ON SMOOTH PERTURBATIONS OF CHEBYSHËV POLYNOMIALS AND
$\bar{\partial}$-RIEMANN-HILBERT METHOD

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ABSTRACT. $\bar{\partial}$-extension of the matrix Riemann-Hilbert method is used to study asymptotics of the polynomials $P_n(z)$ satisfying orthogonality relations

$$\int_{-1}^{1} x^l P_n(x) \frac{\rho(x)dx}{\sqrt{1-x^2}} = 0, \quad l \in \{0, \dots, n-1\},$$

where $\rho(x)$ is a positive m times continuously differentiable function on $[-1, 1]$, $m \geqslant 3$.

1. MAIN RESULTS

In this note we are interested in the asymptotic behavior of monic polynomials $P_{n,i}(x)$, $\deg(P_{n,i}) = n$, dependent on a parameter $i \in \{1, 2, 3, 4\}$, satisfying orthogonality relations

$$(1) \qquad \int_{-1}^{1} x^l P_{n,i}(x) \frac{\rho(x)|v_i(x)|dx}{\sqrt{1-x^2}} = 0, \quad l \in \{0, \dots, n-1\},$$

where $\rho(x)$ is a positive and smooth function on $[-1, 1]$ and

$$v_1(z) \equiv 1, \quad v_2(z) = z^2 - 1, \quad v_3(z) = z + 1, \quad \text{and} \quad v_4(z) = z - 1.$$

That is, $P_{n,i}(z)$ are smooth perturbations of the Chebyshëv polynomials of the i-th kind. Besides polynomials themselves, we are also interested in the asymptotic behavior of their recurrence coefficients. That is, numbers $a_{n,i} \in [0, \infty)$ and $b_{n,i} \in (-\infty, \infty)$ such that

$$xP_{n,i}(x) = P_{n+1,i}(x) + b_{n,i}P_{n,i}(x) + a_{n,i}^2 P_{n-1,i}(x).$$

To describe the results, let $w(z) := \sqrt{z^2 - 1}$ be the branch analytic in $\mathbb{C}\backslash[-1, 1]$ such that $w(z)/z \to 1$ as $z \to \infty$. The Szegő function of the weight $\rho(x)$ is defined by

$$(2) \qquad S(z) := \exp\left\{\frac{w(z)}{2\pi i} \int_{-1}^{1} \frac{\log \rho(x)}{z - x} \frac{dx}{w_+(x)}\right\}, \quad z \in \overline{\mathbb{C}}\backslash[-1, 1],$$

which is an analytic and non-vanishing function in the domain of its definition satisfying

$$(3) \qquad S_+(x)S_-(x) = \rho^{-1}(x), \quad x \in [-1, 1].$$

Since $\rho(x)$ is positive, it holds that $S_+(x) = \overline{S_-(x)}$ for $x \in [-1, 1]$, and, utilizing the full power of Plemelj-Sokhotski formulae, (3) can be strengthen to

$$(4) \qquad \sqrt{\rho(x)}S_\pm(x) = e^{\pm i\theta(x)}, \qquad \theta(x) := \frac{\sqrt{1-x^2}}{2\pi} \fint_{-1}^{1} \frac{\log\rho(t)}{t-x} \frac{dt}{\sqrt{1-t^2}},$$

where $\fint$ is the integral in the sense of the principal value. Further, let

$$(5) \qquad \varphi(z) := z + w(z)$$

be the conformal map of $\overline{\mathbb{C}}\setminus[-1,1]$ onto $\mathbb{C}\setminus\{z : |z| \geqslant 1\}$ such that $\varphi(z)/z \to 2$ as $z \to \infty$. One can readily verify that

$$(6) \qquad \varphi_\pm(x) = x \pm i\sqrt{1-x^2} = e^{\pm i\arccos(x)}, \quad x \in [-1,1].$$

Finally, we explicitly define the Szegő functions of the weights $|v_i(x)|$. Namely, set

$$(7) \qquad \begin{cases} S_1(z) :\equiv 1, & S_3(z) := \big(\varphi(z)/(z+1)\big)^{1/2}, \\ S_2(z) := \varphi(z)/w(z), & S_4(z) := \big(\varphi(z)/(z-1)\big)^{1/2}, \end{cases}$$

where the square roots are principal and one needs to notice that the images of $\overline{\mathbb{C}}\setminus[-1,1]$ under $(z+1)/\varphi(z)$ and $(z-1)/\varphi(z)$ are domains symmetric with respect to conjugation whose intersections with the real line are equal to $(0,2)$ (so the square roots are indeed well defined). These functions satisfy

$$(8) \qquad S_{i+}(x)S_{i-}(x) = |S_{i\pm}(x)|^2 = 1/|v_i(x)|, \quad x \in (-1,1).$$

Observe also that $S_1(\infty) = 1$, $S_2(\infty) = 2$, and $S_3(\infty) = S_4(\infty) = \sqrt{2}$. Moreover, one can readily deduce from (6) and (8) that

$$(9) \quad S_{i\pm}(x) = \frac{e^{\pm i\theta_i(x)}}{\sqrt{|v_i(x)|}}, \quad \begin{cases} \theta_1(x) :\equiv 0, & \theta_2(x) := \arccos(x) - \frac{\pi}{2}, \\ \theta_3(x) := \frac{1}{2}\arccos(x), & \theta_4(x) := \frac{1}{2}\arccos(x) - \frac{\pi}{2}. \end{cases}$$

