graph of a pattern is the graph of the piecewise linear function $f : [1, n] \rightarrow [1, n]$ which “connects the dots” between the points $(i, \pi(i))$, for $1 \leq i \leq n$. Graphs of patterns are minimal realizations of the forcing relation.

Theorem 1.5 ([ALM00]). If f is the graph of the pattern π_1 , and f exhibits a periodic orbit with pattern π_2 , then $\pi_1 \rightarrow \pi_2$.

The non-specificity of the Sharkovskii theorem noted above may be overcome, but it comes at a price. The ordering of Theorem 1.4 is far from being a total ordering, and attempts to make sense of the global picture quickly become a nasty business (see a pictorial representation of the partial ordering on all cycles of period ≤ 5 in [Bal87][Fig. 4]).

The picture becomes somewhat more manageable if we restrict the analysis to unimodal maps and patterns.

Definition 1.6. A map $f : I \rightarrow I$ is unimodal if it has the following properties:

U1: f is continuous.

U2: There is a unique turning point $c \in \text{int}(I)$ such that f is strictly increasing on $[a, c]$ and strictly decreasing on $(c, b]$.

U3: $f(x) \geq x$ on $[a, c]$, with strict inequality except possibly at the point a .

The condition **U3** is not essential to what is presented below, but is included here in order to remove some annoying special cases.

Definition 1.7. A pattern π is k -modal if $\#\{i | 1 < i < n \text{ and } (\pi(i) - \pi(i-1))(\pi(i+1) - \pi(i)) < 0\}$ is k .

The points i in the previous definition for which the product is negative are the turning points of the graph of π . A pattern π is unimodal if $k = 1$. In this instance, the graph of π is clearly unimodal, its single turning point either a global minimum or maximum. For any forcing between two unimodal patterns with global maxima, there is a symmetric forcing between two unimodal patterns with global minima (see [Bal87, ALM00]). For that reason, it suffices to analyze the forcing relation on exactly one of these types, and we let U denote the collection of all unimodal patterns with a global maximum.

Theorem 1.8 ([Bal87]). The relation \rightarrow is a linear ordering on U .

Definition 1.9. A pattern π is primary if it does not force any other pattern of the same length. The n -periodic primary pattern will be denoted M_n .

The primary patterns M_n are unimodal, their form well documented in the literature (see [Š77, ALM00, BC91, CE80]). This gives a natural embedding of the Sarkovskii ordering into the unimodal patterns $\langle U, \rightarrow \rangle$.

In the current paper, we take this type of forcing to a finer resolution through the use of the rotation number as a pattern invariant (see [Blo94, Blo95a, Blo95b]).

Definition 1.10. The rotation number of a unimodal pattern π is $\rho(\pi) = \frac{p}{q}$, where q is the length of π and $p = p(\pi) = \#\{1 \leq i \leq n | \pi(i) < i\}$. If $x \in I$ is periodic for a continuous map f with pattern π , then the rotation number of x is defined $\rho(x) = \rho(\pi)$.

Clearly the rotation number of a pattern is a rational number $0 < \frac{p}{q} \leq \frac{1}{2}$.

Definition 1.11. A pattern π with rotation number $\rho(\pi) = \frac{p}{q}$ is twist if it does not force any other pattern of the same length. The twist pattern of rotation number $\frac{p}{q}$ will be denoted $T_{\frac{p}{q}}$.

Our first order of business will be to prove the following:

Theorem 1.12. Given two distinct rational numbers $0 < \mu_1, \mu_2 < \frac{p}{q}$, and unimodal twist patterns T_{μ_1} and T_{μ_2} , $\mu_1 < \mu_2$ if and only if $T_{\mu_2} \rightarrow T_{\mu_1}$.

Next, we would like to extend the notion of rotation to include points which are not necessarily periodic. Note that if f is a unimodal map, then there is a unique fixed point x_R on $[c, b]$ (i.e. $f(x_R) = x_R$). The set of points $x \in I$ for which $f(x) < x$ is then precisely the interval $(x_R, b]$. Let χ_0 be the modified indicator function

$$\chi_0(x) = \begin{cases} 0, & \text{if } x < x_R \\ \frac{1}{2}, & \text{if } x = x_R \\ 1, & \text{if } x > x_R. \end{cases}$$

Definition 1.13. If $f : I \rightarrow I$ is a unimodal map and $x \in I$, then the rotation set of x is

$$\rho(x) = \bigcap_{j \geq 1} \bigcup_{n \geq j} \overline{\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \chi_0(f^i(x)) \right\}}. \quad (1.1)$$

The rotation set of a point x is a closed interval [Ito80]. If the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_0(f^i(x)) \quad (1.2)$$

exists, then the interval $\rho(x)$ collapses to a point $\{\mu\}$, and we say that x has rotation number $\rho(x) = \mu$. If $f^n(x) = x_R$ for some $n \geq 0$, then $f^j(x) = x_R$ for all $j \geq n$ and clearly x has rotation number $\rho(x) = \frac{1}{2}$. If $f^n(x) \neq x_R$ for any $n \geq 0$, and if x has a rotation number, then $\rho(x)$ is the ergodic average of the intersection of the orbit of the point x with the set $(x_R, b]$. If x is periodic with pattern π , then it is a simple matter to see that the rotation set collapses and the rotation number of x is $\rho(x) = \rho(\pi)$, justifying the consistent notation used in definitions 1.10 and 1.13. The rotation set of the map f is the set

$$\rho(f) = \bigcup_{x \in I} \rho(x). \quad (1.3)$$

We utilize the kneading theory of Milnor and Thurston for unimodal maps in the analysis of rotation sets. In the following sections we introduce the μ -twist (ν_μ) and μ -twist homoclinic ($\nu_{\mu, \text{hom}}$) itineraries for any real number $0 < \mu < \frac{1}{2}$.

Theorem 1.14. The rotation set, $\rho(f)$, of a unimodal map f is an interval of the form $[\mu, \frac{1}{2}]$. The rotation set $\rho(f)$ is equal to $[\mu, \frac{1}{2}]$ if and only if $\nu_\mu \preceq K(f) \preceq \nu_{\mu, \text{hom}}$.

We remark that a statement very similar to this Theorem appears as Theorem 2.8 in the MSRI preprint [Blo94]. The upper bound for the kneading sequence presented there is eventually constant, however, in contrast to the value $\nu_{\mu, \text{hom}}$ presented here which is either preperiodic or aperiodic. We also remark that in

Proposition A6 of [BM97], the rotation interval for a unimodal map f is correlated with the entropy $h(f)$ of the map. In particular, the rotation interval is shown to have a rational left endpoint μ if and only if $h(f)$ lies in an interval $[\lambda_\mu, \widetilde{\lambda}_\mu]$, where the numbers λ_μ and $\widetilde{\lambda}_\mu$ satisfy certain algebraic conditions. The values of λ_μ and $\widetilde{\lambda}_\mu$ can be obtained from the kneading sequences ν_μ and $\nu_{\mu, \text{hom}}$ without too much difficulty, giving a proof of Proposition A6 of [BM97] using very different methods.