Recall that the modulus of continuity of a continuous function $f(x)$ on $[-1,1]$ is given by

$$\omega(f;h) := \max_{|x-y| \leqslant h, \ x,y \in [-1,1]} |f(x) - f(y)|.$$

Theorem 1

Assume that $\rho(x)$ is a strictly positive m times continuously differentiable function on $[-1,1]$ for some $m \geqslant 3$. Set

$$\varepsilon_n := \frac{\log n}{n^m} \omega\left((1/\rho)^{(m)}; 1/n\right).$$

Then it holds for any $i \in \{1,2,3,4\}$ that

$$P_{n,i}(z) = (1 + O(\varepsilon_n)) \frac{(S_i S)(z)}{(S_i S)(\infty)} \left(\frac{\varphi(z)}{2}\right)^n$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus[-1,1]$ and

$$P_{n,i}(x) = \frac{\cos\left(n\arccos(x) + \theta(x) + \theta_i(x)\right) + O(\varepsilon_n)}{2^{n-1}(S_i S)(\infty)\sqrt{\rho(x)|v_i(x)|}}$$

uniformly on $[-1, 1]$. Moreover, it also holds for any $i \in \{1, 2, 3, 4\}$ that
$$a_{n,i} = 1/2 + O(\varepsilon_n) \quad \text{and} \quad b_{n,i} = O(\varepsilon_n).$$

The above results are not entirely new. It is well known [18, Theorem 11.5] that perturbed first and second kind Chebyshëv polynomials can be expressed via orthogonal polynomials on the unit circle with respect to the weight $\rho(\frac{1}{2}(\tau + 1/\tau))$. Then using [17, Corollary 5.2.3], that in itself is an extension of ideas from [5], and Geronimus relations, see [17, Theorem 13.1.7], one can show that
$$\sum (n+1)^{\gamma} \left(|a_{n,1} - 1/2| + |b_{n,1}| \right) < \infty$$

for any $\gamma \in (0, m-1)$ and $m \geqslant 2$, which is consistent with Theorem 1. What is novel in this note is the method of proof. While the Baxter-Simon argument relies on the machinery of Banach algebras, we follow the approach of Fokas, Its, and Kitaev [11, 12] connecting orthogonal polynomials to matrix Riemann-Hilbert problems and then utilizing the non-linear steepest descent method of Deift and Zhou [9]. The main advantages of this approach are the ability to get full asymptotic expansions for analytic weights of orthogonality [8, 15] and its indifference to positivity of such weights [1, 6, 2]. However, here we deal with non-analytic densities by elaborating on the idea of extensions with controlled $\bar{\partial}$-derivative introduced by Miller and McLaughlin [16] and adapted to the setting of Jacobi-type polynomials by Baratchart and the author [4].

2. WEIGHT EXTENSION

Given $r > 1$, let $E_r := \{z : |\varphi(z)| < r\}$. The boundary ∂E_r is an ellipse with foci ± 1.

Proposition 1

Let $\rho(x)$ and ε_n be as in Theorem 1. For each $r > 1$ and $n > 2m$ there exists a continuous function $\ell_{n,r}(z) = l_n(z) + L_{n,r}(z)$, $z \in \mathbb{C}$, such that
$$\ell_{n,r}(x) = l_n(x), \quad x \in [-1, 1],$$
where $l_n(z)$ is a polynomial of degree at most n satisfying
$$\sup_{x \in [-1,1]} |l_n(x)| \leqslant C'_{\rho}$$
for some constant C'_{ρ} independent of n, while $L_{n,r}(z)$ and $\bar{\partial} L_{n,r}(z)$ are continuous functions in $\mathbb{C}$ supported by $\overline{E}_r$ (in particular, $L_{n,r}(z) = 0$ for $z \notin E_r$) and
$$\frac{|\bar{\partial} L_{n,r}(z)|}{\sqrt{|1 - z^2|}} \leqslant C''_{\rho} \frac{n \varepsilon_n}{\log n}, \quad z \in \overline{E}_r,$$
for some constant C''_{ρ} independent of n and r, where $\bar{\partial} := (\partial_x + i\partial_y)/2$, $z = x + iy$.

Proof. It follows from [14, Theorem 9] that for each $n > 2m$ there exists a polynomial $l_n(z)$ of degree at most n such that
$$\left| \left(\rho^{-1}(x) \right)^{(k)} - l_n^{(k)}(x) \right| \leqslant C_{m,k} (1 - x^2)^{\frac{m-k}{2}} n^{k-m} E_{n-m} \left(\left(\rho^{-1} \right)^{(m)} \right)$$
for all $x \in [-1, 1]$ and each $k \in \{0, \dots, m\}$, where $C_{m,k}$ is a constant that depends only m and k and $E_j(f)$ is the error of best uniform approximation on the interval $[-1, 1]$ of

a continuous function $f(x)$ by algebraic polynomials of degree at most j. Furthermore, it was shown by Timan, see [14, Equation (3)], that

$$
\begin{aligned}
E_{n-m}(f) &\leqslant C_1\omega\left(f;\frac{\sqrt{1-x^2}}{n-m}+\frac{1}{(n-m)^2}\right) \leqslant C_1\omega\left(f;\frac{2}{n-m}\right) \\
&\leqslant C_1\omega\left(f;\frac{4}{n}\right) \leqslant 4C_1\omega\left(f;\frac{1}{n}\right)
\end{aligned}
$$

for some absolute constant C_1, where we used that $n > 2m$ and $\omega(f;2h) \leqslant 2\omega(f;h)$ (in what follows, we understand that all constants C_j might depend on $\rho(x)$, but are independent of n). Set

$$
\lambda_n(x) := \frac{\rho^{-1}(x)-l_n(x)}{\sqrt{1-x^2}}, \quad x \in [-1,1].
$$

It then holds that $\lambda_n(x)$ is a continuous function on $[-1,1]$ that satisfies $\|\lambda_n\| \leqslant C_3\varepsilon_n/\log n$, where $\|\cdot\|$ is the uniform norm on $[-1,1]$. Since $m \geqslant 3$, it also holds that

$$
\lambda_n'(x) = \frac{(\rho^{-1}(x))'-l_n'(x)}{\sqrt{1-x^2}}+x\frac{\rho^{-1}(x)-l_n(x)}{\sqrt{(1-x^2)^3}}
$$

is a continuous function on $[-1,1]$ that satisfies $\|\lambda_n'\| \leqslant C_4 n\varepsilon_n/\log n$ (this is exactly the place where condition $m \geqslant 3$ is used). Extend $\lambda_n(x)$ by zero to the whole real line. As the numerator of $\lambda_n(x)$ together with its first and second derivatives vanishes at ± 1, $\lambda_n'(x)$ also extends continuously by zero to the whole real line. The following construction is standard, see [10, Proof of Theorem 3.67]. Define

$$
\Lambda_n(z) := \frac{1}{|y|}\int_0^{|y|}\lambda_n(x+t)dt, \quad z = x+\mathrm{i}y,
$$

which, due to continuity of $\lambda_n(x)$, is a continuous function in $\mathbb{C}$ satisfying $\Lambda_n(x) = \lambda_n(x)$ on the real line and $|\Lambda_n(z)| \leqslant \|\lambda_n\|$ in the complex plane. Similarly,

$$
\left|\partial_x\Lambda_n(z)\right| = \left|\frac{1}{|y|}\int_0^{|y|}\lambda_n'(x+t)dt\right| \leqslant \|\lambda_n'\|
$$

and the function $\partial_x\Lambda_n(z)$, which is given by the integral within the absolute value in the above equation, is also continuous in $\mathbb{C}$. Furthermore, we have that

$$
\begin{aligned}
\left|\partial_y\Lambda_n(z)\right| &= \left|\frac{1}{y^2}\int_0^{|y|}\left(\lambda_n(x+t)-\lambda_n(x+|y|)\right)dt\right| \\
&\leqslant \|\lambda_n'\|\int_0^{|y|}\frac{|y|-t}{y^2}dt = \frac{\|\lambda_n'\|}{2}
\end{aligned}
$$

and is also a continuous function in $\mathbb{C}$. Altogether, since $\bar{\partial} = (\partial_x+\mathrm{i}\partial_y)/2$, it holds that $\bar{\partial}\Lambda_n(z)$ is a continuous function in $\mathbb{C}$ that satisfies $|\bar{\partial}\Lambda_n(z)| \leqslant \|\lambda_n'\|$ in the complex plane. Let $\psi_r(z)$ be any real-valued continuous function with continuous partial derivatives that is equal to one on $[-1,1]$ and is equal to zero in the complement of E_r. Define

$$
L_{n,r}(z) := \mathrm{i}w(z)\Lambda_n(z)\psi_r(z)\begin{cases}-1, & \mathrm{Im}(z) \geqslant 0, \\ 1, & \mathrm{Im}(z) < 0.\end{cases}
$$

Since $w_\pm(x) = \pm\mathrm{i}\sqrt{1-x^2}$ for $x \in [-1,1]$ and $\Lambda_n(x) = 0$ for $x \notin (-1,1)$, it holds that $L_{n,r}(z)$ is a continuous function in $\mathbb{C}$ that is supported by $\overline{E}_r$ and is equal to $\rho^{-1}(x)-l_n(x)$

for $x \in [-1, 1]$. Furthermore, since $\bar{\partial}(\Lambda_n(z)\psi_n(z))$ is continuous in $\mathbb{C}$ and vanishes for $z = x \notin (-1, 1)$ while $w_+(x) = -w_-(x)$ for $x \in (-1, 1)$, $\bar{\partial}L_{n,r}(z)$ is also continuous in $\mathbb{C}$. Moreover, it holds that

$$
\begin{aligned}
|\bar{\partial}L_{n,r}(z)| &= \sqrt{|1 - z^2|} \, |\bar{\partial}(\Lambda_n(z)\psi_r(z))| \\
&\leqslant C_5\sqrt{|1 - z^2|} \left(|\Lambda_n(z)| + |\bar{\partial}\Lambda_n(z)|\right) \\
&\leqslant C_6\sqrt{|1 - z^2|} \, \frac{n\varepsilon_n}{\log n}, \quad z \in \overline{E}_r.
\end{aligned}
$$

Finally, observe that polynomials $l_n(x)$ approximate $\rho^{-1}(x)$ on $[-1, 1]$ and therefore have uniformly bounded above uniform norms. The claim of the proposition now follows by setting $\ell_{n,r}(z) := l_n(z) + L_{n,r}(z)$ for $l_n(z)$ and $L_{n,r}(z)$ as above. $\square$

3. Proof of Theorem 1

3.1. **Initial Riemann-Hilbert Problem.** Notice that the functions $v_i(x)$ and $|v_i(x)|$ are either equal to each other or differ by a sign when $x \in [-1, 1]$. So, we can equally use $v_i(x)$ in (1) without changing the polynomials $P_{n,i}(x)$.