Please see [Can, Wen] for a discussion of material related to that presented here.

2. Kneading Theory. The current investigation into the ordering $\langle U, \rightarrow \rangle$ will use the Kneading Theory of Milnor and Thurston. For convenience, we provide a brief summary here. Detailed expositions may be found in [MT88, dMvS93] for example, but we will follow more closely the slightly different flavor presented in [CE80].

Let f be a fixed unimodal mapping. We associate to any $x \in I$ an itinerary $\underline{i}(x)$, which is a finite or infinite sequence of symbols $\{L, R, C\}$ determined as follows:

- I1:** $\underline{i}(x)$ is either an infinite sequence of L 's and R 's, or a finite (perhaps empty) sequence of L 's and R 's terminating in a single C .
- I2:** If $f^k(x) \neq c$ for any $k \geq 0$, then $\underline{i}(x) = x_0 x_1 x_2 \cdots$ where $x_i = L$ if $f^i(x) < c$ and $x_i = R$ if $f^i(x) > c$.
- I3:** If $f^k(x) = c$ for some $k \geq 0$, and j is the smallest such, then $\underline{i}(x) = x_0 x_1 x_2 \cdots x_j$, where $x_i = L$ if $f^i(x) < c$ and $x_i = R$ if $f^i(x) > c$ for $0 \leq i < j$ and $x_j = C$.

Sequences of the form dictated by **I1** are called itineraries, the set of which is denoted Σ . Every itinerary $\underline{\alpha} \in \Sigma$ has associated to it an extended itinerary $E(\underline{\alpha})$ which is infinite. If $\underline{\alpha}$ is itself infinite, then $E(\underline{\alpha}) = \underline{\alpha}$. If $\underline{\alpha} = \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n C$ is finite, then $E(\underline{\alpha}) = \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n \underline{i}(c)$ if $\underline{i}(c)$ is infinite and $E(\underline{\alpha}) = \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n (\underline{i}(c))^\infty$ otherwise, where w^∞ denotes the infinite concatenation of copies of a finite word w . The space Σ_E of extended itineraries equipped with the product topology is a compact space. The shift map $\sigma : \Sigma_E \rightarrow \Sigma_E$ is the continuous map $\sigma(\alpha_0 \alpha_1 \alpha_2 \cdots) = \alpha_1 \alpha_2 \alpha_3 \cdots$. It is clear that the shift map acts so that $\underline{i}(f(x)) = \sigma(\underline{i}(x))$.

The sets Σ and Σ_E have a total ordering, the parity lexicographical ordering, defined as follows: A finite word is a string of symbols $\alpha_{i,j} = \alpha_i \alpha_{i+1} \cdots \alpha_j$ where $i \leq j$ and $\alpha_k \in \{L, R, C\}$ for all $i \leq k \leq j$. The word $\alpha_{i,j}$ is said to be odd (even) if $\#\{i \leq k \leq j \mid \alpha_k = R\}$ is an odd (even) number. For two itineraries $\underline{\alpha} \neq \underline{\beta}$, let n be the smallest non-negative integer such that $\alpha_n \neq \beta_n$. If we set $L < C < \overline{R}$, then we say that $\underline{\alpha} \prec \underline{\beta}$ if $\alpha_{0,n-1}$ is even and $\alpha_n < \beta_n$, or if $\alpha_{0,n-1}$ is odd and $\alpha_n > \beta_n$. An extended itinerary $\underline{\alpha}$ is *shift maximal* if $\sigma^i(\underline{\alpha}) \preceq \underline{\alpha}$ for all $i \geq 0$. It is a simple matter to see that the only shift maximal extended itinerary which does not begin with R is L^∞ .

Definition 2.1. *The admissible itineraries for a unimodal map f is the set $A = A(f) = \{\underline{\alpha} \in \Sigma \mid \underline{\alpha} = \underline{i}(x) \text{ for some } x \in I\}$.*

The parity lexicographical ordering restricted to A is faithful to that on I .

Theorem 2.2. *Let x and y be distinct points in I with itineraries $\underline{\alpha} = \underline{i}(x)$ and $\underline{\beta} = \underline{i}(y)$. if $\underline{\alpha} \prec \underline{\beta}$, then $x < y$, and if $x < y$ then $\underline{\alpha} \preceq \underline{\beta}$.*

The kneading sequence of f is the shift maximal itinerary $K(f) = \underline{i}(f(c))$. The set of admissible itineraries can be established with an analysis of the kneading sequence.

Theorem 2.3. *Suppose that f is unimodal and $\underline{\alpha} \in \Sigma$ is a sequence satisfying*

A1: $\underline{\alpha} \succeq \underline{i}(a)$.

A2: *If $K(f)$ does not contain any C 's, then $\sigma^k(\underline{\alpha}) \preceq K(f)$ for all $k \geq 0$.*

A3: *If $K(f) = (wC)^\infty$, then $\sigma^k(\underline{\alpha}) \preceq \inf((wL)^\infty, (wR)^\infty)$ for all $k \geq 0$.*

Then there is an $x \in I$ such that $\underline{i}(x) = \underline{\alpha}$.

Definition 2.4. *A sequence $\underline{\alpha}$ is allowable for a shift maximal sequence $\underline{\beta}$ if $\sigma(\underline{\beta}) \preceq \sigma^i(\underline{\alpha}) \preceq \underline{\beta}$ for all $i \geq 0$.*

Note that if $\underline{\alpha}$ and $\underline{\beta}$ are distinct shift maximal sequences, neither being L^∞ , then $\alpha_0 = \beta_0 = R$. If, in this case, $\underline{\alpha} \preceq \underline{\beta}$, then $\sigma(\underline{\beta}) \preceq \sigma(\underline{\alpha})$ and it follows that $\underline{\alpha}$ is allowable for $\underline{\beta}$. In particular, suppose that $\underline{\alpha}$ and $\underline{\beta}$ are periodic shift maximal sequences, neither including C , and $\underline{\alpha}$ is allowable for $\underline{\beta}$. If f is any unimodal map with $K(f) \succeq \underline{\beta}$, then not only is $\underline{\beta}$ admissible for $K(f)$, but $\underline{\alpha}$ is as well. In this way, the notion of allowability for periodic itineraries reflects the forcing relation on periodic orbits with those itineraries.