Denote by $R_{n,i}(z)$ the function of the second kind associated with $P_{n,i}(z)$. That is,

$$
(10) \qquad R_{n,i}(z) := \frac{1}{2\pi i} \int_{-1}^{1} \frac{P_{n,i}(x)}{x - z} \frac{\rho(x)v_i(x)dx}{w_+(x)},
$$

which is a holomorphic function in $\overline{\mathbb{C}}\setminus[-1, 1]$. It follows from Plemelj-Sokhotski formulae, [13, Chapter I.4.2], that

$$
R_{n,i+}(x) - R_{n,i-}(x) = P_{n,i}(x)\frac{\rho(x)v_i(x)}{w_+(x)}, \quad x \in (-1, 1),
$$

and, see [13, Chapter I.8.4], that

$$
R_{n,i}(z) = O\left(|z - a|^{\alpha_{a,i}}\right) \quad \text{as} \quad \mathbb{C}\setminus[-1, 1] \ni z \to a \in \{-1, 1\},
$$

where $\alpha_{a,i} = 0$ if $v_i(a) = 0$ and $\alpha_{a,i} = -1/2$ otherwise. Moreover, we get from (1) that

$$
R_{n,i}(z) = \frac{1}{m_{n,i}z^n} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty
$$

for some finite constant $m_{n,i}$. Consider the following Riemann-Hilbert problem for 2×2 matrix functions (RHP-Y):

(a) $Y(z)$ is analytic in $\mathbb{C}\setminus[-1, 1]$ and $\lim_{z\to\infty} Y(z)z^{-n\sigma_3} = I$;

(b) $Y(z)$ has continuous traces on $(-1, 1)$ that satisfy

$$
Y_+(x) = Y_-(x)\begin{pmatrix} 1 & \frac{\rho(x)v_i(x)}{w_+(x)} \\ 0 & 1 \end{pmatrix};
$$

(c) $Y(z)$ behaves like

$$
Y(z) = O\begin{pmatrix} 1 & |z - a|^{\alpha_{a,i}} \\ 1 & |z - a|^{\alpha_{a,i}} \end{pmatrix} \quad \text{as} \quad \mathbb{C}\setminus[-1, 1] \ni z \to a \in \{-1, 1\}.
$$

The following lemma is well known [15, Theorem 2.4].

> **Lemma 1**
>
> RHP-Y is uniquely solvable by
>
> (11) $$Y(z) = \begin{pmatrix} P_{n,i}(z) & R_{n,i}(z) \\ m_{n-1,i} P_{n-1,i}(z) & m_{n-1,i} R_{n-1,i}(z) \end{pmatrix}.$$

3.2. **Opening of the Lens.** Fix $1 < r < R$ and orient ∂E_R clockwise. Set

$$
(12) \qquad X(z) := \begin{cases} Y(z) \begin{pmatrix} 1 & 0 \\ -\frac{w(z)\ell_{n,r}(z)}{v_i(z)} & 1 \end{pmatrix}, & \text{in} \quad E_R \setminus [-1,1], \\ Y(z), & \text{in} \quad \mathbb{C} \setminus \overline{E}_R, \end{cases}
$$

where $\ell_{n,r}(z)$ is the extension of $\rho^{-1}(x)$ constructed in Proposition 1. Observe that

$$\ell_{n,r}(s) = l_n(s), \quad s \in \partial E_R, \quad \text{and} \quad \bar\partial \ell_{n,r}(z) = \bar\partial L_{n,r}(z), \quad z \in \overline{E}_r,$$

since $L_{n,r}(z)$ is supported by $\overline{E}_r$ and $l_n(z)$ is analytic (in fact, is a polynomial). It is trivial to verify that $X(z)$ solves the following $\bar\partial$-Riemann-Hilbert problem ($\bar\partial$RHP-X):

(a) $X(z)$ is continuous in $\mathbb{C} \setminus ([-1,1] \cup \partial E_R)$ and $\lim_{z\to\infty} X(z) z^{-n\sigma_3} = I$;

(b) $X(z)$ has continuous traces on $(-1,1) \cup \partial E_R$ that satisfy

$$
X_+(s) = X_-(s) \begin{cases} \begin{pmatrix} 0 & \frac{\rho(s) v_i(s)}{w_+(s)} \\ -\frac{w_+(s)}{\rho(s) v_i(s)} & 0 \end{pmatrix} & \text{on} \quad s \in (-1,1), \\ \begin{pmatrix} 1 & 0 \\ \frac{w(s) l_n(s)}{v_i(s)} & 1 \end{pmatrix} & \text{on} \quad s \in \partial E_R; \end{cases}
$$

(c) $X(z)$ has the same behavior near ± 1 as $Y(z)$, see RHP-Y(c);

(d) $X(z)$ deviates from an analytic matrix function according to

$$\bar\partial X(z) = X(z) \begin{pmatrix} 0 & 0 \\ -\frac{w(z)\bar\partial L_{n,r}(z)}{v_i(z)} & 0 \end{pmatrix}.$$

One can readily verified that the following lemma holds, see [4, Lemma 6.4].