3. Twist Itineraries. The analysis below will involve itineraries of a quite specific form. We introduce the necessary notation here (see [BD99]). A finite or infinite sequence of non-negative integers $n_1 n_2 \cdots n_q$ ($q = \infty$ if the sequence is infinite) is said to have initial sums maximal if

$$(\text{ISM}): \sum_{i=1}^l n_i - \sum_{i=1}^l n_{i+j} \geq 0 \text{ for all } j, l \text{ such that } 1 \leq 1+j \leq l+j \leq q.$$

The sequence satisfies the finite sum condition if

$$(\text{FSC}): \left| \sum_{i=1}^l n_i - \sum_{i=1}^l n_{i+l} \right| \leq 1 \text{ for all } j, l \text{ such that } 1 \leq 1+j \leq l+j \leq q.$$

An itinerary has property $(*)$ if it is of the form

$$(*): RL^{n_1} RRL^{n_2} RRL^{n_3} R \cdots$$

An itinerary with property $(*)$ satisfies **(ISM)** (resp. **(FSC)**) if the infinite sequence $n_1 n_2 n_3 \cdots$ satisfies **(ISM)** (resp. **(FSC)**). An itinerary satisfying all three of these properties will be said to have property **(T)**. A note on convention: Many arguments in this section will involve sums of the form $\sum_{i=j}^k n_i$. If $k < j$, then the sum is taken to be 0.

In [BD99], Barge and Diamond analyze itineraries with property **(T)**. We collect several of their results in the following lemma.

Lemma 3.1 ([BD99]). *Any itinerary $\underline{\alpha}$ with property **(T)** has the following properties:*

1. *The itinerary $\underline{\alpha}$ is shift maximal (i.e. $\sigma^k(\underline{\alpha}) \preceq \underline{\alpha}$ for all $k \geq 0$).*
2. *If $\underline{\alpha} = (RL^{n_1} RRL^{n_2} R \cdots RL^{n_p} R)^\infty$ is periodic, then the sequence of numbers $n_1, n_2, \dots, n_{p-1}, n_p + 1$ is symmetric.*
3. *If $\underline{\alpha} = (RL^{n_1} RRL^{n_2} R \cdots RL^{n_p} R)^\infty$ is periodic, then*

$$(RL^{n_p} RRL^{n_{p-1}} R \cdots RL^{n_1} R)^\infty = (RL^{n_k} RRL^{n_{k+1}} R \cdots RL^{n_p} RRL^{n_1} R \cdots \\ \cdots RL^{n_{k-1}} R)^\infty$$

for some $k > 0$.

$$\nu_\mu(i) = \begin{cases} R, & \text{if } \xi_\mu^{i+1}(0) \in A_j \text{ for some } j \geq 0 \\ L, & \text{if } \xi_\mu^{i+1}(0) \in B_j \text{ for some } j \geq 0 \end{cases} \quad (3.4)$$

Proposition 3.2. *The itineraries ν_μ have property (T) for any $\mu \in (0, \frac{1}{2})$.*

If $[a, a + r)$ is a half open interval of length $r > 0$ for some $a \geq \mu$, then the cardinality $\#\{[a, a + r) \cap W\}$ is maximized when $a \in W$. It is not difficult to see that

for any $a, b \geq \mu$. The intervals $B_j = [(j-1) + 2\mu, j)$ introduced in the definition of ν_μ are each of length $1 - 2\mu$. Thus for any $j \geq 1, k \geq 0$,

Therefore, $\sum_{i=1}^j n_i - \sum_{i=1}^j n_{i+k} = \# \{W \cap [2\mu, j)\} - \# \{W \cap [2\mu + k, j + k)\} \geq 0$ as the left endpoint of $[2\mu, j)$ is an element of W . So ν_μ has initial sums maximal.

$$\left| \sum_{i=1}^j n_i - \sum_{i=1}^j n_{i+k} \right| = |\#\{W \cap [2\mu, j)\} - \#\{W \cap [2\mu + k, j + k)\}| \leq 1. \quad (3.7)$$
$$\nu_\mu = (RL^{n_1}R \cdots RL^{n_p}R)^\infty, \quad (3.8)$$
$$\nu_{\mu, \text{hom}} = (RL^{n_p+1}RRL^{n_{p-1}}R \cdots RL^{n_1}R)(RL^{n_p}RRL^{n_{p-1}}R \cdots RL^{n_1}R)^\infty. \quad (3.9)$$
$$\nu_{\mu, \text{hom}} = (RL^{n_1} RRL^{n_2} R \cdots RL^{n_p+1} R)(RL^{n_1-1} RRL^{n_2} R \cdots RL^{n_p+1} R)^\infty \quad (3.10)$$

$$\nu_{\mu, \text{hom}} = (RL^{n_p+1}RRL^{n_{p-1}}R \cdots RL^{n_1}R) \quad (3.11)$$

$$(RL^{n_k} RRL^{n_{k+1}} R \dots RL^{n_p} R)(RL^{n_1} RRL^{n_2} R \dots RL^{n_p} R)^\infty.$$

A proof of the following proposition appears in [BD99, Lemma 3.5], but is presented here in the local language.

Proposition 3.3. $\nu_{\mu, \text{hom}}$ has property **(T)**.

Proof. If μ is irrational, then $\nu_{\mu, \text{hom}} = \nu_\mu$, and we have seen that ν_μ satisfies the conditions of the proposition. If $\mu = \frac{p}{q}$ is rational, then $\nu_{\mu, \text{hom}}$ is of the form given in (3.9) and thus has property $(*)$ by definition. We relabel

$$\nu_{\mu, \text{hom}} = RL^{m_1}RRL^{m_2}RRL^{m_3}RRL^{m_4}R \cdots. \quad (3.12)$$

For any integer $l \geq 1$, there exists non-negative integers $k \geq 0$ and $0 \leq r < p$ such that $l = kp + r$. We note the following identities:

If $k = 0$, then $l = r$ and

$$\sum_{i=1}^l m_i = \sum_{i=1}^l n_i \quad (3.13)$$

while for any $j > 0$,

$$\sum_{i=1}^l m_{i+j} = \sum_{i=s+1}^{s+l} n_i, \quad (3.14)$$

for some $0 \leq s < q$. Because ν_μ has property **(T)**,

$$\sum_{i=1}^l m_i - 1 \leq \sum_{i=1}^l m_{i+j} \leq \sum_{i=1}^l m_i. \quad (3.15)$$

If $k > 0$, then

$$\begin{aligned} \sum_{i=1}^l m_i &= \sum_{i=1}^{kp+r} m_i \\ &= 1 + k \sum_{i=1}^p n_i + \sum_{i=kp+1}^{kp+r} m_i \\ &= 1 + k \sum_{i=1}^p n_i + \sum_{i=0}^{r-1} n_{p-i} \\ &= \begin{cases} k \sum_{i=1}^p n_i + \sum_{i=1}^r n_i & \text{if } r \neq 0 \\ k \sum_{i=1}^p n_i + 1 & \text{if } r = 0 \end{cases} \end{aligned} \quad (3.16)$$

and for any $j > 0$,

$$\begin{aligned} \sum_{i=1}^l m_{i+j} &= \sum_{i=1}^{kp+r} m_{i+j} \\ &= k \sum_{i=1}^p n_i + \sum_{i=kp+1}^{kp+r} m_{i+j} \\ &= k \sum_{i=1}^p n_i + \sum_{i=s+1}^{s+r} n_i, \text{ for some } s \end{aligned} \quad (3.17)$$

From (3.16) and (3.17), we see that for all positive integers l, j ,

$$\sum_{i=1}^l m_i - 1 \leq \sum_{i=1}^l m_{i+j} \leq \sum_{i=1}^l m_i. \quad (3.18)$$

Thus $\nu_{\mu, \text{hom}}$ has properties **(ISM)** and **(FSC)**. \square

Proposition 3.4. *For any $\mu \in (0, \frac{1}{2})$, there are no itineraries strictly between ν_μ and $\nu_{\mu, \text{hom}}$ satisfying property **(T)**.*

Proof. The statement is trivially true for irrational μ . So let μ be rational,

$$\nu_\mu = (RL^{n_1} RRL^{n_2} R \cdots RL^{n_p} R)^\infty \quad (3.19)$$

$$= RL^{n'_1} RRL^{n'_2} RRL^{n'_3} RRL^{n'_4} R \cdots \text{ and}$$

$$\nu_{\mu, \text{hom}} = RL^{n_1} RRL^{n_2} R \cdots RL^{n_p+1} R(RL^{n_1-1} RRL^{n_2} R \cdots RL^{n_p+1} R)^\infty \quad (3.20)$$

$$= RL^{n''_1} RRL^{n''_2} RRL^{n''_3} RRL^{n''_4} R \cdots$$

Suppose that there were a word

$$v = RL^{m_1} RRL^{m_2} RRL^{m_3} R \cdots \quad (3.21)$$

satisfying property **(T)** and $\nu_\mu \prec v \prec \nu_{\mu, \text{hom}}$. Because $n'_i = n_i = n''_i$ for $1 \leq i \leq p-1$, it must necessarily hold that $m_i = n_i$ for $1 \leq i \leq p-1$ and $n'_p \leq m_p \leq n''_p = n'_p + 1$.

We first note that $m_p = n''_p = n_p + 1$. If this were not the case, then $m_p = n'_p = n_p$ and we let $l > p$ be the smallest integer such that $m_l \neq n'_l$. Of course $m_l > n'_l$, as $v \succ \nu_\mu$, and it follows that

$$\sum_{i=1}^l m_i > \sum_{i=1}^l n'_i. \quad (3.22)$$

Expressing $l = kp + r$ for $k \geq 1$ and $0 < r \leq p$, we see that

$$\begin{aligned} \sum_{i=1}^{kp+r} m_i &> \sum_{i=1}^{kp+r} n'_i, \text{ and} \\ \sum_{i=kp+1}^{kp+r} m_i &> \sum_{i=kp+1}^{kp+r} n'_i = \sum_{i=1}^r n'_i = \sum_{i=1}^r m_i, \end{aligned} \quad (3.23)$$

violating property **(ISM)** for v , and thus $m_p = n''_p = n_p + 1$.

If $v \prec \nu_{\mu, \text{hom}}$, and $l > p$ is the smallest integer such that $m_l < n''_l$, then $m_l = m_1 - 1 = n_1 - 1$ while $n''_l = n_1$. We again express $l = kp + r$ for some $k \geq 1$ and $1 < r < p$ ($l \neq kp$ as $n''_l = n_1$).

$$\begin{aligned} \sum_{i=1}^r m_{kp+i} &= \sum_{i=1}^r n''_{kp+i} - 1 \\ &\leq \sum_{i=1}^r n''_i - 2 \\ &= \sum_{i=1}^r m_i - 2 \end{aligned} \quad (3.24)$$

violating property **(FSC)** for v . Therefore, no such v exists. \square

Proposition 3.5. *If $0 < \mu_1 < \mu_2 < \frac{1}{2}$, then $\nu_{\mu_2} \preceq \nu_{\mu_2, \text{hom}} \prec \nu_{\mu_1} \preceq \nu_{\mu_1, \text{hom}}$.*

Proof. It is easily seen that $\nu_{\mu_i} \preceq \nu_{\mu_i, \text{hom}}$ for $i = 1, 2$. Indeed, if μ_i is irrational, then $\nu_{\mu_i} = \nu_{\mu_i, \text{hom}}$, and if μ_i is rational, then this follows by comparing ν_{μ_i} with $\nu_{\mu_i, \text{hom}}$ as given in (3.10).

Proposition 3.4 asserts that there are no itineraries with property **(T)** between ν_{μ_2} and $\nu_{\mu_2, \text{hom}}$. The result will follow then if we show that $\nu_{\mu_2} \prec \nu_{\mu_1}$. Denote the itineraries

$$\begin{aligned}\nu_{\mu_1} &= RL^{n_1} RRL^{n_2} RRL^{n_3} R \dots \\ \nu_{\mu_2} &= RL^{m_1} RRL^{m_2} RRL^{m_3} R \dots\end{aligned}\tag{3.25}$$

Let j be the smallest positive integer such that $n_j \neq m_j$. The result follows if $n_j > m_j$. Suppose, to the contrary, that $n_j < m_j$. Recall that $n_j = \#\{\cup_{l \geq 1} \{\xi_{\mu_i}^l(0)\} \cap [j-1+2\mu_1, j)\}$. If $K = 2j-1 + \sum_{i=1}^j m_i$, then $\xi_{\mu_1}^K(0), \xi_{\mu_2}^K(0) < j$, while $\xi_{\mu_2}^{K+1}(0) < j \leq \xi_{\mu_1}^{K+1}(0)$, contradicting the fact that $\xi_{\mu_1}^i(0) < \xi_{\mu_2}^i(0)$ for all $i \geq 1$. It must therefore be the case that $n_j > m_j$. \square

If $\alpha \neq R^\infty$ is an itinerary with property **(T)** which does not lie in an interval $(\nu_\mu, \nu_{\mu, \text{hom}})$ for any $\mu \in (0, \frac{1}{2})$, then it follows from Proposition 3.5 that α is bounded below by $\lim_{\mu \rightarrow \mu_0^-} \nu_\mu$ and above by $\lim_{\mu \rightarrow \mu_0^+} \nu_\mu$ for some $\mu_0 \in (0, \frac{1}{2})$. Proposition 3.6 shows that this is not possible, and thus the itineraries with property **(T)** are precisely the set $\{R^\infty\} \cup \bigcup_{\mu \in (0, \frac{1}{2})} \{\nu_\mu \cup \nu_{\mu, \text{hom}}\}$.