> **Lemma 2**
>
> $\bar\partial$RHP-X and RHP-Y are simultaneously solvable and the solutions are connected by (12).

3.3. **Model Riemann-Hilbert Problem.** In this subsection we present the solution of the following Riemann-Hilbert problem (RHP-N):

(a) $N(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$ and $\lim_{z\to\infty} N(z) z^{-n\sigma_3} = I$;

(b) $N(z)$ has continuous traces on $(-1,1)$ that satisfy

$$N_+(x) = N_-(s) \begin{pmatrix} 0 & \frac{\rho(x) v_i(x)}{w_+(x)} \\ -\frac{w_+(x)}{\rho(x) v_i(x)} & 0 \end{pmatrix};$$

(c) $N(z)$ has the same behavior near ± 1 as $Y(z)$, see RHP-Y(c).

Recall the definition of the functions $S_i(z)$ in (7). Define $S_*(z) := S_i(z)$ when $i \in \{1,3\}$ and $S_*(z) := iS_i(z)$ when $i \in \{2,4\}$. Then it follows from (8) that

$$S_{*+}(x)S_{*-}(x) = 1/v_i(x), \quad x \in (-1,1).$$

Let $S(z)$ and $\varphi(z)$ be given by (2) and (5), respectively. It follows from (3) and (6) that

$$(S_*S\varphi^n)_-^{\sigma_3}(x) \begin{pmatrix} 0 & \frac{\rho(x)v_i(x)}{w_+(x)} \\ -\frac{w_+(x)}{\rho(x)v_i(x)} & 0 \end{pmatrix} (S_*S\varphi^n)_+^{-\sigma_3}(x) = \begin{pmatrix} 0 & 1/w_+(x) \\ -w_+(x) & 0 \end{pmatrix}$$

for $x \in (-1,1)$. It also can be readily verified with the help of (6) that

$$\begin{pmatrix} 1 & \frac{1}{w_+(x)} \\ \frac{1}{2\varphi_+(x)} & \frac{\varphi_+(x)}{2w_+(x)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{w_-(x)} \\ \frac{1}{2\varphi_-(x)} & \frac{\varphi_-(x)}{2w_-(x)} \end{pmatrix} \begin{pmatrix} 0 & 1/w_+(x) \\ -w_+(x) & 0 \end{pmatrix}$$

for $x \in (-1,1)$. Therefore, RHP-N is solved by $N(z) = CM(z)$, where

$$(13) \qquad C := (2^n S_* S)^{-\sigma_3}(\infty) \quad \text{and} \quad M(z) := \begin{pmatrix} 1 & \frac{1}{w(z)} \\ \frac{1}{2\varphi(z)} & \frac{\varphi(z)}{2w(z)} \end{pmatrix} (S_* S\varphi^n)^{\sigma_3}(z).$$

3.4. Analytic Approximation.

To solve $\bar{\partial}$RHP-X, we first solve its analytic version. That is, consider the following Riemann-Hilbert problem (RHP-A):

(a) $A(z)$ is analytic in $\mathbb{C}\backslash([-1,1] \cup \partial E_R)$ and $\lim_{z\to\infty} A(z)z^{-n\sigma_3} = I$;
(b,c) $A(z)$ satisfies $\bar{\partial}$RHP-X(b,c).

Lemma 3

For all n large enough there exists a matrix $Z(z)$, analytic in $\overline{\mathbb{C}}\backslash\partial E_R$ and satisfying
$$Z(z) = I + O\left(R_*^{-n}\right)$$
uniformly in $\overline{\mathbb{C}}$ for any $r < R_* < R$, such that $A(z) = CZ(z)M(z)$ solves RHP-A.

Proof. Assume that there exists a matrix $Z(z)$ that is analytic in $\overline{\mathbb{C}}\backslash\partial E_R$, is equal to I at infinity, and satisfies

$$Z_+(s) = Z_-(s)M(s) \begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix} M^{-1}(s), \quad s \in \partial E_R.$$

It can be readily verified that $A(z) = CZ(z)M(z)$ solves RHP-A. To show that such $Z(z)$ does indeed exist, observe that

$$\det M(z) = \frac{\varphi(z)}{2w(z)} - \frac{1}{2\varphi(z)w(z)} \equiv 1$$

in the entire complex plane and that

$$v_i(z)S_*^2(z) = (-1)^{i-1}\varphi^{k_i}(z), \quad z \notin [-1,1],$$

straight by the definition of $S_i(z)$ in (7), where $k_1 = 0$, $k_2 = 2$, and $k_3 = k_4 = 1$. Thus,

$$(14) \quad M(s) \begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix} M^{-1}(s) = I + \frac{(-1)^{i-1}l_n(s)}{w(s)S^2(s)\varphi^{2n+k_i}(s)} \begin{pmatrix} \frac{1}{2}\varphi(s) & -1 \\ \frac{1}{4}\varphi^2(s) & -\frac{1}{2}\varphi(s) \end{pmatrix}$$

for $s \in \partial E_R$. It follows from the very definition of E_R that $|\varphi(s)| = R$ for $s \in \partial E_R$. Moreover, since $\deg(l_n) \leqslant n$ and the uniform norms on $[-1, 1]$ of these polynomials are bounded by C'_ρ, see Proposition 1, it holds that