Proposition 3.6. *For any $\mu_0 \in (0, \frac{1}{2})$*

1. $\lim_{\mu \rightarrow \mu_0^+} \nu_\mu = \nu_{\mu_0}$
2. $\lim_{\mu \rightarrow \mu_0^-} \nu_\mu = \nu_{\mu_0, \text{hom}}$

The proof of this Proposition requires the following two rather technical lemmas.

Lemma 3.7. *If $\alpha \in \Sigma_2^+$ is a periodic itinerary satisfying property **(T)**, then α is a μ -twist itinerary for some rational $\mu \in (0, \frac{1}{2})$.*

Proof. Suppose that α satisfies property **(T)** with $\alpha = (RL^{n_1} RRL^{n_2} R \dots RL^{n_p} R)^\infty$. Let $q = 2p + \sum_{i=1}^p n_i$ and $\nu_\mu = (RL^{m_1} RRL^{m_2} R \dots RL^{m_p} R)^\infty$ be the μ -twist itinerary for $\mu = \frac{p}{q}$. We will show that $\alpha = \nu_\mu$ by demonstrating that $m_i = n_i$ for all $1 \leq i \leq p$. The sequences $m_1, m_2, \dots, m_p + 1$ and $n_1, n_2, \dots, n_p + 1$ are both symmetric by Lemma 3.1. Suppose there were a smallest k such that $m_k \neq n_k$. Without loss of generality, $m_k < n_k$ and $\sum_{i=1}^k m_i < \sum_{i=1}^k n_i$. It follows from **(ISM)** and **(FSC)** that

$$\sum_{i=1}^k m_{i+s} \leq \sum_{i=1}^k n_{i+s} \quad \text{for any } 0 \leq s \leq p-k\tag{3.26}$$

Let $p = lk + r$ for some $l \geq 1$ and $0 \leq r < k$. By symmetry we see that

$$1 + \sum_{i=lk+1}^{lk+r} m_i = \sum_{i=1}^r m_i = \sum_{i=1}^r n_i = 1 + \sum_{i=lk+1}^{lk+r} n_i.\tag{3.27}$$

Therefore

$$\sum_{i=1}^p m_i = \sum_{i=1}^{lk+r} m_i < \sum_{i=1}^{lk} m_i + \sum_{i=lk+1}^{lk+r} m_i = \sum_{i=1}^{lk+r} n_i = \sum_{i=1}^p n_i.\tag{3.28}$$

This contradicts the fact that

$$\sum_{i=1}^p m_i = q - 2p = \sum_{i=1}^p n_i. \quad (3.29)$$

Thus there can be no such k and $\underline{\alpha} = \nu_\mu$. \square

Lemma 3.8. *Let $\beta_1, \beta_2, \dots, \beta_q$ be a finite sequence of non-negative integers, symmetric and satisfying properties **(ISM)** and **(FSC)**. If*

$$\underline{\alpha} = (RL^{\beta_1} RRL^{\beta_2} R \dots RL^{\beta_q-1} R)^\infty \quad (3.30)$$

then $\underline{\alpha} = \nu_\mu$ for some $\mu \in (0, \frac{1}{2})$.

Proof. For simplicity of notation, we relabel:

$$\underline{\alpha} = RL^{n_1} RRL^{n_2} RRL^{n_3} R \dots \quad (3.31)$$

In light of Lemma 3.7, it suffices to show that the infinite sequence n_1, n_2, n_3, \dots satisfies both **(ISM)** and **(FSC)** (that is $0 \leq \sum_{i=1}^m n_i - \sum_{i=1}^m n_{j+i} \leq 1$ for any $j, m \geq 1$). Any non-negative integer may be expressed as $m = kq + r$ where $k \geq 0$ and $0 \leq r < q$.

$$\begin{aligned} \sum_{i=1}^m n_i - \sum_{i=1}^m n_{j+i} &= \sum_{i=1}^{k_1 q + r_1} n_i - \sum_{i=1}^{k_1 q + r_1} n_{k_2 q + r_2 + i} \\ &= \sum_{i=1}^{k_1 q + r_1} n_i - \sum_{i=1}^{k_1 q + r_1} n_{r_2 + i} \\ &= \sum_{i=k_1 q + 1}^{k_1 q + r_1} n_i - \sum_{i=k_1 q + 1}^{k_1 q + r_1} n_{r_2 + i} \\ &= \sum_{i=1}^{r_1} n_i - \sum_{i=1}^{r_1} n_{r_2 + i} \end{aligned} \quad (3.32)$$

If $r_1 + r_2 \leq q$ then we are done as the sequence n_1, \dots, n_q has initial sums maximal and satisfies the finite sum condition. On the other hand, if $r_1 + r_2 > q$ then

$$\begin{aligned} \sum_{i=1}^{r_1} n_i - \sum_{i=1}^{r_1} n_{r_2 + i} &= \sum_{i=1}^{q-r_2} n_i - \sum_{i=1}^{q-r_2} n_{r_2 + i} + \sum_{i=q-r_2+1}^{r_1} n_i - \sum_{i=q-r_2+1}^{r_1} n_{r_2 + i} \\ &= 1 + \sum_{i=q-r_2+1}^{r_1} n_i - \sum_{i=1}^{r_1+r_2-q} n_i \\ &= 1 \text{ or } 0. \end{aligned} \quad (3.33)$$

This proves the claim. \square

Proof. of Proposition 3.6

case i: $\mu_0 = \frac{p}{q}$ is rational.

In order to prove the first statement of the proposition, it suffices to show that there are μ -twist itineraries $\nu_\mu \prec \nu_{\mu_0}$ which agree with ν_{μ_0} on arbitrarily

long initial words. By Lemma 3.1, there is a k such that the μ_0 -twist itinerary may be expressed as

$$\nu_{\mu_0} = (RL^{n_1}R \cdots RL^{n_p}R)^m (RL^{n_1}R \cdots RL^{n_{k-1}}R) \\ (RL^{n_k}R \cdots RL^{n_p}RRL^{n_1}R \cdots RL^{n_{k-1}}R)^\infty \quad (3.34)$$

for any $m \geq 0$. The finite sequence of non-negative integers

$$(n_1, n_2, \dots, n_p,)^m n_1, \dots, n_{k-1} \quad (3.35)$$

is symmetric and satisfies **(ISM)** and **(FSC)**. For any $m > 0$, the word

$$\nu_{\mu_m} = ((RL^{n_1}R \cdots RL^{n_p}R)^m RL^{n_1}R \cdots RL^{n_{k-1}-1}R)^\infty \quad (3.36)$$

is μ_m -twist periodic for some rational $\mu \in (0, \frac{1}{2})$ by Lemma 3.8. Further, we observe that for any $m > 0$, $\nu_{\mu_m} \prec \nu_{\mu_{m+1}} \prec \nu_{\mu_0}$, $\mu_m > \mu_{m+1} > \mu_0$, and that $\lim_{m \rightarrow \infty} \nu_{\mu_m} = \nu_{\mu_0}$.

As for the second statement, we have seen (3.11) that

$$\nu_{\mu, \text{hom}} = (RL^{n_p+1}R \cdots RL^{n_1}R)(RL^{n_k}RRL^{n_{k+1}}R \cdots RL^{n_p}R) \\ (RL^{n_1}RRL^{n_2}R \cdots RL^{n_p}R)^\infty$$

for some k . The finite sequence of integers

$$n_p + 1, n_{p-1}, \dots, n_1, (n_k, n_{k+1}, \dots, n_p,)(n_1, n_2, \dots, n_p,)^m n_1, n_2, \dots, n_p + 1 \quad (3.37)$$

is symmetric and satisfies **(ISM)** and **(FSC)**. We again appeal to Lemma 3.8 and conclude that

$$\nu_{\mu_m} = ((RL^{n_p+1}RRL^{n_{p-1}}R \cdots RL^{n_1}R)(RL^{n_k}RRL^{n_{k+1}}R \cdots RL^{n_p}R) \\ (RL^{n_1}RRL^{n_2}R \cdots RL^{n_p}R)^m RL^{n_1}RRL^{n_2}RRL^{n_p}R)^\infty \quad (3.38)$$

is μ_m -twist periodic for some rational $\mu \in (0, \frac{1}{2})$. As above, we observe that $\nu_{\mu_0, \text{hom}} \prec \nu_{\mu_{m+1}} \prec \nu_{\mu_m}$, $\mu_0 > \mu_{m+1} > \mu_m$ for all $m > 0$ and that $\lim_{m \rightarrow \infty} \nu_{\mu_m} = \nu_{\mu_0, \text{hom}}$.

case ii: μ_0 is irrational.

Let $\nu_{\mu_0} = RL^{n_1}RRL^{n_2}RRL^{n_3}R \cdots$. Recall that if $W(\mu_0) = \cup_{i \geq 1} \{\xi_{\mu_0}^i(0)\}$ and $B_j(\mu_0) = [j - 1 + 2\mu_0, j)$, then $n_j = \#\{W(\mu_0) \cap B_j(\mu_0)\}$. It follows from the irrationality of μ_0 that if $\xi_{\mu_0}^i(0) \in B_j(\mu_0)$ for some $i > 2$, then $\xi_{\mu_0}^i(0) \in \text{int}(B_j(\mu_0))$. For any finite $M \geq 1$, there exists a $\delta > 0$ such that for any $|\mu - \mu_0| < \delta$ and $1 \leq i \leq M$, $\xi_{\mu}^i(0) \in B_j(\mu)$ if and only if $\xi_{\mu_0}^i(0) \in B_j(\mu_0)$. Therefore $\nu_{\mu}(i) = \nu_{\mu_0}(i)$ for all $1 \leq i \leq M$ whenever $|\mu - \mu_0| < \delta$. Noting that $\nu_{\mu_0} = \nu_{\mu_0, \text{hom}}$, the result follows as in the previous case. \square

The rotation set of a point $x \in I$ may be recovered from its itinerary. If $\chi_0(x) = 1$, then $\underline{i}(x) \succ R^\infty$, and if $\chi_0(x) = \frac{1}{2}$, then $\underline{i}(x) = R^\infty$. We introduce the modified indicator $\chi : \Sigma \rightarrow \{0, \frac{1}{2}, 1\}$,

$$\chi(\underline{\alpha}) = \begin{cases} 1, & \text{if } \underline{\alpha} \succ .R^\infty \\ \frac{1}{2}, & \text{if } \underline{\alpha} = .R^\infty \\ 0, & \text{if } \underline{\alpha} \prec .R^\infty, \end{cases} \quad (3.39)$$

If $\underline{\alpha} = \underline{i}(x)$, then

$$\rho(x) = \rho(\underline{\alpha}) = \bigcap_{j \geq 1} \bigcup_{n \geq j} \overline{\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \chi(\sigma^i(\underline{\alpha})) \right\}}. \quad (3.40)$$

In light of this observation, we see that an alternate description of the rotation set of a unimodal map f is given by

$$\rho(f) = \bigcup_{\underline{\alpha} \in A(f)} \rho(\underline{\alpha}). \quad (3.41)$$

Proposition 3.9. *For any $0 < \mu < \frac{1}{2}$, the sequences ν_μ and $\nu_{\mu, \text{hom}}$ each have rotation number $\rho(\nu_\mu) = \rho(\nu_{\mu, \text{hom}}) = \mu$.*

Proof. Let $\nu_\mu = RL^{n_1}RRL^{n_2}RRL^{n_3}R \cdots$. First note that $\nu_\mu \neq \cdots R^\infty$, so $\chi(\sigma^i(\nu_\mu)) \neq \frac{1}{2}$ for all $i \geq 0$. Also, $\sigma^i(\nu_\mu) \succ R^\infty$ if and only if $\sigma^i(\nu_\mu) = R^m L \cdots$, where m is odd. It follows from the definition of ν_μ that $\sigma^i(\nu_\mu) \succ R^\infty$ if and only if $\xi_\mu^{i+1}(0) \in [j + \mu, j + 2\mu)$ for some $j \geq 0$. Therefore, $\sum_{i=0}^{k-1} \chi(\sigma^i(\nu_\mu)) = j$ if and only if $(j - 1) + \mu \leq \xi_\mu^k(0) < j + \mu$. Thus $(k - 1)\mu \leq j \leq (k - 1)\mu + 1$ and

$$(k - 1)\mu \leq \sum_{i=0}^{k-1} \chi(\sigma^i(\nu_\mu)) \leq (k - 1)\mu + 1. \quad (3.42)$$

It follows easily that

$$\rho(\nu_\mu) = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \chi(\sigma^i(\nu_\mu)) = \mu. \quad (3.43)$$