$$|l_n(s)| \leqslant C'_\rho |\varphi(s)|^n = C'_\rho R^n, \quad s \in \partial E_R,$$

by the Bernstein-Walsh inequality. Hence, we can conclude that the jump of $\boldsymbol{Z}(z)$ on ∂E_R can be estimated as $\boldsymbol{I} + \boldsymbol{O}(R^{-n})$. It now follows from [7, Theorem 7.18 and Corollary 7.108] that such $\boldsymbol{Z}(z)$ does exist, is unique, and has continuous traces on ∂E_R whose L^2-norms with respect to the arclength measure are of size $O(R^{-n})$. This yields the desired pointwise estimate of $\boldsymbol{Z}(z)$ locally uniformly in $\overline{\mathbb{C}} \backslash \partial E_R$. Next, observe that the jump of $\boldsymbol{Z}(s)$ is analytic around ∂E_R and therefore we can vary the value of R. Since the solutions corresponding to different values of R are necessarily analytic continuations of each other, the desired uniform estimate follows from the locally uniform ones for any fixed $R_* < R$ and $R' > R$.   □

3.5. **An Auxiliary Estimate.** Denote by dA the area measure and by $\mathcal{K}$ the Cauchy area operator acting on integrable functions on $\mathbb{C}$, i.e.,

$$(15) \qquad \mathcal{K}f(z) = \frac{1}{\pi} \iint \frac{f(s)}{z - s} dA.$$

> **Lemma 4**
>
> Let $u(z)$ be a bounded function supported on $\overline{E}_r$. Then
>
> $$\left\| \mathcal{K}\big(u|\varphi|^{-2n}\big) \right\| \leqslant C_r \frac{\log n}{n} \|u\|,$$
>
> where $\| \cdot \|$ is the essential supremum norm and the constant C_r is independent of n.

Proof. Observe that the integrand is a bounded compactly supported function and therefore its Cauchy area integral is Hölder continuous in $\mathbb{C}$ with any index $\alpha < 1$, see [3, Theorem 4.3.13]. Moreover, since the integral is analytic in $\overline{\mathbb{C}} \backslash \overline{E}_r$, the maximum of its modulus is achieved on $\overline{E}_r$. Notice also that it is enough to prove the claim of the lemma only for $u(z) = \chi_{E_r}(z)$, the indicator function of E_r.

Let $z \in \overline{E}_r$. Observe that $\varphi(s) = \tau$ when $s = \frac{1}{2}(\tau + 1/\tau)$. Write $z = \frac{1}{2}(\xi + 1/\xi)$. Then

$$\left| \mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}} \right)(z) \right| \leqslant \frac{1}{\pi} \iint_{E_r} \frac{1}{|z - s|} \frac{dA}{|\varphi(s)|^{2n}}$$

$$= \frac{1}{\pi} \iint_{1 < |\tau| < r} \frac{|\tau^2 - 1|^2}{|(\xi - \tau)(1 - 1/(\tau\xi))|} \frac{dA}{|\tau|^{2n+4}}.$$

Partial fraction decomposition now yields

$$\left| \mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}} \right)(z) \right| \leqslant \frac{1}{\pi} \iint_{1 < |\tau| < r} \left| \frac{\xi}{\tau - \xi} + \frac{\tau}{\tau - 1/\xi} \right| \frac{|\tau^2 - 1|}{|\tau|^{2n+4}} dA$$

$$\leqslant \frac{2r^3}{\pi} \iint_{1 < |\tau| < r} \left(\frac{1}{|\tau - \xi|} + \frac{1}{|\tau - 1/\xi|} \right) \frac{dA}{|\tau|^{2n+4}}.$$

Write $\tau = \varrho e^{i\theta}$ and $\xi = \varrho_* e^{i\theta_*}$. Then

$$
\begin{aligned}
|\tau - \xi| &= \sqrt{(\varrho - \varrho_*)^2 + 4\varrho\varrho_* \sin^2\left(\frac{\theta - \theta_*}{2}\right)} \\
&\geqslant \frac{1}{\sqrt{2}}\left(|\varrho - \varrho_*| + \sqrt{\varrho\varrho_*}\left|2\sin\left(\frac{\theta - \theta_*}{2}\right)\right|\right) \\
&\geqslant C\left(|\varrho - \varrho_*| + |\theta - \theta_*|\right)
\end{aligned}
$$

for some constant $C < 1/\sqrt{2}$, where on the last step we used inequalities $\varrho\varrho_* \geqslant 1$ and $\min_{[-\pi/2,\pi/2]}|\sin x/x| > 0$. Since $\varrho/\varrho_* \geqslant 1/r$, the constant C can be adjusted so that

$$
|\tau - 1/\xi| \geqslant C\left(|\varrho - 1/\varrho_*| + |\theta + \theta_*|\right) \geqslant C\left(|\varrho - \varrho_*| + |\theta + \theta_*|\right)
$$

is true as well. By going to polar coordinates and applying the above estimates we get that

$$
\begin{aligned}
\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leqslant \frac{4r^3}{\pi C}\int_1^r\left(\int_0^\pi \frac{d\theta}{|\varrho - \varrho_*| + \theta}\right)\frac{d\varrho}{\varrho^{2n+3}} \\
&= \frac{4r^3}{\pi C}\left(\int_{I_1} + \int_{I_2}\right)\log\left(1 + \frac{\pi}{|\varrho - \varrho_*|}\right)\frac{d\varrho}{\varrho^{2n+3}} =: S_1 + S_2,
\end{aligned}
$$