It is easily seen that $\rho(\nu_{\mu, \text{hom}}) = \mu$ for any $\mu \in (0, \frac{1}{2})$. Indeed, if μ is irrational, then $\nu_\mu = \nu_{\mu, \text{hom}}$, and if μ is rational, then this can be deduced from (3.11). \square

Proposition 3.10. *If $\underline{\alpha} \in \Sigma_2^+$ is an itinerary such that $\inf(\rho(\underline{\alpha})) < \frac{p}{q} \leq \frac{1}{2}$, then $\underline{\alpha}$ is not an allowable itinerary for the $\frac{p}{q}$ -twist itinerary $\nu_{\frac{p}{q}}$.*

Proof. If $\inf(\rho(\underline{\alpha})) < \frac{p}{q} \leq \frac{1}{2}$, then $\underline{\alpha} \neq \cdots R^\infty$ and there must be an integer $j \geq 0$ such that

$$\sum_{i=j}^{q+j-1} \chi(\sigma^i(\underline{\alpha})) < p. \quad (3.44)$$

Without loss of generality we may assume that $j = 0$.

The itinerary $\underline{\alpha}$ may be expressed in exactly one of three ways (1, 2 and 3 below), the parentheses there contain the initial word W of length q . We make some comments on notation to make the exposition more understandable. A finite word B of the form

$$RL^{m_1}RRL^{m_2}RRL^{m_3}RR \cdots RRL^{m_k}, \quad m_i \geq 0 \quad (3.45)$$

will be called an odd block. Note that if $m_i = 0$ for $1 \leq i \leq k$, then $B = R^{2k+1}$. Within an odd block, consecutive blocks of L 's are separated by an even number of R 's, while if two odd blocks B_1 and B_2 are concatenated

$$(RL^{m_1}RR \cdots RRL^{m_{k_2}})(RL^{n_1}RR \cdots RRL^{n_{k_2}}), \quad (3.46)$$

and $m_i, n_j > 0$ for some $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, then the number of R 's between the last L in B_1 and the first in B_2 is odd. The words 1, 2 and 3 below have decorations at their beginning and end. The decoration L^{m_0} appearing at

the beginning of 1 and 2 and $L^{m_{11}}$ in 3 allow for an initial (possibly empty) string of L 's of arbitrary length. If this string is empty, then the word begins with R . The terminating decoration $R^{o_1}R^{2r}$ allows for an arbitrary length string (possibly empty) of trailing R 's. In the counting arguments below, it is only the parity of o_2 which matters.

1. $\underline{\alpha}$ has no odd blocks.

$$\underline{\alpha} = (L^{m_0} R^{o_1} R^{2r}) R^{o_2} L \dots \quad (3.47)$$

$$m_0, r, o_2 \geq 0, \quad 0 \leq o_1 \leq 1 \quad (3.48)$$

$$q = m_0 + o_1 + 2r$$

2. An odd number of R 's follow L^{m_0} .

$$\underline{\alpha} = (L^{m_0} R L^{m_{11}} R R L^{m_{12}} \dots R R L^{m_{1k_1}} \quad (3.49)$$

$$R L^{m_{21}} R R L^{m_{22}} \dots R R L^{m_{2k_2}}$$

$$\vdots$$

$$R L^{m_{l1}} R R L^{m_{l2}} \dots R R L^{m_{lk_l}} R^{o_1} R^{2r}) R^{o_2} L$$

$$l, k_j \geq 1, \quad m_0, m_{ji}, r, o_2 \geq 0, \quad 0 \leq o_1 \leq 1 \quad (3.50)$$

$$q = m_0 + \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} + 2 \sum_{j=1}^l k_j - l + o_1 + 2r$$

3. An even number of R 's follow $L^{m_{11}}$.

$$\underline{\alpha} = (L^{m_{11}} R R L^{m_{12}} \dots R R L^{m_{1k_1}} \quad (3.51)$$

$$R L^{m_{21}} R R L^{m_{22}} \dots R R L^{m_{2k_2}}$$

$$\vdots$$

$$R L^{m_{l1}} R R L^{m_{l2}} \dots R R L^{m_{lk_l}} R^{o_1} R^{2r}) R^{o_2} L$$

$$l, k_j \geq 1, \quad m_{ji}, r, o_2 \geq 0, \quad 0 \leq o_1 \leq 1 \quad (3.52)$$

$$q = \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} + 2 \sum_{j=1}^l k_j - (l+1) + o_1 + 2r$$

We let $p' = \sum_{i=0}^{q-1} \chi(\sigma^i(\underline{\alpha}))$ and assume that $p' - p < 0$. We show that in each of the three cases, $\underline{\alpha}$ is not allowable for $\nu_{\frac{p}{q}}$.

If $\underline{\alpha}$ is of form 1 then

$$p' = \begin{cases} r+1, & \text{if } o_2 \text{ is even and } o_1 = 1, \\ r, & \text{otherwise.} \end{cases} \quad (3.53)$$

We know that

$$2p + \sum_{i=1}^p n_i = q = m_0 + o_1 + 2r \quad (3.54)$$

$$0 = m_0 - \sum_{i=1}^p n_i + 2(r-p) + o_1 \quad (3.55)$$

Suppose that $m_0 \leq n_1$. Then $m_0 - \sum_{i=1}^p n_i \leq 0$ and it follows that $2(r-p) + o_1 \geq 0$ and $r-p \geq 0$. but this contradicts the fact that $r-p \leq p'-p < 0$. Therefore $m_0 > n_1$ and $\underline{\alpha}$ is not allowable for $\nu_{\frac{p}{q}}$.

If $\underline{\alpha}$ is of form 2 then

$$p' = \begin{cases} \sum_{j=1}^l k_j + r + 1, & \text{if } o_2 \text{ is even and } o_1 = 1, \\ \sum_{j=1}^l k_j + r, & \text{otherwise.} \end{cases} \quad (3.56)$$

Suppose that $m_0 \leq n_1$ and $\sum_{i=r+1}^{r+s} m_{ji} \leq \sum_{i=1}^s n_i$ for any $1 \leq j \leq l$ and $1 \leq r+1 \leq r+s \leq k_j$. Then

$$m_0 + \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} \leq n_1 + \sum_{j=1}^l \sum_{i=1}^{k_j} n_i \leq \sum_{i=1}^{1+\sum_{j=1}^l k_j} n_i + l \leq \sum_{i=1}^p n_i + l, \quad (3.57)$$

and therefore

$$m_0 + \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} - \sum_{i=1}^p n_i - l \leq 0. \quad (3.58)$$

We know that

$$2p + \sum_{i=1}^p n_i = q = m_0 + \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} + 2 \sum_{j=1}^l k_j - l + o_1 + 2r \quad (3.59)$$

and thus

$$0 = m_0 + \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} - \sum_{i=1}^p n_i - l + 2 \left(\sum_{j=1}^l k_j + r - p \right) + o_1. \quad (3.60)$$

Comparing (3.58) with (3.60), we see that $2(\sum_{j=1}^l k_j + r - p) + o_1 \geq 0$ and thus $\sum_{j=1}^l k_j + r - p \geq 0$, contradicting the fact that $p' - p < 0$. Therefore $m_0 > n_1$ or $\sum_{i=r+1}^{r+s} m_{ji} > \sum_{i=1}^s n_i$ for some $1 \leq j \leq l$ and $1 \leq r+1 \leq r+s \leq k_j$. In either case $\underline{\alpha}$ is not allowable for $\nu_{\frac{p}{q}}$.