where $I_1 = (1, r) \cap \{\varrho : |\varrho - \varrho_*| < \pi/n\}$ and $I_2 = (1, r)\backslash I_1$. Then

$$
\begin{aligned}
S_1 &\leqslant \frac{8r^3}{\pi C}\int_0^{\pi/n}\log\left(1 + \frac{\pi}{\varrho}\right)d\varrho = \frac{8r^3}{C}\int_{n+1}^\infty \frac{\log t\,dt}{(t-1)^2} \\
&= \frac{8r^3}{C}\left(\frac{\log(n+1)}{n} + \int_{n+1}^\infty \frac{dt}{t(t-1)}\right) \leqslant \frac{8r^3}{C}\frac{\log(n+1) + 1}{n}.
\end{aligned}
$$

Finally, it holds that

$$
S_2 \leqslant \frac{8r^3\log(n+1)}{\pi C}\int_1^\infty \frac{d\varrho}{\varrho^{2n+3}} = \frac{4r^3}{\pi C}\frac{\log(n+1)}{n+1},
$$

which finishes the proof of the lemma. $\qquad\square$

3.6. $\bar{\partial}$-Problem. Consider the following $\bar{\partial}$-problem ($\bar{\partial}$P-D):

(a) $D(z)$ is a continuous matrix function on $\overline{\mathbb{C}}$ and $D(\infty) = I$;
(b) $D(z)$ satisfies $\bar{\partial}D(z) = D(z)W(z)$, where

$$
W(z) := Z(z)M(z)\begin{pmatrix} 0 & 0 \\ -w(z)\bar{\partial}L_{n,r}(z)/v_i(z) & 0 \end{pmatrix}M^{-1}(z)Z^{-1}(z).
$$

Notice that $W(z)$ is supported by $\overline{E}_r$ and therefore $D(z)$ is necessarily analytic in the complement of $\overline{E}_r$.

> **Lemma 5**
>
> The solution of $\bar{\partial}$P-D exists for all n large enough and it holds uniformly in $\overline{\mathbb{C}}$ that
> $$
> D(z) = I + O(\varepsilon_n).
> $$

Proof. As explained in [4, Lemma 8.1], solving $\bar{\partial}$P-D is equivalent to solving an integral equation

$$
I = (\mathcal{I} - \mathcal{K}_W)D(z)
$$

in the space of bounded matrix functions on $\mathbb{C}$, where $\mathcal{I}$ is the identity operator and $\mathcal{K}_W$ is the Cauchy area operator (15) acting component-wise on the product $\boldsymbol{m}(s)\boldsymbol{W}(s)$ for a bounded matrix function $\boldsymbol{m}(z)$. If $\|\|\mathcal{K}_W\|\|$, the operator norm of $\mathcal{K}_W$, is less than $1 - \epsilon$, $\epsilon \in (0, 1)$, then $(\mathcal{I} - \mathcal{K}_W)^{-1}$ exists as a Neumann series and

$$\boldsymbol{D}(z) = (\mathcal{I} - \mathcal{K}_W)^{-1}\boldsymbol{I} = \boldsymbol{I} + \boldsymbol{O}_\epsilon\left(\|\|\mathcal{K}_W\|\|\right)$$

uniformly in $\overline{\mathbb{C}}$ (it also holds that $\boldsymbol{D}(z)$ is Hölder continuous in $\mathbb{C}$). It follows from Lemma 4 that to estimate $\|\|\mathcal{K}_W\|\|$, we need to estimate L^∞-norms of the entries of $\boldsymbol{W}(z)$. To this end, similarly to (14), we get that

$$\boldsymbol{W}(z) = \frac{(-1)^i \bar{\partial} L_{n,r}(z)}{w(z)S^2(z)\varphi^{2n+k_i}(z)}\boldsymbol{Z}(z)\begin{pmatrix} \frac{1}{2}\varphi(z) & -1 \\ \frac{1}{4}\varphi^2(z) & -\frac{1}{2}\varphi(z) \end{pmatrix}\boldsymbol{Z}^{-1}(z), \quad z \in \overline{E}_r.$$

Using Proposition 1 and Lemma 3 we can conclude that entries of $\boldsymbol{W}(z)$ are continuous functions on $\mathbb{C}$ supported by $\overline{E}_r$ with absolute values bounded above by $C_\rho|\varphi(z)|^{-2n}n\varepsilon_n/\log n$ for some constant C_ρ independent of n. Hence, $\|\|\mathcal{K}_W\|\| = O(\varepsilon_n)$ as claimed. $\qquad\square$

3.7. Asymptotic Formulae.

It readily follows from RHP-$\boldsymbol{A}$ and $\bar{\partial}$P-$\boldsymbol{D}$ as well as Lemmas 3 and 5 that $\bar{\partial}$RHP-$\boldsymbol{X}$ is solved by

$$\boldsymbol{X}(z) = \boldsymbol{C}\boldsymbol{D}(z)\boldsymbol{Z}(z)\boldsymbol{M}(z).$$