If $\underline{\alpha}$ is of form 3 then

$$p' = \begin{cases} \sum_{j=1}^l k_j + r + 1, & \text{if } o_2 \text{ is even and } o_1 = 1, \\ \sum_{j=1}^l k_j + r, & \text{otherwise.} \end{cases} \quad (3.61)$$

Suppose that $\sum_{i=r+1}^{r+s} m_{ji} \leq \sum_{i=1}^s n_i$ for all $1 \leq j \leq l$ and $1 \leq r+1 \leq r+s \leq k_j$. Then

$$\sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} \leq \sum_{j=1}^l \sum_{i=1}^{k_j} n_i \leq \sum_{i=1}^{\sum_{j=1}^l k_j} n_i + (l-1) \leq \sum_{i=1}^p n_i + (l-1), \text{ and} \quad (3.62)$$

$$\sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} - \sum_{i=1}^p n_i - l + 1 \leq 0 \quad (3.63)$$

We know

$$2p + \sum_{i=1}^p n_i = q = \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} + 2 \sum_{j=1}^l k_j - (l+1) + o_1 + 2r. \quad (3.64)$$

Thus,

$$0 = \sum_{j=1}^l \sum_{i=1}^{k_j} m_{ji} - \sum_{i=1}^p n_i - l + 1 + 2 \left(\sum_{j=1}^l k_j + r - p \right) + o_1 - 2 \quad (3.65)$$

Comparing (3.63) with (3.65), we see that

$$2 \left(\sum_{j=1}^l k_j + r - p \right) + o_1 - 2 \geq 0, \quad (3.66)$$

and, as $0 \leq o_1 \leq 1$,

$$\sum_{j=1}^l k_j + r - p \geq 1, \quad (3.67)$$

which again contradicts the fact that $p' - p < 0$. Therefore $\sum_{i=r+1}^{r+s} m_{ji} > \sum_{i=1}^s n_i$ for some $1 \leq j \leq l$ and $1 \leq r+1 \leq r+s \leq k_j$ and we conclude that $\underline{\alpha}$ is not allowable for $\nu_{\frac{p}{q}}$. \square

And now for the main theorem:

Theorem 3.11. *The rotation set, $\rho(f)$, of a unimodal map f is an interval of the form $[\mu, \frac{1}{2}]$. The rotation set $\rho(f)$ is equal to $[\mu, \frac{1}{2}]$ if and only if $\nu_{\mu} \preceq K(f) \preceq \nu_{\mu, \text{hom}}$.*

Proof. Suppose that $\nu_{\mu} \preceq K(f) \preceq \nu_{\mu, \text{hom}}$. For any $\mu \leq \mu_1 \leq \frac{1}{2}$, $\nu_{\mu_1} \in A(f)$ by Proposition 3.5 and $\rho(\nu_{\mu_1}) = \mu_1$ by Proposition 3.9. Therefore $\rho(f_a) \supset [\mu, \frac{1}{2}]$. If $\mu_2 \in \rho(f)$ with $\mu_2 < \mu$, then there must be an $\underline{\alpha} \in A(f)$ with $\liminf \rho(\underline{\alpha}) \leq \mu_2 < \mu$. The itinerary $\underline{\alpha}$, however, is not admissible for any ν_{μ_3} if $\mu_3 > \mu_2$ by Proposition 3.10. In particular, if we fix $\mu_3 \in (\mu_2, \mu)$, then $\nu_{\mu_3} \succ \nu_{\mu, \text{hom}} \succeq \nu_{\mu}$ and $\underline{\alpha}$ is not allowable for $\nu_{\mu, \text{hom}}$, contradicting the fact that $\underline{\alpha} \in A(f_a)$ and we conclude that $\rho(f_a) \subset [\mu, \frac{1}{2}]$.

If $\rho(f) = [\mu, \frac{1}{2}]$, then there exists an $\underline{\alpha} \in A(f)$ such that $\liminf \rho(\underline{\alpha}) = \mu$. $\underline{\alpha}$ is not admissible for any twist itinerary ν_{μ_1} for any $\mu_1 > \mu$ (Proposition 3.10). Therefore $K(f) \succ \nu_{\mu_1}$ for all $\mu_1 > \mu$ and $K(f) \succeq \lim_{\mu_1 \rightarrow \mu^+} \nu_{\mu_1} = \nu_{\mu}$. On the other hand, if $K(f) \succ \nu_{\mu, \text{hom}}$, then $K(f) \succeq \nu_{\mu_1}$ for some $\mu_1 < \mu$ (by Proposition 3.6). Therefore ν_{μ_1} is admissible for f and $\mu > \mu_1 \in \rho(f)$, contradicting the fact that $\rho(f) = [\mu, \frac{1}{2}]$. We conclude that $K(f) \preceq \nu_{\mu, \text{hom}}$. \square

We make a final remark regarding renormalization associated with the μ -twist periodic orbits (see [CE80, dMvS93, Dev89] for a discussion of renormalization. In particular, the $*$ notation is taken from [CE80]). If

$$\nu_{\mu} * RL^{\infty} = RL^{n_1} RRL^{n_2} R \cdots RL^{n_p+1} (RL^{n_1} R \cdots RL^{n_p} R)^{\infty}, \quad (3.68)$$

and $\nu_{\mu} \prec K(f) \preceq \nu_{\mu} * RL^{\infty}$, then the map f is renormalizable, where the renormalization intervals are bounded on one side by a twist periodic orbit with itinerary ν_{μ} . It is a simple matter to see that $\nu_{\mu} * RL^{\infty} \prec \nu_{\mu, \text{hom}}$ for any rational $\mu \in (0, \frac{1}{2})$. Thus, the “renormalization window” $[\nu_{\mu}, \nu_{\mu} * RL^{\infty}]$ is contained within the larger intervals $[\nu_{\mu}, \nu_{\mu, \text{hom}}]$.

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