Given a closed set $B \subset \overline{\mathbb{C}}\setminus[-1, 1]$, we can choose r amd R so that $\overline{E}_R \cap B = \varnothing$. Then it holds that $\boldsymbol{Y}(z) = \boldsymbol{X}(z)$ for $z \in B$ by (12). Write

$$\boldsymbol{D}(z)\boldsymbol{Z}(z) = \boldsymbol{I} + \begin{pmatrix} \upsilon_{n1}(z) & \upsilon_{n2}(z) \\ \upsilon_{n3}(z) & \upsilon_{n4}(z) \end{pmatrix}.$$

It follows from Lemmas 3 and 5 that $|\upsilon_{nj}(z)| = O(\varepsilon_n)$ uniformly in $\overline{\mathbb{C}}$ and that $\upsilon_{nj}(\infty) = 0$. Then we get from (11) and (13) that

$$P_n(z) = \left(1 + \upsilon_{n1}(z) + \frac{\upsilon_{n2}(z)}{2\varphi(z)}\right)\frac{(S_*S)(z)}{(S_*S)(\infty)}\left(\frac{\varphi(z)}{2}\right)^n, \quad z \in B.$$

Since $S_*(z)/S_*(\infty) = S_i(z)/S_i(\infty)$, the first claim of the theorem follows. Next, notice that the first column of $\boldsymbol{Y}(z)$ is entire and is equal to the first column of

$$\boldsymbol{X}_+(x)\begin{pmatrix} 1 & 0 \\ w_+(x)/(\rho(x)v_i(x)) & 1 \end{pmatrix}$$

for $x \in [-1, 1]$ by (12) and Proposition 1. Since the functions $\upsilon_{ni}(z)$ are continuous across $[-1, 1]$ and $S_{*\pm}(x)/S_*(\infty) = S_{i\pm}(x)/S_i(\infty)$, we deuce from (3), (6), (8), and (13) that

$$P_n(x) = (1 + \upsilon_{n1}(x))\frac{(S_iS\varphi^n)_+(x) + (S_iS\varphi^n)_-(x)}{2^n(S_iS)(\infty)} +$$

$$\upsilon_{n2}(x)\frac{(S_iS\varphi^{n-1})_+(x) + (S_iS\varphi^{n-1})_-(x)}{2^{n+1}(S_iS)(\infty)}$$

for any $x \in [-1, 1]$. It now follows from (4), (6), and (8) that

$$(S_iS\varphi^k)_+(x) + (S_iS\varphi^k)_-(x) = \frac{2\cos\left(k\arccos(x) + \theta(x) + \theta_i(x)\right)}{\sqrt{\rho(x)|v_i(x)|}}, \quad x \in [-1, 1].$$

The last two formulae now yield the second claim of the theorem. Finally, it is known, see [15, Equations (9.6) and (9.7)], that

$$\begin{cases} a_{n,i}^2 & = \lim_{z\to\infty} z^2 [Y(z)]_{12}[Y(z)]_{21}, \\ b_{n,i} & = \lim_{z\to\infty} (z - P_{n+1,i}(z)[Y(z)]_{22}), \end{cases}$$

where $Y(z)$ corresponds to the index n. As in the first part of the proof, we get that

$$[Y(z)]_{12} = [X(z)]_{12} = \frac{1}{w(z)} \frac{1 + \upsilon_{n1}(z) + \upsilon_{n2}(z)\varphi(z)/2}{2^n (S_*S)(\infty)(S_*S)(z)\varphi^n(z)}$$

and

$$[Y(z)]_{21} = [X(z)]_{21} = \left(\upsilon_{n3}(z) + \frac{1 + \upsilon_{n4}(z)}{2\varphi(z)} \right) 2^n (S_*S)(\infty)(S_*S)(z)\varphi^n(z)$$

for all z large. Since $\upsilon_{nj}(\infty) = 0$, it holds that

$$a_{n,i}^2 = \frac{1}{4} + \lim_{z\to\infty} z\upsilon_{n3}(z)(1 + z\upsilon_{n2}(z)) = \frac{1}{4} + O(\varepsilon_n)$$

by the maximum modulus principle for holomorphic functions. Similarly, we have that

$$[Y(z)]_{22} = [X(z)]_{22} = \left(\upsilon_{n3}(z) + \frac{1}{2}(1 + \upsilon_{n4}(z))\varphi(z) \right) \frac{1}{w(z)} \frac{2^n (S_*S)(\infty)}{(S_*S)(z)\varphi^n(z)}$$

for all z large. Hence,

$$P_{n+1,i}(z)[Y(z)]_{22} = \frac{\varphi^2(z)}{4w(z)} \left(1 + \upsilon_{n+11}(z) + \frac{\upsilon_{n+12}(z)}{2\varphi(z)} \right) \left(1 + \upsilon_{n4}(z) + 2\frac{\upsilon_{n3}(z)}{\varphi(z)} \right)$$

in this case. It can be readily verified that

$$\frac{\varphi^2(z)}{4w(z)} = z + \frac{z}{2w(z)(z + w(z))} - \frac{1}{4w(z)} = z + O\left(\frac{1}{z} \right)$$

as $z \to \infty$. Therefore,

$$b_{n,i} = - \lim_{z\to\infty} z\big(\upsilon_{n+11}(z) + \upsilon_{n4}(z)\big) = O(\varepsilon_n)$$

again, by the maximum modulus principle for holomorphic functions. This finishes the proof of the theorem.

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